Canad. Math. Bull. Vol. 14 (4), 1971

FINITE SETS OF INTEGERS WITH EQUAL POWER SUMS

BY
D. Ž. DJOKOVIĆ[1]

This note originated in an attempt to generalize the assertion of Problem 164 which was proposed by Moser [2].

Let us first state our generalization. We fix positive integers $m \geq 2$ and n. If i is an integer and

$$0 \leq i \leq m^n - 1$$

then i can be written, in a unique way, as

$$i = \sum_{r=0}^{n-1} a_r m^r$$

where a_r are suitable integers which satisfy

$$0 \leq a_r \leq m-1 \quad (0 \leq r \leq n-1).$$

We define

$$\sigma(m, i) = \sum_{r=0}^{n-1} a_r.$$

Let E be the set of integers $0, 1, 2, \ldots, m^n - 1$. If

$$E_k = \{i \in E \mid \sigma(m, i) \equiv k \pmod{m}\}$$

then (E_k), $0 \leq k \leq m-1$ is a partition of E. We shall agree that $0^0 = 1$.

Then we have the following:

THEOREM 1. *With the above notations we have*

(i) *the equality*

$$m \sum_{i \in E_k} i^s = \sum_{i \in E} i^s \tag{1}$$

is valid for all $0 \leq s \leq n-1$ *and all* $0 \leq k \leq m-1$;

(ii) *if* $\epsilon_m \neq 1$ *is an mth root of unity in the complex field then*

$$\sum_{i \in E} \epsilon_m^{\sigma(m, i)} i^s = \begin{cases} 0 & \text{for } 0 \leq s \leq n-1 \\ S & \text{for } \quad s = n \end{cases} \tag{2}$$

where

$$S = n! m^n (\epsilon_m - 1)^{-n} m^{\binom{n}{2}}. \tag{3}$$

---

[1] This work was supported in part by National Research Council of Canada Grant A-5285.

In order to obtain the assertion of Problem 164 from this theorem we take $m=2$. One can easily check that the coefficients e_i defined in [2] are given by

$$e_i = -(-1)^{\sigma(2,i)}.$$

Then the second assertion of Theorem 1 coincides with that of Problem 164.

The first assertion of Theorem 1 is in fact a particular case of a theorem of Lehmer [1]. Two other proofs of Lehmer's theorem were published by Wright [3]. We refer to [1] and [3] for the references to earlier results connected with this theorem. I am grateful to Professor J. W. S. Cassels for bringing to my attention the work of E. M. Wright.

It turned out that our proof of Theorem 1 applies, without any change, to Lehmer's theorem. Moreover we have a result for $s=n$ which does not appear in [1] or [3].

THEOREM 2. *Let $m\geq 2$ and $m\geq 1$ be integers and let z_r $(0\leq r\leq n-1)$ be any complex numbers. Let $E(k)$ $(0\leq k\leq m-1)$ be the set of all sequences*

$$(a_i) = (a_0, a_1, \ldots, a_{n-1})$$

such that a_i's are integers, $0\leq a_i\leq m-1$, and

$$a_0+a_1+\cdots+a_{n-1} \equiv k \pmod m.$$

Then

(i) *if $0\leq s\leq n-1$ is an integer the sum*

$$\sum_{(a_i)\in E(k)} (a_0z_0+a_1z_1+\cdots+a_{n-1}z_{n-1})^s \tag{4}$$

does not depend on k;

(ii) *if $\epsilon_m\neq 1$ is an mth root of 1 then*

$$\sum_{k=0}^{m-1} \epsilon_m^k \sum_{(a_i)\in E(k)} (a_0z_0+a_1z_1+\cdots+a_{n-1}z_{n-1})^s = \begin{cases} 0 & \text{for } 0\leq s\leq n-1 \\ S & \text{for } \quad s=n \end{cases} \tag{5}$$

where

$$S = n!m^n(\epsilon_m-1)^{-n}\left(\prod_{r=0}^{n-1} z_r\right). \tag{6}$$

Proof. Let I be the ideal of $Z[X]$ generated by $1+X+X^2+\cdots+X^{m-1}$ and let ξ be the image of X under the canonical mapping $Z[X]\to Z[X]/I$.

Then

$$\xi^m = 1, \tag{7}$$

$$1+\xi+\xi^2+\cdots+\xi^{m-1} = 0, \tag{8}$$

$$(\xi-1)\sum_{a=0}^{n-1} a\xi^a = m. \tag{9}$$

https://doi.org/10.4153/CMB-1971-100-3 Published online by Cambridge University Press

Let

$$R = \sum_{k=0}^{m-1} \xi^k \sum_{(a_i)\in E(k)} (a_0z_0+a_1z_1+\cdots+a_{n-1}z_{n-1})^s.$$

If $E=\bigcup_{k=0}^{m-1} E(k)$ then

$$R = \sum_{(a_i)\in E} \xi^{a_0+\cdots+a_{n-1}}(a_0z_0+\cdots+a_{n-1}z_{n-1})^s$$

$$= \sum_{(a_i)\in E} \sum_{f\in F} \xi^{a_0+\cdots+a_{n-1}} \prod_{r=1}^{s} a_{f(r)}z_{f(r)}$$

where F is the set of all mappings $\{1, 2, \ldots, s\} \to \{0, 1, 2, \ldots, n-1\}$.

If $0 \le s \le n-1$ then (8) implies that

$$\sum_{(a_i)\in E} \xi^{a_0+\cdots+a_{n-1}} \prod_{r=1}^{s} a_{f(r)}z_{f(r)} = 0 \tag{10}$$

for each $f\in F$. Hence, in that case $R=0$ which proves the first assertion of the theorem.

If $s=n$ then (10) is valid for those $f\in F$ which are not bijective. Hence, in that case we have

$$R = n! \left(\prod_{r=0}^{n-1} z_r\right) \left(\sum_{a=0}^{m-1} a\xi^a\right)^n.$$

Using (9) we get

$$(\xi-1)^n R = n!\, m^n \left(\prod_{r=0}^{n-1} z_r\right). \tag{11}$$

We have a homomorphism $Z[\xi] \to Z[\epsilon_m]$ which sends ξ to ϵ_m. This homomorphism transforms (11) into (5) for $s=n$ with S given by (6). Of course, formula (5) for $0 \le s \le n-1$ follows from the first assertion of the theorem.

This completes the proof of Theorem 2.

If we choose

$$z_r = m^r, \quad 0 \le r \le n-1$$

then we obtain Theorem 1 from Theorem 2.

Note that formula (5) for $s=n$ implies that the sums (4) for $s=n$ and $k=0, 1, 2, \ldots, m-1$ cannot be all equal.

References

1. D. H. Lehmer, *The Tarry–Escott problem*, Scripta Math. **13** (1947), 37–41.
2. L. Moser, *Problem 164*, Canad. Math. Bull. (1) **13** (1970), p. 153.
3. E. M. Wright, *Equal sums of like powers*, Proc. Edinburgh Math. Soc. (2) **8** (1949), 138–142.

University of Waterloo,
Waterloo, Ontario

https://doi.org/10.4153/CMB-1971-100-3 Published online by Cambridge University Press