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THE TOPOLOGICAL EXTENSION OF A PRODUCT

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1. **Introduction.** If E is a topological space, then according to Mrowka in [7], a space X is E-completely regular if X can be embedded as a subspace of a topological product of copies of E, and X is E-compact if X can be embedded as a closed subspace of a product of copies of E. The following is [7, Theorem 4.14].

1.1 THEOREM. *For every E-completely regular space X there exists a superspace $\beta_E X$ of X in which X is dense, such that $\beta_E X$ is E-compact and every continuous function $f: X \to E$ admits a continuous extension $f^E: \beta_E X \to E$.*

According to Herrlich and van der Slot in [6], and Woods in [11], if $\mathscr{P}$ is a topological property such that:

i) all compact Hausdorff spaces satisfy $\mathscr{P}$;

ii) $\mathscr{P}$ is closed hereditary; and

iii) $\mathscr{P}$ is preserved under the formation of topological products, then every completely regular Hausdorff space X has a maximal $\mathscr{P}$- extension; i.e. for every c.r.H. (completely regular Hausdorff) space X there exists a superspace $\mathscr{P}X$ such that X is dense in $\mathscr{P}X$, $\mathscr{P}X$ satisfies $\mathscr{P}$, and every continuous map $f: X \to Y$ where Y satisfies $\mathscr{P}$, admits a continuous extension $f^{\mathscr{P}}: \mathscr{P}X \to Y$. Furthermore $X \subseteq \mathscr{P}X \subseteq \beta X$ where βX is the Stone-Čech compactification of X.

In this paper we characterize, for certain extension properties $\mathscr{P}$, and for $E=\{0, 1\}$, the two point discrete space, those spaces $X \times Y$ for which the relations $\mathscr{P}(X \times Y) = \mathscr{P}X \times \mathscr{P}Y$ and $\beta_E(X \times Y) = \beta_E X \times \beta_E Y$ hold. Some results for infinite products are also obtained. In particular we show that for a large family of extension properties $\mathscr{P}$, $\mathscr{P}(X \times Y) = \mathscr{P}X \times \mathscr{P}Y$ iff $X \times Y$ is pseudocompact. In section 2 it is shown that for $E=\{0, 1\}$, $\beta_E(\prod_\alpha X_\alpha) = \prod_\alpha \beta_E(X_\alpha)$ iff $\prod_\alpha X_\alpha$ is pseudocompact. Also an example is given (assuming the continuum hypothesis) of an extension property $\mathscr{P}$ such that $\mathscr{P}(X \times Y) = \mathscr{P}X \times \mathscr{P}Y$ iff $X \times Y$ is pseudocompact but the pseudocompactness of $\prod_\alpha X_\alpha$ does not imply $\mathscr{P}(\prod_\alpha X_\alpha) = \prod_\alpha \mathscr{P}X_\alpha$.

The following two theorems are relied on for many of the results of this paper. The first is [3, Theorem 2.1] due to Frolik, and the second is [5, Theorem 1] due to Glicksberg.

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1.2 Theorem. *The following conditions on infinite c.r.H. spaces X and Y are equivalent.*

1) *The topological product $X\times Y$ is pseudocompact.*

2) $\beta(X\times Y)=\beta X\times\beta Y$, *i.e. every bounded continuous real-valued function on $X\times Y$ admits a continuous extension to $\beta X\times\beta Y$.*

1.3 Theorem. *Let $\{X_\alpha\}_{\alpha\in A}$ be a set of c.r.H. spaces and suppose $\prod_{\alpha\neq\alpha_0}X_\alpha$ is infinite for every $\alpha_0\in A$. Then the following are equivalent.*

1) *The topological product $\prod X_\alpha$ is pseudocompact.*

2) $\beta(\prod_{\alpha\in A}X_\alpha)=\prod_{\alpha\in A}\beta X_\alpha$.

All spaces discussed are assumed to be c.r.H. The notation will be that of [4] except that by stating that a space is 0-dimensional, it is meant that the space has a base of clopen sets. Also, N will denote the set of natural numbers.

2. **The result for β_0.** If $E=\mathbf{2}=\{0, 1\}$, the two point discrete space, then the E-completely regular spaces are the c.r.H. 0-dimensional spaces, and the E-compact spaces are the compact Hausdorff O-dimensional spaces. For notational convenience, β_2X will be denoted by β_0X for a O-dimensional space X. Note that β_0X is the Stone space of the Boolean algebra of clopen subsets of X.

In this section it is shown that Theorems 1.2 and 1.3 in the introduction remain valid, in the case of O-dimensional spaces, with β replaced by β_0. The following is [3, Lemma 1.2] of Frolik, which will be required, and is stated without proof.

2.1 Lemma. *Let X and Y be infinite spaces. If the topological product $X\times Y$ is not pseudocompact then there exists a locally finite sequence $\{U_n\times V_n\}$ of non-empty canonical open subsets of $X\times Y$ such that the sequences $\{U_n\}$ and $\{V_n\}$ are disjoint in X and Y respectively.*

2.2 Theorem. *The following conditions on infinite O-dimensional spaces X and Y are equivalent.*

1) *The topological product $X\times Y$ is pseudocompact.*

2) $\beta_0(X\times Y)=\beta_0X\times\beta_0Y$, *that is every continuous $\{0, 1\}$-valued function on $X\times Y$ admits a continuous extension to $\beta_0X\times B_0Y$.*

Proof. 1) implies 2). This implication easily follows from [3, Lemma 1.4]. However a proof is given here, the techniques of which will be required in the proof of the theorem in the infinite product case.

If $i:X\to X$ is the identity map, then there is an extension $i^\beta:\beta X\to\beta_0X$. Let $f:X\to\{0, 1\}$ be a continuous map. Then there exist continuous extensions $f^\beta:\beta X\to\{0, 1\}$ and $f^0:\beta_0X\to\{0, 1\}$. Clearly $f^\beta=f^0\circ i^\beta$. If $p\in\beta_0X$, then $f^\beta(i^\beta\leftarrow(p))=f^0\circ i^\beta(i^\beta\leftarrow(p))=f^0(p)$, and thus f^β is constant on $i^\beta\leftarrow(p)$ for all $p\in\beta_0X$. If i_X^β: $\beta X\to\beta_0X$ and $i_Y^\beta:\beta Y\to\beta_0Y$ (where i_X, i_Y are the identity maps on X and Y respectively) then $h=i_X^\beta\times i_Y^\beta:\beta X\times\beta Y\to\beta_0X\times\beta_0Y$ is a quotient map. Let $f:X\times Y\to\{0, 1\}$ be continuous. By hypothesis f has an extension $f^\beta:\beta X\times\beta Y\to\{0, 1\}$. Let

$z=(p,q)\in\beta_0X\times\beta_0Y$. Suppose (p_1,q_1), $(p_2,q_2)\in h^{\leftarrow}(z)=i_X^{\beta\leftarrow}(p)\times i_Y^{\beta\leftarrow}(q)$. Since $\beta X\times\{q_1\}$ is homeomorphic to $\beta X, f^\beta$ must be constant on $i_X^{\beta\leftarrow}(p)\times\{q_1\}$ as remarked above. Thus $f^\beta(p_1,q_1)=f^\beta(p_2,q_1)$. Similarly, f^β is constant on $\{p_2\}\times i_Y^{\beta\leftarrow}(q)$ and $f^\beta(p_2,q_1)=f^\beta(p_2,q_2)$, and thus f^β is constant on $h^{\leftarrow}(z)$ for all $z\in\beta_0X\times\beta_0Y$. So $f^0:\beta_0X\times\beta_0Y\to\{0,1\}$ defined by $f^0(p,q)=f^\beta(i_X^{\beta\leftarrow}(p)\times i_Y^{\beta\leftarrow}(q))$ is well-defined. In addition f^0 is continuous as h is a quotient map. Clearly f^0 is an extension of f.

2) implies 1). Suppose $X\times Y$ is not pseudocompact. By 2.1 above there is a locally finite sequence $\{U_n\times V_n\}$ of canonical open subsets of $X\times Y$ such that $\{U_n\}$ and $\{V_n\}$ are pairwise disjoint sequences. Since X and Y are O-dimensional, U_n and V_n can be taken to be clopen for all n. Let $U=\bigcup_{n\in N}U_n\times V_n$. Clearly U is open. U is also closed since $\{U_n\times V_n\}$ is a locally finite sequence and each $U_n\times V_n$ is clopen.

$\beta_0X\times\beta_0Y$ is pseudocompact, so the sequence $\{\mathrm{cl}_{\beta_0X}U_n\times\mathrm{cl}_{\beta_0Y}V_n\}$ must have a cluster point (p,q) in $\beta_0X\times\beta_0Y$. Let $V=(X\times Y)-U$. Suppose $A\times B$ is a nbhd. of (p,q) in $\beta_0X\times\beta_0Y$. Then there are $n_1, n_2\in N$ such that $n_1\neq n_2$ and $A\cap U_{n_i}\neq\varnothing$, $B\cap V_{n_i}\neq\varnothing$ for $i=1,2$. Let $x_i\in A\cap U_{n_i}$, $y_i\in B\cap V_{n_i}$ for $i=1,2$. Then $(x_1,y_2)\in A\times B$. But $(x_1,y_2)\notin U$, for if $(x_1,y_2)\in U$, then there is an $n\in N$ such that $(x_1,y_2)\in U_n\times V_n$. Hence $x_1\in U_n\cap U_{n_1}$, $y_2\in V_n\cap V_{n_2}$. Thus $n_1=n=n_2$ which is a contradiction to the assumption $n_1\neq n_2$. So $(x_1,y_2)\notin U$, and hence $(p,q)\in\mathrm{cl}_{\beta_0X\times\beta_0Y}V$.

Since U is clopen in $X\times Y$, the function $f:X\times Y\to\{0,1\}$ defined by $f(U)=\{0\}$, $f(V)=\{1\}$ is continuous. But $(p,q)\in(\mathrm{cl}\,U)\cap(\mathrm{cl}\,V)$, hence f cannot extend continuously to (p,q). Thus $\beta_0(X\times Y)\neq\beta_0X\times\beta_0Y$ contradicting 2). Therefore $X\times Y$ is pseudocompact. □

We are now in a position to prove the following theorem.

2.3 Theorem. *Let $\{X_\alpha\}_{\alpha\in A}$ be a set of O-dimensional spaces and suppose that $\prod_{\alpha\neq\alpha_0}X_\alpha$ is infinite for every $\alpha_0\in A$. Then the following are equivalent.*

1) *The topological product $\prod_{\alpha\in A}X_\alpha$ is pseudocompact.*

2) $\beta_0(\prod_{\alpha\in A}X_\alpha)=\prod_{\alpha\in A}\beta_0X_\alpha$.

Proof. 2) implies 1). In view of 2.2, the proof of this implication is identical to that given by Glicksberg in [5, Theorem 1] for the β case.

1) implies 2). As in 2.2, there is a map $i_\alpha^\beta:\beta X_\alpha\to\beta_0X_\alpha$ for each $\alpha\in A$. Let $h=\prod_{\alpha\in A}i_\alpha^\beta:\prod_{\alpha\in A}\beta X_\alpha\to\prod_{\alpha\in A}\beta_0X_\alpha$. Suppose $f:\prod X_\alpha\to\{0,1\}$ is a continuous map. By hypothesis and 1.3 there is a continuous extension $f^\beta:\prod\beta X_\alpha\to\{0,1\}$. Let $(p_\alpha)_{\alpha\in A}\in\prod\beta_0X_\alpha$. Then $h^{\leftarrow}((p_\alpha)_\alpha)=\prod_{\alpha\in A}i_\alpha^{\beta\leftarrow}(p_\alpha)$. Suppose $(q_\alpha)_{\alpha\in A}$, $(q'_\alpha)_{\alpha\in A}\in h^{\leftarrow}((p_\alpha)_\alpha)$. Then $i_\alpha^\beta(q_\alpha)=i_\alpha^\beta(q'_\alpha)=p_\alpha$ for all $\alpha\in A$. Fix $\alpha_0\in A$. As in 2.2 f^β is constant on $[i_\alpha^{\beta\leftarrow}(p_{\alpha_0})]\times\prod_{\alpha\neq\alpha_0}\{q_\alpha\}$. Thus $f^\beta((q_\alpha)_\alpha)=f^\beta((q'_{\alpha_0}\times\prod_{\alpha\neq\alpha_0}q_\alpha))$. By induction on the number of coordinates, it is easily seen that for any finite number of coordinates $\alpha_1,\ldots,\alpha_n\in A$, $f^\beta((q_\alpha)_\alpha)=f^\beta(q'_{\alpha_1}\times\ldots\times q'_{\alpha_n}\times\prod_{\substack{\alpha\neq\alpha_i\\ i=1\ldots,n}}q_\alpha)$. Since the points of form

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on the right of this equation form a net in $\prod \beta X_\alpha$ converging to $(q'_\alpha)_{\alpha\in A}$, by continuity of f^β we must have $f^\beta((q'_\alpha)_\alpha)=f^\beta((q_\alpha)_\alpha)$. Thus f^β is constant on $h^\leftarrow(z)$ for every $z\in\prod\beta_0X_\alpha$. If $f^0:\prod\beta_0X_\alpha\to\{0, 1\}$ is defined by $f^0((p_\alpha)_\alpha)=f^\beta(h^\leftarrow((p_\alpha)_\alpha))$, then f^0 is well-defined. In addition, f^0 is continuous as h is a quotient map. Clearly f^0 is an extension of f. □

As is shown by Pierce in [8, p. 375], there is nothing to be gained by trying to replace pseudocompactness with a O-dimensional analogue in 2.2 and 2.3.

If E is any compact Hausdorff space, then a simple modification of the proof of 1) implies 2) in 2.2 and 2.3 shows that the implication remains correct if β_0 is replaced by β_E, and O-dimensionality is replaced by E-complete regularity.

3. **Further results.** In this section extension properties of the type mentioned in the introduction are dealt with. Recall that if $\mathscr{P}$ is an extension property then $X\subseteq \mathscr{P}X\subseteq\beta X$. Note that if $\mathscr{P}$ and $\mathscr{Q}$ are extension properties such that $\mathscr{P}$ is contained in $\mathscr{Q}$ (i.e. every space which satisfies $\mathscr{P}$ also satisfies $\mathscr{Q}$) then $X\subseteq\mathscr{Q}X\subseteq \mathscr{P}X\subseteq\beta X$. As an example, if $\mathscr{P}$ = compactness and $\mathscr{Q}$ = realcompactness, then $\mathscr{P}X=\beta X$, $\mathscr{Q}X=\nu X$ and $X\subseteq\nu X\subseteq\beta X$.

The following result which is mentioned by Comfort in [2] has some interesting consequences.

3.1 Proposition. *Suppose $\mathscr{P}$ and $\mathscr{Q}$ are extension properties such that $\mathscr{P}$ is contained in $\mathscr{Q}$. If for two spaces X and Y, $\mathscr{P}(X\times Y)=\mathscr{P}X\times\mathscr{P}Y$, then $\mathscr{Q}(X\times Y)=\mathscr{Q}X\times\mathscr{Q}Y$. (By $\mathscr{P}(X\times Y)=\mathscr{P}X\times\mathscr{P}Y$ it is meant that every continuous map from $X\times Y$ to a space satisfying $\mathscr{P}$ admits an extension to $\mathscr{P}X\times\mathscr{P}Y$. Equivalently, it means there is a homeomorphism between $\mathscr{P}(X\times Y)$ and $\mathscr{P}X\times\mathscr{P}Y$ which fixes $X\times Y$ pointwise.)*

Proof. Let $f:X\times Y\to Z$ be a continuous map where Z satisfies $\mathscr{Q}$. Let $f_y=f\mid X\times\{y\}$ for each $y\in Y$. Since $\mathscr{P}Z$ satisfies $\mathscr{P}$, there is an extension $f^{\mathscr{P}}:\mathscr{P}X\times\mathscr{P}Y\to\mathscr{P}Z$ by hypothesis. Since $X\times\{y\}$ is homeomorphic to X for every $y\in Y$, every f_y has an extension $f_y^{\mathscr{Q}}:\mathscr{Q}X\times\{y\}\to Z$. It is clear by continuity of maps that $f^{\mathscr{P}}\mid\mathscr{Q}X\times\{y\}=f_y^{\mathscr{Q}}$ for all $y\in Y$.

Thus $f^{\mathscr{P}}(\mathscr{Q}X\times Y)\subseteq Z$. By a repetition of this argument for points in $\mathscr{Q}X$, it is seen that $f^{\mathscr{P}}(\mathscr{Q}X\times\mathscr{Q}Y)\subseteq Z$. Thus every continuous map $f:X\times Y\to Z$ where Z satisfies $\mathscr{Q}$, admits a continuous extension to $\mathscr{Q}X\times\mathscr{Q}Y$. That is $\mathscr{Q}(X\times Y)=\mathscr{Q}X\times\mathscr{Q}Y$. □

One consequence of this proposition is the following:

3.2 Theorem. *Let $\mathscr{P}$ be an extension property contained in pseudocompactness. Then the following are equivalent for infinite spaces X and Y.*

1) *$X\times Y$ is pseudocompact.*
2) *$\mathscr{P}(X\times Y)=\mathscr{P}X\times\mathscr{P}Y$.*

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Proof. 1) implies 2). If $X\times Y$ is pseudocompact, then by 1.2 $\beta(X\times Y)=\beta X\times\beta Y$. Since β is the extension related to the property of compactness, and every compact space satisfies $\mathscr{P}$ (i.e. $X\subseteq\mathscr{P}X\subseteq\beta X$), we can invoke 3.1 to get $\mathscr{P}(X\times Y)=\mathscr{P}X\times\mathscr{P}Y$.

2) implies 1). If $\mathscr{P}(X\times Y)=\mathscr{P}X\times\mathscr{P}Y$ then by hypothesis $\mathscr{P}X\times\mathscr{P}Y$ is pseudocompact. Hence by 1.2 $\beta(\mathscr{P}X\times\mathscr{P}Y)=\beta(\mathscr{P}X)\times\beta(\mathscr{P}Y)=\beta X\times\beta Y$. But $\beta(\mathscr{P}X\times\mathscr{P}Y)=\beta(\mathscr{P}(X\times Y))=\beta(X\times Y)$. Hence $\beta(X\times Y)=\beta X\times\beta Y$. Again, by 1.2. $X\times Y$ must be pseudocompact. □

It also follows by a proof similar to that of 3.1, that, if $\mathscr{P}$ is any extension property, then $\mathscr{P}(X\times Y)=\mathscr{P}X\times\mathscr{P}Y$ iff $X\times Y$ is C^*-embedded in $\mathscr{P}X\times\mathscr{P}Y$.

It is clear by the method of proof that 3.1 remains true for any finite product of spaces. Thus 3.2 is true for any finite product of spaces. Moreover if $\{X_\alpha\}_{\alpha\in A}$ is any family of spaces such that $\prod_{\alpha\neq\alpha_0}X_\alpha$ is infinite fo every $\alpha_0\in A$, then the implication 2) implies 1) in 3.2 with $X\times Y$ replaced by $\prod_{\alpha\in A}X_\alpha$, can be proved in a manner identical to that given in 3.2. However, as the following example shows, the implication 1) implies 2) in 3.2 (and hence the result of 3.1) is not true in general for arbitrary products.

3.3. Example. Under the assumption of the continuum hypothesis, Rudin in [9, Theorem 4.2] has shown that $\beta N-N$ has 2^c P-points (a point $x\in X$ is a P-point of X if every G_δ containing x is a nbhd. of x). If we denote by A the set of P-points of $\beta N-N$, then for every $p\in A$, $\beta N-\{p\}$ is pseudocompact and locally compact (this follows from [4, 6J]). Thus by [5, Theorem 4] $\prod_{p\in A}(\beta N-\{p\})$ is pseudocompact. Let $\mathscr{P}$ be the property of $\aleph_0$-boundedness (A space X is $\aleph_0$-bounded if every countable subset of X has compact closure in X). $\mathscr{P}$ is easily checked to be an extension property. Woods has shown in [10, Theorem 1.3] that the $\aleph_0$-bounded extension of a space X is the set of points of βX in the βX-closure of some countable subset of X. So it is clear that the $\aleph_0$-bounded extension of $\beta N-\{p\}$ is βN. Thus $\prod_{p\in A}\mathscr{P}(\beta N-\{p\})=\prod_{p\in A}\beta N$. We will show that the point $(x_p)_{p\in A}\in\prod_{p\in N}\beta N$ defined by $x_p=p$, is not in the closure of any countable subset of $\prod_{p\in A}(\beta N-\{p\})$. Hence $\mathscr{P}(\prod_{p\in A}(\beta N-\{p\}))\subsetneqq\prod_{p\in A}\beta N=\prod_{p\in A}\mathscr{P}(\beta N-\{p\})$. Since every $\aleph_0$-bounded space is countably compact, hence pseudocompact, this will show that 1) implies 2) of 3.2 is false for arbitary products. We shall require some preliminary remarks.

In the space βN, the points of $\beta N-N$ are the free ultrafilters of subsets of N (see [4, chapter 6] for a detailed construction of βX in terms of z-ultrafilters). In what follows, a free ultrafilter $\mathscr{D}$ of subsets of N will be referred to both as an ultrafilter on N and as a point of $\beta N-N$.

Let $\{x_n\}_{n\in N}$ be a sequence in a space X, and let $\mathscr{D}$ be a free ultrafilter on N. According to Bernstein in [1], a point $x\in X$ is a $\mathscr{D}$-limit point of the sequence $\{x_n\}$ if given a nbhd. U of x, the set $\{n \mid x_n\in U\}$ is a member of the ultrafilter $\mathscr{D}$. It is straight forward to check that if $f:X\to Y$ is a continuous map and x is the $\mathscr{D}$-limit in X of

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$\{x_n\}_{n\in N}$, then $f(x)$ is the $\mathscr{D}$-limit in Y of $\{f(x_n)\}_{n\in N}$. (We speak of x being *the* $\mathscr{D}$-limit of $\{x_n\}$ because, as is easily checked, in a Hausdorff space $\mathscr{D}$-limits are unique when they exist.) If $x\in \mathrm{cl}_X(\{x_n\}_{n\in N})-(\{x_n\}_{n\in N})$ for a space X and a sequence $\{x_n\}\subseteq X$, then there exists a free ultrafilter $\mathscr{D}$ on N such that x is the $\mathscr{D}$-limit of $\{x_n\}_{n\in N}$ (we can take the sets $B_U=\{n\in N \mid x_n\in U\}$ where U is a nbhd. of x, as the base for a free ultrafilter on N). Suppose that $\mathscr{D}$ is a free ultrafilter on N. Then there are at most $c=2^{\aleph_0}$ points x in $\beta N-N$ such that x is the $\mathscr{D}$-limit of some sequence from N. This is true since there are $2^{\aleph_0}$ sequences of elements of N and $\mathscr{D}$-limits are unique in the Hausdorff space βN.

Suppose, then, that the point $(x_p)_{p\in A}\in\prod_{p\in A}\beta N$ defined by $x_p=p$ is in the closure of some countable subset of $\prod_{p\in A}(\beta N-\{p\})$. Let this countable set be $\{x_n\}_{n\in N}$. Then there exists a free ultrafilter $\mathscr{D}$ on N such that $(x_p)_{p\in A}$ is the $\mathscr{D}$-limit of $\{x_n\}_{n\in N}$. By the continuity of the projection map we must have for each $p\in A$, p is the $\mathscr{D}$-limit in βN of the sequence $\{\Pi_p(x_n)\}_{n\in N}$ (where $\Pi_p:\prod_{p\in A}\beta N\to\beta N$ is the p-th projection map). But we know that there are at most c points q in $\beta N-N$ such that q is the $\mathscr{D}$-limit of a sequence in N. Since A has cardinality 2^c we can find a point $p_0\in A$ such that p_0 is not the $\mathscr{D}$-limit of any sequence in N. As p_0 is a P-point of $\beta N\to N$, and $\{\Pi_{p_0}(x_n)\}_{n\in N}$ is a countable set not containing p_0, there exists a nbhd. U of p_0 in βN such that $U\cap(\beta N-N)\cap\{\Pi_{p_0}(x_n)\}_{n\in N}=\varnothing$. Let us define a sequence $\{y_n\}_{n\in N}$ in N by

$$y_n = 1 \quad \text{if} \quad \Pi_{p_0}(x_n)\in\beta N-N$$
$$y_n = \Pi_{p_0}(x_n) \quad \text{otherwise.}$$

Let V be a nbhd. of p_0 in βN. Then $V\cap U\cap(\beta N-\{1\})$ is a nbhd in βN of p_0. Since p_0 is the $\mathscr{D}$-limit of $\{\Pi_{p_0}(x_n)\}_{n\in N}$, the set $S=\{n \mid \Pi_{p_0}(x_n)\in V\cap U\cap(\beta N-\{1\})\}\in\mathscr{D}$. But $S\subseteq\{n \mid y_n\in V\}$, hence $\{n\mid y_n\in V\}\in\mathscr{D}$ (as an ultrafilter of sets is directed by definition). Thus p_0 is the $\mathscr{D}$-limit point of $\{y_n\}_{n\in N}$, a sequence in N. This contradicts the way in which p_0 was chosen. Thus $(x_p)_{p\in A}$ cannot be in the closure of any countable subset of $\prod_{p\in A}(\beta N-\{p\})$, and the example is completed.

Note that if $\prod_{\alpha\in B}X_\alpha$ is pseudocompact and $|B|\le c$, then $\mathscr{P}(\prod_{\alpha\in B}X_\alpha)=\prod_{\alpha\in B}\mathscr{P}(X_\alpha)$ where $\mathscr{P}$ is $\aleph_0$-boundedness. This is true because any product of c separable spaces is separable.

3.2 showed that for many extension properties the functor $\mathscr{P}$ commutes over finite products iff the product is pseudocompact. The following result shows exactly which extension property functors $\mathscr{P}$ commute under these conditions.

3.4. PROPOSITION. *Let $\mathscr{P}$ be an extension property. The following statements are equivalent.*

1) *$\mathscr{P}(X\times Y)=\mathscr{P}X\times\mathscr{P}Y$ iff $X\times Y$ is pseudocompact.*

2) *$\mathscr{P}$ is contained in pseudocompactness.*

Proof. 2) implies 1). This is the statement of 3.2.

1) implies 2). If $\mathscr{P}$ is not contained in pseudocompactness then N satisfies $\mathscr{P}$.

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Thus $\mathscr{P}(N\times N)=N\times N=\mathscr{P}N\times\mathscr{P}N$, but $N\times N$ is not pseudocompact. This contradicts 1). □

Woods in [11] describes O-dimensional extension properties. If $\mathscr{P}$ is an extension property then the property $\mathscr{P}_0$ defined by "X has $\mathscr{P}_0$ iff X has $\mathscr{P}$ and is O-dimensional" is an extension property in the category of O-dimensional spaces and continuous maps. Every O-dimensional space X has an extension $\mathscr{P}_0X$ such that $X\subseteq\mathscr{P}_0X\subseteq\beta_0X$. Furthermore $\mathscr{P}_0X$ satisfies $\mathscr{P}_0$ and any continuous map from X to a space with $\mathscr{P}_0$ admits an extension to $\mathscr{P}_0X$. In view of 2.2 it is clear that 3.1, 3.2 and 3.4 remain true for O-dimensional spaces X and Y and O-dimensional extension properties of the form $\mathscr{P}_0$. It is worth pointing out here that $\mathscr{P}X\neq\mathscr{P}_0X$ in general for O-dimensional spaces X. If X is the space Δ_1 of [4, 16 M] then $\beta X\neq\beta_0X$.

By the above remarks it is clear that if X and Y are O-dimensional spaces and $\mathscr{P}$ is an extension property contained in pseudocompactness then $\mathscr{P}(X\times Y)=\mathscr{P}X\times\mathscr{P}Y$ iff $\mathscr{P}_0(X\times Y)=\mathscr{P}_0X\times\mathscr{P}_0Y$. The following unanswered question arises. Is it true for any extension property $\mathscr{P}$ that $\mathscr{P}(X\times Y)=\mathscr{P}X\times\mathscr{P}Y$ iff $\mathscr{P}_0(X\times Y)=\mathscr{P}_0X\times\mathscr{P}_0Y$?

Woods in [11] introduces the concept of $\mathscr{P}$-pseudocompactness. A space X is $\mathscr{P}$-pseudocompact if $\mathscr{P}X=\beta X$. The property pseudocompactness is precisely $\mathscr{P}$-pseudocompactness where $\mathscr{P}$ = realcompactness. In general however, $X\times Y$ being $\mathscr{P}$-pseudocompact does not imply $\mathscr{P}(X\times Y)=\mathscr{P}X\times\mathscr{P}Y$. Let $\mathscr{P}=\aleph_0$-boundedness. Then N is $\mathscr{P}$-pseudocompact and so in $N\times N$. However $\mathscr{P}(N\times N)=\beta(N\times N)\neq\beta N\times\beta N=\mathscr{P}N\times\mathscr{P}N$.

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