CORRIGENDUM: VON NEUMANN ALGEBRAS AND EXTENSIONS OF INVERSE SEMIGROUPS

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Definition 2.1 of [1] is incomplete and should be replaced by the following.

Corrected Definition 2.1. *Let $\mathcal{L}$ be a Boolean algebra. A representation of $\mathcal{L}$ is a map $\pi : \mathcal{L} \to proj(\mathcal{B})$ of $\mathcal{L}$ into the projection lattice of a unital C^*-algebra $\mathcal{B}$ such that for every $s, t \in \mathcal{L}$,*

$$\pi(s \wedge t) = \pi(s)\pi(t) \quad and \quad \pi(\neg s) = I - \pi(s).$$

With this corrected definition, the statement and proof of [1, Proposition 2.2] are unchanged with the exception that the C^*-algebra $\mathcal{B}$ is to be assumed unital:

Corrected Proposition 2.2. *Let $\mathcal{L}$ be a Boolean algebra with character space $\widehat{\mathcal{L}}$. For each $s \in \mathcal{L}$, let $\widehat{s} \in C(\widehat{\mathcal{L}})$ be the Gelfand transform, $\widehat{s}(\rho) = \rho(s)$. Then $C(\widehat{\mathcal{L}})$ has the following universal property: if $\mathcal{B}$ is a unital C^*-algebra and $\theta : \mathcal{L} \to \mathcal{B}$ is a representation such that $\theta(\mathcal{L})$ generates $\mathcal{B}$ as a C^*-algebra, then there exists a unique $*$-epimorphism $\alpha : C(\widehat{\mathcal{L}}) \to \mathcal{B}$ such that for every $s \in \mathcal{L}$,*

$$\theta(s) = \alpha(\widehat{s}).$$

The discussion on [1, Page 63] uses [1, Proposition 2.2] in preparation for [1, Definition 2.9]. This discussion is unaffected by the corrections given above.

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Unfortunately, the incomplete definition of representation given in [1, Definition 2.1] leads to a gap in the work presented in [1]. To describe the issue, given a Cartan inverse monoid $\mathcal{S}$, [1, Section 4.2] defines $\mathcal{D} := C(\widehat{\mathcal{E}(\mathcal{S})})$, constructs a $\mathcal{D}$-valued reproducing kernel $K : \mathcal{S} \times \mathcal{S} \to \mathcal{D}$ ([1, Defintion 4.7]), and constructs a right Hilbert $\mathcal{D}$-module $\mathfrak{A}$ ([1, Proposition 4.12]). Further [1, Theorem 4.16] shows there is a map $\lambda : \mathcal{G} \to \mathcal{L}(\mathfrak{A})$ so that for $v \in \mathcal{G}$ and $s \in \mathcal{S}$,

$$\lambda(v)k_s = k_{q(v)s}\sigma(v, s),$$

where (as in [1, Definition 4.13]),

$$\sigma(v, s) = j(q(v)s)^\dagger v j(s) = j(s^\dagger q(v^\dagger)) v j(s)$$

and $j : \mathcal{S} \to \mathcal{G}$ is an order-preserving section (see [1, Definition 4.1 and Proposition 4.6]). Also, the proof of [1, Theorem 4.16] establishes:

(a) λ is one-to-one;
(b) for $v, w \in \mathcal{G}$,

$$\lambda(v)^* = \lambda(v^\dagger) \quad \text{and} \quad \lambda(vw) = \lambda(v)\lambda(w).$$

To this point, all is well. However, on page 84 of [1] we wrote,
"Let

$$\mathfrak{B} = \overline{\operatorname{span}}\{k_e : e \in \mathcal{E}(\mathcal{S})\} \subseteq \mathfrak{A}.$$

Note that $\mathfrak{B}$ is a right Hilbert $\mathcal{D}$-submodule of $\mathfrak{A}$. Proposition 2.2 shows that $\lambda|_{\mathcal{E}(\mathcal{G})}$ extends to a $*$-monomorphism $\alpha_\ell : \mathcal{D} \to \mathcal{L}(\mathfrak{A})$."

There is a gap here, because we did not establish that $\lambda|_{\mathcal{E}(\mathcal{G})} : \mathcal{E}(\mathcal{G}) \to \mathcal{D}$ is a representation in the sense of Corrected Definition 2.1 given above, and therefore, we have not shown that we can apply Corrected Proposition 2.2. To fill this gap, we must establish the following fact.

Lemma 1. *For $e \in \mathcal{E}(\mathcal{G})$,*

$$\lambda(\neg e) = I - \lambda(e). \tag{2}$$

Before verifying this, we make some preliminary remarks. Observe that since
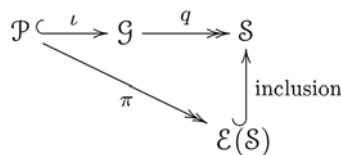
is an idempotent separating extension, $\mathcal{E}(\mathcal{G}) = \iota(\mathcal{E}(\mathcal{P}))$. Therefore, we may identify $\mathcal{E}(\mathcal{G})$ with $\mathcal{E}(\mathcal{P})$ via the map ι.

Since $q \circ \iota = \pi$, we find $q|_{\mathcal{E}(\mathcal{G})} : \mathcal{E}(\mathcal{G}) \to \mathcal{E}(\mathcal{S})$ is an isomorphism. As $q \circ j = \mathrm{id}|_{\mathcal{S}}$,

$$j|_{\mathcal{E}(\mathcal{S})} = \left(q|_{\mathcal{E}(\mathcal{G})}\right)^{-1} = \left(\pi|_{\mathcal{E}(\mathcal{P})}\right)^{-1}.$$

Hence for $e \in \mathcal{E}(\mathcal{G})$ and $s \in \mathcal{S}$, [1, Lemma 4.2] gives

$$\sigma(e, s) = j(s^{\dagger}q(e))ej(s) = j(s^{\dagger})j(q(e))ej(s) = j(s^{\dagger})ej(s) = j(s^{\dagger}q(e)s).$$

Thus, for $e \in \mathcal{E}(\mathcal{G})$ and $s \in \mathcal{S}$,

$$\lambda(e)k_s = k_{q(e)s}j(s^{\dagger}q(e)s) \overset{[1, \text{ Cor. } 4.9]}{=} k_{q(e)s(s^{\dagger}q(e)s)} = k_{q(e)s}.$$

By [1, Proposition 4.12], $\{k_s : s \in \mathcal{S}\}$ has dense span in $\mathfrak{A}$. Thus to establish (2), it suffices to show that for every $s \in \mathcal{S}$ and $e \in \mathcal{E}(\mathcal{S})$,

$$k_{es} + k_{(\neg e)s} = k_s. \tag{3}$$

To establish (3), we require the following.

Fact 4. Let $\mathcal{S}$ be a Boolean inverse monoid. For $r, s, t \in \mathcal{S}$ with s and t orthogonal,

$$(s \vee t) \wedge r = (s \wedge r) \vee (t \wedge r). \tag{5}$$

Proof. Note that $s \wedge r \leq (s \vee t) \wedge r$ because $s \leq s \vee t$; a similar inequality holds for t. Therefore,

$$(s \wedge r) \vee (t \wedge r) \leq (s \vee t) \wedge r. \tag{6}$$

Since $(s \vee t) \wedge r \leq s \vee t$, there is $f \in \mathcal{E}(\mathcal{S})$ such that $(s \vee t) \wedge r = (s \vee t)f$. Now multiplication distributes over finite orthogonal joins in a Boolean inverse monoid [2, p. 386], so

$$(s \vee t) \wedge r = sf \vee tf.$$

As the left side is a meet and the right a join, $r \geq sf$ and $r \geq tf$, and so

$$sf \leq s \wedge r \quad \text{and} \quad tf \leq t \wedge r.$$

Hence

$$(s \vee t) \wedge r = sf \vee tf \leq (s \wedge r) \vee (t \wedge r). \tag{7}$$

Combining (6) and (7) gives (5). $\qquad\square$

We now complete the proof of Lemma 1. As noted, it suffices to establish (3), and this is what we shall do.

Fix $s \in \mathcal{S}$ and $e \in \mathcal{E}(\mathcal{S})$. Again using the fact that multiplication distributes over orthogonal joins in a Boolean inverse monoid, we see that for $t \in \mathcal{S}$,

$$s^\dagger t = (s^\dagger et) \vee (s^\dagger (\neg e) t).$$

Fact 4 gives

$$((s^\dagger et) \wedge 1) \vee (s^\dagger (\neg e) t) \wedge 1) = ((s^\dagger et) \vee (s^\dagger (\neg e) t)) \wedge 1 = s^\dagger t \wedge 1.$$

Notice that $(s^\dagger et) \wedge 1$ and $(s^\dagger (\neg e) t) \wedge 1$ are orthogonal idempotents in $\mathcal{E}(\mathcal{S})$. Since $j|_{\mathcal{E}(\mathcal{S})}$ is a Boolean algebra isomorphism onto $\mathcal{E}(\mathcal{P}) = \mathrm{proj}(\mathcal{D})$, we get

$$j(s^\dagger t \wedge 1) = j((s^\dagger et) \wedge 1) \vee j((s^\dagger (\neg e) t) \wedge 1) = j((s^\dagger et) \wedge 1) + j((s^\dagger (\neg e) t) \wedge 1).$$

In other words, this shows that for $t \in \mathcal{S}$,

$$k_s(t) = k_{es}(t) + k_{(\neg e)s}(t),$$

which is (3).

References

(1)   A. P. Donsig, A. H. Fuller and D. R. Pitts, Von Neumann algebras and extensions of inverse semigroups, *Proc. Edinb. Math. Soc. (2)* **60**(1): (2016), 57–97.
(2)   M. V. Lawson, A noncommutative generalization of Stone duality, *J. Aust. Math. Soc.* **88**(3): (2010), 385–404.