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RESEARCH ARTICLE

Barlow and Proschan principle for coherent systems with statistically dependent component and redundancy lifetimes

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For coherent systems with components and active redundancies having heterogeneous and dependent lifetimes, we prove that the lifetime of system with redundancy at component level is stochastically larger than that with redundancy at system level. In particular, in the setting of homogeneous components and redundancy lifetimes linked by an Archimedean survival copula, we develop sufficient conditions for the reversed hazard rate order, the hazard rate order and the likelihood ratio order between two system lifetimes, respectively. The present results substantially generalize some related results in the literature. Several numerical examples are presented to illustrate the findings as well.

1. Introduction

As one of the most effective and direct methods, redundancy is widely adopted to improve system reliability. An *active* redundancy, also called *hot standby*, is put in parallel to a component and starts functioning once the component is initiated, and a *standby* redundancy is put in standby and starts up once the component fails. In the prominent monograph [1], it was put forward that *the active redundancy at the component level is generally more reliable than the redundancy at the system level in the sense of the usual stochastic ordering for mutually independent components and redundancy lifetimes*. From then on, this milestone conclusion was known as *Barlow–Proschan* (BP) principle in reliability theory.

In the literature, there are bunch of typical references on active redundancy allocation in the setting of independent component and redundancy lifetimes. Take for example, for coherent systems of *independent and identically distributed* (i.i.d.) component and redundancy lifetimes, [5] derived the necessary and sufficient conditions to extend the BP principle from the usual stochastic comparison to the hazard rate order, [34] further developed this principle in terms of the likelihood ratio order for k-out-of-n systems, [15, 24] respectively brought forth two sufficient conditions to upgrade the BP principle to the reversed hazard rate order, and [24] also derived the necessary and sufficient condition on BP principle in the sense of the likelihood ratio order. In addition, for coherent systems with i.i.d. component and redundancy lifetimes, [16] developed sufficient conditions for the BP principle in the sense of shifted reversed hazard rate order, the hazard rate order, and the likelihood ratio order, respectively. [18] built the BP principle for k-out-of-n systems in the sense of the stochastic precedence order. Recently, for coherent systems with dependent component lifetimes, [19] examined the condition sufficient for the BP principle in the context of the relevation transform. For more research on the BP principle, please also refer to [6, 7, 9, 18, 25, 41].

In traditional theory of reliability, component and redundancy lifetimes are usually assumed independent for tractability in mathematics. However, such a rigid assumption discards the potential statistical dependence among component lifetimes and thus is far away from the truth for many complicated engineering systems due to the common stresses bore by components and redundancies and the cross impact of each other. Motivated by the pressing demand on reliability and safety of complicated engineering systems, several authors made impressive progresses in the attempt to incorporate statistical dependence into component and redundancy lifetimes of those simple coherent systems. The past decade has witnessed a stream of research in this line. For example, [2, 3, 11, 12, 28, 29, 36–38], just to name a few. Remarkably, for coherent systems with *dependent but identically distributed* component and redundancy lifetimes, [14] developed necessary and sufficient conditions for the BP principle in the sense of likelihood ratio, hazard rate, reversed hazard rate, and the usual stochastic order, respectively.

In this study, we examine the BP principle for coherent systems with dependent and heterogeneous component and redundancy lifetimes. The rest of this paper rolls out as follows: Section 2 reviews stochastic orders, copula functions, minimal cut decomposition of coherent systems, and two technical lemmas. In Section 3, by numerical examples, we illustrate that necessary and sufficient conditions of [14] are of less merit due to unrealistic assumptions. In Sections 4, we derived the BP principle in the sense of the usual stochastic order for coherent systems with components and redundancies having dependent and heterogeneous lifetimes. Further, for coherent systems with component and redundancy lifetimes linked by an Archimedean copula, we develop in Section 5 necessary and sufficient conditions for the BP principle in the sense of the likelihood ratio order, the reversed hazard rate order, and the hazard rate order, respectively. In Section 6, through several corollaries, we show that characterizations in Section 5 substantially extend the key results of [5, 24, 34].

Throughout the remaining sections, the random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ and real vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ represent component or redundancy lifetimes and their realizations, respectively. For convenience, we denote $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, and $\boldsymbol{x} \vee \boldsymbol{y} = (x_1 \vee y_1, \ldots, x_n \vee y_n)$. By convention, the terms "increasing" and "decreasing" stand for "nondecreasing" and "nonincreasing," respectively.

2. Some preliminaries

For ease of reference, in this section, we review some related notions and also introduce two technical lemmas, which are useful in developing the main results in Sections 4 and 5.

2.1. Stochastic orders

Well developed in the past decades, stochastic orders have been widely utilized to conduct nonparametric comparison on random variables such as system or component lifetimes in engineering reliability, operating times in operations management, running times of algorithms in computer science, potential rewards of portfolios in financial and quantitative risks, etc.

Let X and Y be random variables with *cumulative distribution function* (cdf) F, G and *probability density function* (pdf) f, g (if absolutely continuous), and denote $\bar{F} = 1 - F$, $\bar{G} = 1 - G$. Then, X is said to be *smaller than* Y in the sense of the

(i) *likelihood ratio* order, denoted as $X \le_{\rm lr} Y$, if $g(x)/f(x)$ is increasing on the support of X;
(ii) *hazard rate* order, denoted as $X \le_{\rm hr} Y$, if $\bar{G}(x)/\bar{F}(x)$ is increasing in x with $\bar{F}(x) > 0$;
(iii) *reversed hazard rate* order, denoted as $X \le_{\rm rh} Y$, if $G(x)/F(x)$ increases in x with $F(x) > 0$;
(iv) *usual stochastic* order, denoted as $X \le_{\rm st} Y$, if $\bar{G}(x) \geqslant \bar{F}(x)$ for any x.

For comprehensive discussions on stochastic orders, we refer readers to [4, 20, 22, 33].

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2.2. *Copula function*

For X_i with cdf F_i, $i = 1, \ldots, n$, if a mapping $\mathcal{L} : [0,1]^n \mapsto [0,1]$ is such that $\boldsymbol{X}$ has the joint cdf

$$\mathrm{P}(X_1 \leqslant x_1, \ldots, X_n \leqslant x_n) = \mathcal{L}(F_1(x_1), \ldots, F_n(x_n)), \qquad \text{for all } x_1, \ldots, x_n,$$

then, $\mathcal{L}(u_1, \ldots, u_n)$ is called the *copula* of $\boldsymbol{X}$, and if there exists a mapping $\mathcal{K} : [0,1]^n \mapsto [0,1]$ such that their joint survival function is represented as

$$\mathrm{P}(X_1 > x_1, \ldots, X_n > x_n) = \mathcal{K}\big(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)\big), \qquad \text{for all } x_1, \ldots, x_n,$$

then, $\mathcal{K}$ is said to be the *survival copula* of $\boldsymbol{X}$.

For a nonempty $C \subseteq \{1, \ldots, n\}$, let $u_{i,C} = u_i I(i \in C) + I(i \notin C)$ for $i = 1, \ldots, n$. The subvector of X_i's with subscripts inside C has survival copula

$$\mathcal{K}_C(u_1, \ldots, u_n) = \mathcal{K}\big(u_{1,C}, \ldots, u_{n,C}\big). \tag{2.1}$$

For example, if (X_1, X_2, X_3, X_4) has copula $\mathcal{K}(u_1, u_2, u_3, u_4)$, then the survival copula of (X_2, X_4) is $\mathcal{K}_{\{2,4\}}(u_1, u_2, u_3, u_4) = \mathcal{K}\big(u_{1,\{2,4\}}, u_{2,\{2,4\}}, u_{3,\{2,4\}}, u_{4,\{2,4\}}\big) = \mathcal{K}(1, u_2, 1, u_4)$.

A function φ on $(0, \infty)$ is said to be *n-monotone* if $(-1)^r \varphi^{(r)}(t) \geqslant 0$ for $r = 0, 1, \ldots, n$ and $t \in (0, \infty)$, where $\varphi^{(r)}(t)$ is the rth order derivative for $r > 0$, and $\varphi^{(0)}(t) \equiv \varphi(t)$ by convention. A function φ is said to be *completely monotone* if $(-1)^r \varphi^{(r)}(t) \geqslant 0$ for $r = 0, 1, \ldots$. Obviously, a n-monotone $\varphi(t)$ is such that $(-1)^r \varphi^{(r)}(t)$ is decreasing for any $r = 0, \ldots, n-1$. For a continuous, strictly decreasing and n-monotone function $\varphi : [0, +\infty) \mapsto (0, 1]$ with $\varphi(0) = 1$ and $\lim_{t \to \infty} \varphi(t) = 0$, let $\psi = \varphi^{-1}$ be the pseudo inverse of φ, then, the function

$$\mathcal{K}_\varphi(u_1, \ldots, u_n) = \varphi\big(\psi(u_1) + \cdots + \psi(u_n)\big)$$

is said to be an n-dimensional *Archimedean* copula associated with the *generator* φ.

Due to the simple form of separating the dependence from the marginal distributions, the theory of copula was rapidly developed and has been successfully applied in many statistics related areas, to name a few, lifetime data analysis, biomedical science, quantitative risk management, actuarial science, etc. In particular, owing to the technical tractability, the *Archimedean* family of copulas became rather popular in real data analysis during the recent two decades. In this study, we employ a general copula to model the statistical dependence among component and redundancy lifetimes. One may refer to [23, 31] for a comprehensive exposition on copula theory and Archimedean copulas, respectively.

2.3. *Coherent systems*

Based on components $\mathsf{D}_1, \ldots, \mathsf{D}_n$ with respective lifetimes $X_1, \ldots, X_n$, a reliability structure with system lifetime $T(X_1, \ldots, X_n)$ is said to be *coherent* if the structure function $T(x_1, \ldots, x_n) \geqslant 0$ is increasing in each x_i and every x_i is relevant. A set of components of coherent system is said to be a *cut* if system fails whenever they all fail, and it is called a *minimal cut* if any its subset is not a cut any more. Let $\mathcal{S} = \{C_1, \ldots, C_r\}$ be the class of all system minimal cuts. As per [1], based on component lifetimes $\boldsymbol{X} = (X_1, \ldots, X_n)$, the coherent system attains lifetime

$$T(\boldsymbol{X}) = \min_{C_i \in \mathcal{S}} \max_{j \in C_i} X_j = \min_{i=1,\ldots,r} \max_{j \in C_i} X_j. \tag{2.2}$$

For more on coherent systems, we refer readers to [1, 10, 13, 27].

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Assume for $X_1, \ldots, X_n$ a common reliability function $\bar{F}(t)$ and a survival copula $\mathcal{K}(u_1, \ldots, u_n)$. By Theorem 2.1 of [28], the system reliability can be represented as

$$\bar{H}(t) = \mathrm{P}(T(X_1, \ldots, X_n) > t) = \bar{h}(\bar{F}(t)), \qquad t \geqslant 0, \tag{2.3}$$

where the *distortion function* $\bar{h}(u) : [0,1] \mapsto [0,1]$ is increasing, continuous and such that $\bar{h}(0) = 0$, $\bar{h}(1) = 1$. Since $T(\boldsymbol{X})$ has the cdf $H(t) = 1 - \bar{H}(t) = 1 - \bar{h}(\bar{F}(t)) = 1 - \bar{h}(1 - F(t))$, the *dual distortion function* $h(u) = 1 - \bar{h}(1 - u)$ is also increasing, continuous and such that $h(0) = 0$, $h(1) = 1$. Note that a distortion function and its dual version both depend only on the system structure and the dependence of component lifetimes.

As a technical tool, the distortion transform is usually employed to modify the survival function of a potential risk so that the tail area of its probability distribution gains more weight. Such a popular practice also plays an critical role in financial and critical risk management. See for example [17, 32]. It should be remarked that [26] took the first to represent system reliability as a multivariate distortion transform of component reliability. Subsequently, this technique was further applied to unify different redundancy forms for coherent systems in [29, 30, 39].

2.4. Active redundancy allocation

In engineering reliability, the active redundancy is a common practice to enhance system reliability. For system components $\mathsf{D}_1, \ldots, \mathsf{D}_n$ having respective lifetimes $X_1, \ldots, X_n$ and active redundancies $\mathsf{R}_1, \ldots, \mathsf{R}_n$ having respective lifetimes $Y_1, \ldots, Y_n$, it is usually assumed that $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are mutually independent. In practice, engineers may allocate R_i to D_i, $i = 1, \ldots, n$, to end up with the redundant system lifetime $T_c = T(\boldsymbol{X} \vee \boldsymbol{Y})$. Also, one may allocate the system of $\mathsf{R}_1, \ldots, \mathsf{R}_n$ with the same structure to the system of $\mathsf{D}_1, \ldots, \mathsf{D}_n$ (i.e., redundancy at system level) to produce the corresponding system lifetime $T_s = T(\boldsymbol{X}) \vee T(\boldsymbol{Y})$. As for the active redundancy allocation, readers may refer to a brief review of [21]. Denote $\bar{F}$ the common reliability function of components and redundancies. [14] identified reliability functions of $T(\boldsymbol{X} \vee \boldsymbol{Y})$ and $T(\boldsymbol{X}) \vee T(\boldsymbol{Y})$ as

$$\bar{H}_c(t) = \bar{h}\big(1 - (1 - \bar{F}(t))^2\big), \qquad \bar{H}_s(t) = 1 - \big[1 - \bar{h}\big(\bar{F}(t)\big)\big]^2, \qquad \text{for any } t \geqslant 0, \tag{2.4}$$

respectively, where h is exactly the distortion function of (2.3).

Although the system structure is fixed, owing to the dependence of component and redundancy lifetimes, the survival copula of $\boldsymbol{X} \vee \boldsymbol{Y}$ corresponding to redundancy at component level is not necessarily that of component lifetimes $\boldsymbol{X}$. Evidently, $\bar{H}_c$ and $\bar{H}_s$ of (2.4) serve as the desired system reliability functions only when component and redundancy lifetimes are independent. Consequently, the main results of [14] and hence [35, 40] hold only for systems with redundancy lifetimes independent of component ones. We will reexamine BP principle for heterogeneous and dependent component and redundancy lifetimes in Sections 4 and 5.

2.5. Two technical lemmas

For convenience, we build the survival function of two-component series system with redundancy at component level, which serves as a building block of redundant systems.

Lemma 2.1. *If X_1, X_2, Y_1, Y_2 are linked by a survival copula $\mathcal{K}$ and have survival functions $\bar{F}_1, \bar{F}_2, \bar{G}_1, \bar{G}_2$, respectively, then, $T_c = (X_1 \vee Y_1) \wedge (X_2 \vee Y_2)$ has the survival function, for $t \geqslant 0$,*

$$\begin{aligned} \bar{H}_c(t) = \; & \mathcal{K}(\bar{F}_1(t), \bar{F}_2(t), 1, 1) + \mathcal{K}(\bar{F}_1(t), 1, 1, \bar{G}_2(t)) + \mathcal{K}(1, \bar{F}_2(t), \bar{G}_1(t), 1) \\ & + \mathcal{K}(1, 1, \bar{G}_1(t), \bar{G}_2(t)) - \mathcal{K}(\bar{F}_1(t), \bar{F}_2(t)), 1, \bar{G}_2(t)) - \mathcal{K}(\bar{F}_1(t), \bar{F}_2(t), \bar{G}_1(t)), 1) \\ & - \mathcal{K}(\bar{F}_1(t), 1, \bar{G}_1(t), \bar{G}_2(t)) - \mathcal{K}(1, \bar{F}_2(t), \bar{G}_1(t), \bar{G}_2(t)) + \mathcal{K}(\bar{F}_1(t), \bar{F}_2(t), \bar{G}_1(t), \bar{G}_2(t)). \end{aligned} \tag{2.5}$$

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Proof. Owing to the minimal path decomposition $T_c = (X_1 \wedge X_2) \vee (X_1 \wedge Y_2) \vee (X_2 \wedge Y_1) \vee (X_2 \wedge Y_2)$, the desired result of (2.5) follows immediately from the inclusion-exclusion formula. □

Also, we present the one preservation property of Archimedean copulas of homogeneous marginals under the taking of maximum pairwise.

Lemma 2.2. *If $(\boldsymbol{X}, \boldsymbol{Y})$ are homogeneous and linked by the $2n$-dimensional Archimedean copula with generator φ, then, $\boldsymbol{X} \vee \boldsymbol{Y}$ are linked by the n-dimensional version of this copula.*

Proof. Let F be the common cdf of X_i's and Y_i's. By assumption, $(\boldsymbol{X}, \boldsymbol{Y})$ has cdf

$$\mathrm{P}\big(X_1 \leqslant x_1, \ldots, X_n \leqslant x_n, Y_1 \leqslant y_1, \ldots, Y_n \leqslant y_n\big) = \varphi\left(\sum_{i=1}^{n} \big[\psi(F(x_i)) + \psi(F(y_i))\big]\right).$$

According to (2.1), the univariate marginal $X_i \vee Y_i$ has cdf $M(x) = \mathrm{P}\big(X_i \vee Y_i \leqslant x\big) = \varphi\big(2\psi(F(x))\big)$, for $i = 1, \ldots, n$. Therefore, $\boldsymbol{X} \vee \boldsymbol{Y}$ attains the cdf

$$\mathrm{P}\big((X_i \vee Y_i) \leqslant x_i, i = 1, \ldots, n\big) = \varphi\left(\sum_{i=1}^{n} 2\psi\big(F(x_i)\big)\right) = \varphi\left(\sum_{i=1}^{n} \psi\big(M(x_i)\big)\right).$$

Now, based on Sklar's theorem, we conclude that $\boldsymbol{X} \vee \boldsymbol{Y}$ attains the copula

$$\mathcal{L}(u_1, \ldots, u_n) = \varphi\big(\psi(u_1) + \cdots + \psi(u_n)\big), \qquad \text{for any } u_i \in (0,1), i = 1, \ldots, n,$$

which is exactly the n-dimensional Archimedean copula associated with the generator φ. □

In general, the dependent structure of $\boldsymbol{X} \vee \boldsymbol{Y}$ is usually different from that of $\boldsymbol{X}$ (see Example 3.1). However, by Lemma 2.2, both $\boldsymbol{X} \vee \boldsymbol{Y}$ and $\boldsymbol{X}$ have the same copula when $(\boldsymbol{X}, \boldsymbol{Y})$ are linked by one Archimedean copula. This is critically important in developing the proof of the main results in Section 5.

3. Several examples

Based on $\bar{H}_c$ of (2.4), [14] conducted stochastic comparisons on coherent systems with redundancy at component and system level. However, as per Example 3.1, *distortion functions of a coherent system and its redundant version are indeed different*. Consequently, the reliability function (2.2) of [14] is usually invalid for coherent systems with redundancy at component level.

Example 3.1. Assume that (X_1, X_2) and (Y_1, Y_2) are independent of each other and they have the common marginal survival function $\bar{F}$. Let (X_1, X_2) and (Y_1, Y_2) both be coupled by the survival copula $\mathcal{K}(u_1, u_2)$. The series system lifetime $T = X_1 \wedge X_2$ has survival function $\bar{H}(t) = \mathcal{K}\big(\bar{F}(t), \bar{F}(t)\big) = \bar{h}\big(\bar{F}(t)\big)$, where the system level distortion function is $\bar{h}(u) = \mathcal{K}(u, u)$ on $(0, 1)$. Based on (2.1), with redundancy at component level, the system has lifetime $T_c = (X_1 \vee Y_1) \wedge (X_2 \vee Y_2)$ and hence survival function

$$\bar{H}_c(t) = \mathrm{P}\big(T_c > t\big) = 2\mathcal{K}\big(\bar{F}(t), \bar{F}(t)\big) + 2\bar{F}^2(t) - 4\bar{F}(t)\mathcal{K}\big(\bar{F}(t), \bar{F}(t)\big) + \mathcal{K}^2\big(\bar{F}(t), \bar{F}(t)\big). \tag{3.1}$$

By contrast, as per (2.4), T_c has the survival function $\bar{h}\big(1 - (1 - \bar{F}(t))^2\big) = \mathcal{K}\big(1 - (1 - \bar{F}(t))^2, 1 - (1 - \bar{F}(t))^2\big)$. Markedly, $\bar{H}_c(t) \neq \bar{h}\big(1 - (1 - \bar{F}(t))^2\big)$. Therefore, the system and its redundant version usually have different distortion functions.

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Example 3.2. Suppose X_1, X_2, Y_1, Y_2 are with respective marginal survival functions $\bar{F}_1, \bar{F}_2, \bar{G}_1$ and $\bar{G}_2$ and linked by a survival copula $\mathcal{K}$. Then, with redundancy at system level the system attains lifetime $T_s = (X_1 \wedge X_2) \vee (Y_1 \wedge Y_2)$ and hence the survival function, for all $t \geqslant 0$,

$$\bar{H}_s(t) = \mathcal{K}(\bar{F}_1(t), \bar{F}_2(t), 1, 1) + \mathcal{K}(1, 1, \bar{G}_1(t), \bar{G}_2(t)) - \mathcal{K}(\bar{F}_1(t), \bar{F}_2(t), \bar{G}_1(t), \bar{G}_2(t)). \tag{3.2}$$

Note that, for all $t \geqslant 0$,

$$\begin{aligned}
\mathrm{P}(X_1 > t, X_2 \leqslant t, Y_2 > t) &= \mathcal{K}(\bar{F}_1(t), 1, 1, \bar{G}_2(t)) - \mathcal{K}(\bar{F}_1(t), \bar{F}_2(t), 1, \bar{G}_2(t)),\\
\mathrm{P}(X_1 \leqslant t, X_2 > t, Y_1 > t) &= \mathcal{K}(1, \bar{F}_2(t), \bar{G}_1(t), 1) - \mathcal{K}(\bar{F}_1(t), \bar{F}_2(t), \bar{G}_1(t), 1),\\
\mathrm{P}(X_1 \leqslant t, X_2 > t, Y_1 > t, Y_2 > t) &= \mathcal{K}(1, \bar{F}_2(t), \bar{G}_1(t), \bar{G}_2(t)) - \mathcal{K}(\bar{F}_1(t), \bar{F}_2(t), \bar{G}_1(t), \bar{G}_2(t)),\\
\mathrm{P}(X_1 > t, X_2 \leqslant t, Y_1 > t, Y_2 > t) &= \mathcal{K}(\bar{F}_1(t), 1, \bar{G}_1(t), \bar{G}_2(t)) - \mathcal{K}(\bar{F}_1(t), \bar{F}_2(t), \bar{G}_1(t), \bar{G}_2(t)).
\end{aligned}$$

Based on (2.5) of Lemma 2.1 and (3.2), we have

$$\begin{aligned}
\bar{H}_c(t) - \bar{H}_s(t) &= \mathcal{K}(\bar{F}_1(t), 1, 1, \bar{G}_2(t)) + \mathcal{K}(1, \bar{F}_2(t), \bar{G}_1(t), 1) - \mathcal{K}(\bar{F}_1(t), \bar{F}_2(t), 1, \bar{G}_2(t))\\
&\quad - \mathcal{K}(\bar{F}_1(t), \bar{F}_2(t), \bar{G}_1(t), 1) - \mathcal{K}(\bar{F}_1(t), 1, \bar{G}_1(t), \bar{G}_2(t))\\
&\quad - \mathcal{K}(1, \bar{F}_2(t), \bar{G}_1(t), \bar{G}_2(t)) + 2\mathcal{K}(\bar{F}_1(t), \bar{F}_2(t), \bar{G}_1(t), \bar{G}_2(t))\\
&= \mathrm{P}(X_1 > t, X_2 \leqslant t, Y_2 > t) - \mathrm{P}(X_1 > t, X_2 \leqslant t, Y_1 > t, Y_2 > t)\\
&\quad + \mathrm{P}(X_1 \leqslant t, X_2 > t, Y_1 > t) - \mathrm{P}(X_1 \leqslant t, X_2 > t, Y_1 > t, Y_2 > t)\\
&= \mathrm{P}(X_1 > t, X_2 \leqslant t, Y_1 \leqslant t, Y_2 > t) + \mathrm{P}(X_1 \leqslant t, X_2 > t, Y_1 > t, Y_2 \leqslant t)\\
&\geqslant 0, \qquad \text{for any } t \geqslant 0.
\end{aligned}$$

This gives rise to $T_c \geq_{\mathrm{st}} T_s$, the usual stochastic order.

As a continuation, we further illustrate that the necessary and sufficient condition in Theorem 4 of [14] doesn't apply to Example 3.3.

Example 3.3. (Clayton survival copula) Assume that (X_1, X_2) and (Y_1, Y_2) are independent of each other and they have the common marginal survival function $\bar{F}$. Let (X_1, X_2) and (Y_1, Y_2) be coupled by the same Clayton survival copula $\mathcal{K}$. As per (3.1) and (3.2), survival functions of $T_s = (X_1 \wedge X_2) \vee (Y_1 \wedge Y_2)$ and $T_c = (X_1 \vee Y_1) \wedge (X_2 \vee Y_2)$ are such that

$$\bar{H}_c(t) - \bar{H}_s(t) = 2\left[\bar{F}(t) - \mathcal{K}(\bar{F}(t), \bar{F}(t))\right]^2 \geqslant 0, \qquad \text{for all } t \geqslant 0. \tag{3.3}$$

This confirms that $T_c \geq_{\mathrm{st}} T_s$.

However, in the context of Clayton survival copula $\mathcal{K}(u_1, u_2) = \left(u_1^{-\alpha} + u_2^{-\alpha} - 1\right)^{-1/\alpha}$, the distortion function $\bar{h}(u) = \mathcal{K}(u, u) = (2u^{-\alpha} - 1)^{-1/\alpha}$, it is easy to check that neither $\bar{h}(2u - u^2) > 2\bar{h}(u) - \bar{h}^2(u)$ nor $\bar{h}(2u - u^2) < 2\bar{h}(u) - \bar{h}^2(u)$ for $u \in (0, 1)$. In accordance with Theorem 4 of [14], there is no usual stochastic order between T_c and T_s. Evidently, this conflict is due to use of the incorrect distortion function there.

According to Example 3.2, for a series system of two components, the active redundancy at component level results in a stochastically larger system lifetime than does the active redundancy at system level, irrespective of the statistical dependence of component and redundancy lifetimes. This motivates us to further explore the BP principle for dependent and heterogeneous component and redundancy lifetimes in the coming Section 4.

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4. Systems with dependent component and redundancy lifetimes

In this section, we develop the BP principle in the sense of the usual stochastic order for dependent component and redundancy lifetimes. For coherent systems, let us denote component lifetimes $\boldsymbol{X} = (X_1, \ldots, X_n)$ and redundancy lifetimes $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$.

Theorem 4.1. *For a coherent system with component and redundancy lifetimes* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$*, respectively,* $T(\boldsymbol{X} \vee \boldsymbol{Y}) \geq_{\mathrm{st}} T(\boldsymbol{X}) \vee T(\boldsymbol{Y})$ *if the statistical dependence structure of* $(\boldsymbol{X}, \boldsymbol{Y})$ *is fixed for any configuration of the system with redundancy at component or system level.*

Proof. Since the structure function $T(\boldsymbol{x})$ of a coherent system is increasing in each x_i, $i = 1, \ldots, n$, we have $T(\boldsymbol{x} \vee \boldsymbol{y}) \geqslant T(\boldsymbol{x})$, $T(\boldsymbol{x} \vee \boldsymbol{y}) \geqslant T(\boldsymbol{y})$, and hence $T(\boldsymbol{x} \vee \boldsymbol{y}) \geqslant T(\boldsymbol{x}) \vee T(\boldsymbol{y})$ for any $\boldsymbol{x}, \boldsymbol{y} \in [0, \infty)^n$. As a result, it holds that, for any increasing function $g(x)$ for which the expectations exist,

$$\begin{aligned} \mathrm{E}[g(T(\boldsymbol{X} \vee \boldsymbol{Y}))] &= \int g(T(\boldsymbol{x} \vee \boldsymbol{y}))\, \mathrm{dP}(\boldsymbol{X} \leq \boldsymbol{x}, \boldsymbol{Y} \leq \boldsymbol{y}) \\ &\geq \int g(T(\boldsymbol{x}) \vee T(\boldsymbol{y}))\, \mathrm{dP}(\boldsymbol{X} \leq \boldsymbol{x}, \boldsymbol{Y} \leq \boldsymbol{y}) \\ &= \mathrm{E}[g(T(\boldsymbol{X}) \vee T(\boldsymbol{Y}))]. \end{aligned}$$

Owing to the arbitrariness of g, this yields $T(\boldsymbol{X} \vee \boldsymbol{Y}) \geq_{\mathrm{st}} T(\boldsymbol{X}) \vee T(\boldsymbol{Y})$. □

In accordance with Theorem 4.1, BP principle actually holds for coherent systems with dependent component and redundancy lifetimes. Additionally, one may wonder whether the usual stochastic order in Theorem 4.1 can be even upgraded to some stronger version, for example, the hazard rate order. In what follows, Example 4.3 serves as a negative answer even though Example 4.2 verifies $T_c \geq_{\mathrm{hr}} T_s$ for series systems with independent component lifetimes.

Example 4.2. (Example 3.2 continued) Consider Example 3.2 again in the setting of the independence copula. Since $\frac{4}{2+u} - \frac{2}{2-u^2}$ is decreasing in $u \in (0, 1)$, as per (2.5) and (3.2),

$$\frac{\bar{H}_c(t)}{\bar{H}_s(t)} = \frac{4\bar{F}^2(t) - 4\bar{F}^3(t) + \bar{F}^4(t)}{2\bar{F}^2(t) - \bar{F}^4(t)} = \frac{4}{2 + \bar{F}(t)} - \frac{2}{2 - \bar{F}^2(t)} - 1$$

is increasing in $t \geqslant 0$. This invokes $T_c \geq_{\mathrm{hr}} T_s$.

Example 4.3. (Example 3.1 continued) One can check that T_s has survival function $\bar{H}_s(t) = 2\mathcal{K}\big(\bar{F}(t), \bar{F}(t)\big) - \mathcal{K}^2\big(\bar{F}(t), \bar{F}(t)\big)$, where Clayton survival copula

$$\mathcal{K}(u_1, u_2) = \big(u_1^{-\alpha} + u_2^{-\alpha} - 1\big)^{-1/\alpha}, \quad \text{for } \alpha > 0.$$

By (3.1), for T_c, the survival function

$$\bar{H}_c(t) = 2\mathcal{K}\big(\bar{F}(t), \bar{F}(t)\big) + 2\bar{F}^2(t) - 4\bar{F}(t)\mathcal{K}\big(\bar{F}(t), \bar{F}(t)\big) + \mathcal{K}^2\big(\bar{F}(t), \bar{F}(t)\big).$$

Set $\bar{F}(t) = e^{-t}$ for $t \geqslant 0$. In Example 3.3, the usual stochastic order $(X_1 \vee Y_1) \wedge (X_2 \vee Y_2) \geq_{\mathrm{st}} (X_1 \wedge X_2) \vee (Y_1 \wedge Y_2)$ is confirmed already. However, for $\alpha = 1$, as is seen in Figure 1, $\bar{H}_c(t)/\bar{H}_s(t)$ is not monotone on $(0, 4)$. Thus, $(X_1 \vee Y_1) \wedge (X_2 \vee Y_2) \geqslant_{\mathrm{hr}} (X_1 \wedge X_2) \vee (Y_1 \wedge Y_2)$ is not true.

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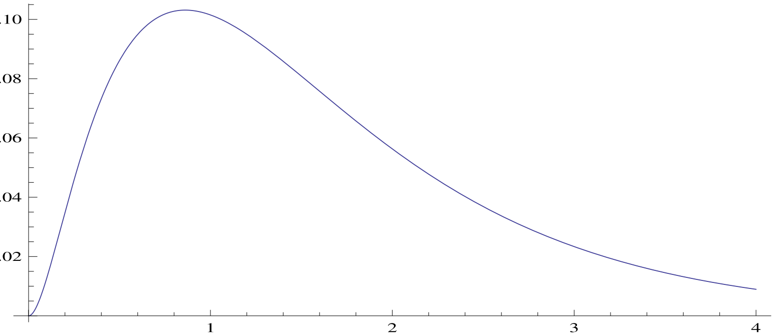

Figure 1. *The curve of the ratio* $\bar{H}_c(t)/\bar{H}_s(t)$ *with* $t \in (0, 4)$.

As suggested by one reviewer, we close this section through making a remark. The ultimate dependence among component lifetimes of engineering systems originates from the interdependence of components only due to system structure and the statistical dependence of component lifetimes due to the common stresses from the environment. Sometimes, with the statistical dependence ignored, one could assume a fixed dependence of component lifetimes for the system irrespective of the type of involved component lifetimes. In particular, this is suitable when redundancies are independent of the components in both options of BP principles and all discussion of [14] can be safely applied. In most of real situations, redundancies and components bear the same stresses due to the common operating environment and thus are of statistically dependent lifetimes, and the two options of BP principle differ only in the way for components and redundancies operate and achieve the mission. Consequently, the dependence and hence survival copula of component and redundancy lifetimes is invariant, and for redundancy at component level, the survival copula of lifetimes of components equipped with active redundancies is different from that of components without redundancies. This exactly corresponds to the more general situation, which is to be further studied in remaining sections. Overall, in practice, engineers ought to select a suitable model based on the real background of the system.

5. Component/redundancy lifetimes with Archimedean copula

In Section 4, we developed the BP principle in the sense of the usual stochastic order irrespective of the dependence structure of component and redundancy lifetimes. To further understand the potential difference between two redundant systems, here we investigate coherent systems with component and redundancy lifetimes linked by an Archimedean copula. With the knowledge of dependence structure, we build the characterization of BP principle in the sense of the reversed hazard rate order, the hazard rate order, and the likelihood rate order, respectively.

Assume that $(\boldsymbol{X}, \boldsymbol{Y})$ has a $2n$-dimensional Archimedean copula $\mathcal{K}_\varphi$. It is plain that $\boldsymbol{X}$ and $\boldsymbol{Y}$ both are linked by the n-dimensional version of this copula. As per Lemma 2.2, $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{X} \vee \boldsymbol{Y}$ have the same Archimedean copula. Thus, there is a dual distortion function h such that $T(\boldsymbol{X})$ and $T(\boldsymbol{Y})$ have common cdf $H(t) = h(F(t))$ and $T_c = T(\boldsymbol{X} \vee \boldsymbol{Y})$ attains the cdf

$$H_c(t) = h\big(\varphi(2\psi(F(t)))\big), \tag{5.1}$$

where $h(u) = H\big(F^{-1}(u)\big)$ for any $u \in (0, 1)$. On the other hand, for the system with redundancy at system level, the cdf of $T_s = T(\boldsymbol{X}) \vee T(\boldsymbol{Y})$ is represented as

$$H_s(t) = \varrho\big(h(F(t))\big), \tag{5.2}$$

where the dual distortion function ϱ is to be determined by (5.4).

Theorem 5.1. *Suppose homogeneous* $(\boldsymbol{X},\boldsymbol{Y})$ *are linked by one Archimedean copula.*

(i) $T(\boldsymbol{X}\vee\boldsymbol{Y})\geq_{\rm rh} T(\boldsymbol{X})\vee T(\boldsymbol{Y})$ *iff* $\frac{h(\varphi(2\psi(u)))}{\varrho(h(u))}$ *is increasing in* $u\in(0,1)$.
(ii) $T(\boldsymbol{X}\vee\boldsymbol{Y})\geq_{\rm hr} T(\boldsymbol{X})\vee T(\boldsymbol{Y})$ *iff* $\frac{1-h(\varphi(2\psi(u)))}{1-\varrho(h(u))}$ *is increasing in* $u\in(0,1)$.
(iii) $T(\boldsymbol{X}\vee\boldsymbol{Y})\geq_{\rm lr} T(\boldsymbol{X})\vee T(\boldsymbol{Y})$ *iff* $\frac{h'(\varphi(2\psi(u)))\varphi'(2\psi(u))\psi'(u)}{\varrho'(h(u))h'(u)}$ *is increasing in* $u\in(0,1)$.

Proof. Recall that $\mathcal{S}=\{C_1,\ldots,C_r\}$ collects all system minimal cuts. Let $A_{i,t}=\bigcap_{j\in C_i}\{X_j\leqslant t\}$ and $B_{i,t}=\bigcap_{j\in C_i}\{Y_j\leqslant t\}$ for any $t\geqslant 0$ and $i=1,2,\ldots,r$. Owing to (2.2), we have

$$\begin{aligned}H_s(t)=\mathrm{P}\big(T(\boldsymbol{X})\vee T(\boldsymbol{Y})\leqslant t\big)&=\mathrm{P}\left(\min_{i=1,\ldots,r}\max_{j\in C_i}X_j\leqslant t,\ \min_{k=1,\ldots,r}\max_{l\in C_k}Y_l\leqslant t\right)\\&=\mathrm{P}\left(\left\{\bigcup_{i=1}^{r}A_{i,t}\right\}\bigcap\left\{\bigcup_{j=1}^{r}B_{j,t}\right\}\right)=\mathrm{P}\left(\bigcup_{i,j=1}^{r}(A_{i,t}B_{j,t})\right).\end{aligned}$$

Taking the Archimedean copula $\mathcal{K}(u_1,\ldots,u_{2n})=\varphi\big(\sum_{i=1}^{2n}\psi(u_i)\big)$ into account, we further have

$$\begin{aligned}H_s(t)&=\sum_{i=1}^{r}\sum_{j=1}^{r}\mathrm{P}(A_{i,t}B_{j,t})-\sum_{i=1}^{r}\sum_{1\leqslant l<k\leqslant r}\mathrm{P}(A_{i,t}B_{l,t}B_{k,t})-\sum_{1\leqslant i<j\leqslant r}\sum_{k=1}^{r}\sum_{l=1}^{r}\mathrm{P}(A_{i,t}A_{j,t}B_{l,t}B_{k,t})\\&\quad+\cdots\cdots+(-1)^{r^2-1}\mathrm{P}\left(\bigcap_{i=1}^{r}A_{i,t}B_{i,t}\right)\\&=\sum_{i=1}^{r}\sum_{j=1}^{r}\mathcal{K}_{C_i\cup C_j}\big(F(t),\ldots,F(t)\big)-\sum_{1\leqslant i<j\leqslant r}\sum_{k=1}^{r}\sum_{l=1}^{r}\mathcal{K}_{C_i\cup C_j\cup C_l\cup C_k}\big(F(t),\ldots,F(t)\big)\\&\quad-\sum_{i=1}^{r}\sum_{1\leqslant l<k\leqslant r}\mathcal{K}_{C_i\cup C_l\cup C_k}\big(F(t),\ldots,F(t)\big)+\cdots+(-1)^{r^2+1}\mathcal{K}\big(F(t),\ldots,F(t)\big).\end{aligned}\tag{5.3}$$

As per (5.2), $\varrho(u)=H_s\big(F^{-1}(h^{-1}(u))\big)$ for $u\in(0,1)$. Now, from (5.3), it follows that

$$\begin{aligned}\varrho(u)&=\sum_{i=1}^{r}\sum_{j=1}^{r}\mathcal{K}_{C_i\cup C_j}\big(h^{-1}(u),\ldots,h^{-1}(u)\big)-\sum_{1\leqslant i<j\leqslant r}\sum_{k=1}^{r}\sum_{l=1}^{r}\mathcal{K}_{C_i\cup C_j\cup C_l\cup C_k}\big(h^{-1}(u),\ldots,h^{-1}(u)\big)\\&\quad-\sum_{i=1}^{r}\sum_{1\leqslant l<k\leqslant r}\mathcal{K}_{C_i\cup C_l\cup C_k}\big(h^{-1}(u),\ldots,h^{-1}(u)\big)+\cdots+(-1)^{r^2+1}\mathcal{K}\big(h^{-1}(u),\ldots,h^{-1}(u)\big).\end{aligned}\tag{5.4}$$

(i) By (5.1) and (5.2), if $h\big(\varphi(2\psi(u))\big)/\varrho\big(h(u)\big)$ is increasing in $u\in(0,1)$, then, $\frac{H_c(t)}{H_s(t)}=\frac{h(\varphi(2\psi(F(t))))}{\varrho(h(F(t)))}$ is increasing in $t\geqslant 0$. That is, $T(\boldsymbol{X}\vee\boldsymbol{Y})\geq_{\rm rh} T(\boldsymbol{X})\vee T(\boldsymbol{Y})$.

(ii) Since $\frac{1-h(\varphi(2\psi(u)))}{1-\varrho(h(u))}$ is increasing, it holds that $\frac{\bar{H}_c(t)}{\bar{H}_s(t)}=\frac{1-h(\varphi(2\psi(F(t))))}{1-\varrho(h(F(t)))}$ is increasing, and this yields $T(\boldsymbol{X}\vee\boldsymbol{Y})\geq_{\rm hr} T(\boldsymbol{X})\vee T(\boldsymbol{Y})$.

(iii) Owing to (5.1) and (5.2), $T(\boldsymbol{X}\vee\boldsymbol{Y})$ and $T(\boldsymbol{X})\vee T(\boldsymbol{Y})$ attains their pdf's

$$H_c'(t)=2h'\big(\varphi(2\psi(F(t)))\big)\varphi'(2\psi(F(t)))\psi'(F(t))F'(t),\quad H_s'(t)=\varrho'\big(h(F(t))\big)h'(F(t))F'(t),$$

respectively. As a result, if $\frac{h'(\varphi(2\psi(u)))\varphi'(2\psi(u))\psi'(u)}{\varrho'(h(u))h'(u)}$ is increasing in $u\in(0,1)$, then,

$$\frac{H_c'(t)}{H_s'(t)}=\frac{2h'\big(\varphi(2\psi(F(t)))\big)\varphi'(2\psi(F(t)))\psi'(F(t))}{\varrho'\big(h(F(t))\big)h'(F(t))}$$

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is increasing in $t \geqslant 0$. That is, $T(\boldsymbol{X} \vee \boldsymbol{Y}) \geq_{\mathrm{lr}} T(\boldsymbol{X}) \vee T(\boldsymbol{Y})$. □

It is worth remarking here that both h and ϱ depend only on the system structure and the dependence structure of component lifetimes. For coherent systems with component and redundancy lifetimes coupled by an Archimedean copula, Theorem 5.1 presents necessary and sufficient conditions based on (5.1) and (5.2) for the BP principle in the sense of the reversed hazard rate order, the hazard rate order, and the likelihood ratio order, respectively. Next, we pay attention to $T_c = (X_1 \vee Y_1) \wedge (X_2 \vee Y_2)$ and $T_s = (X_1 \wedge Y_1) \vee (X_2 \wedge Y_2)$, two versions of the series system $T(X_1, X_2) = X_1 \wedge X_2$ with redundancies Y_1, Y_2.

Corollary 5.2. *Suppose that X_1, X_2, Y_1, Y_2 are homogeneous and linked by Archimedean copula $\mathcal{K}_\varphi$. Then,* (i) $T_c \geq_{\mathrm{rh}} T_s$ *if* $\frac{3\varphi(4t)-4\varphi(3t)}{2\varphi(2t)-\varphi(4t)}$ *is increasing, and* (ii) $T_c \geq_{\mathrm{lr}} T_s$ *if* $\frac{\varphi'(4t)-\varphi'(3t)}{\varphi'(2t)-\varphi'(4t)}$ *is increasing.*

Proof. Denote F the common cdf of X_1, X_2, Y_1, Y_2. The system lifetime $T(\boldsymbol{X}) = X_1 \wedge X_2$ gets cdf $H(t) = \mathrm{P}(X_1 \wedge X_2 \leqslant t) = 2F(t) - \varphi\big(2\psi(\bar{F}(t))\big)$. Thus, the distortion function

$$h(u) = 2u - \varphi\big(2\psi(u)\big), \qquad \text{for } u \in (0,1). \tag{5.5}$$

Since $T(\boldsymbol{X})$ has only two minimal cut sets $C_1 = \{1\}$ and $C_2 = \{2\}$. From (5.3), it follows that

$$H_s(t) = \mathrm{P}\left(\bigcup_{i=1}^{2}\bigcup_{j=1}^{2}\{X_i \leqslant t, Y_j \leqslant t\}\right) = 4\varphi\big(2\psi(F(t))\big) - 4\varphi\big(3\psi(F(t))\big) + \varphi\big(4\psi(F(t))\big), \quad t \geqslant 0,$$

and thus, we have, for $t \geqslant 0$,

$$\varrho(u) = H_s\big(F^{-1}\big(h^{-1}(u)\big)\big) = 4\varphi\big(2\psi\big(h^{-1}(u)\big)\big) - 4\varphi\big(3\psi\big(h^{-1}(u)\big)\big) + \varphi\big(4\psi\big(h^{-1}(u)\big)\big). \tag{5.6}$$

(i) By (5.6) and (5.5), we have

$$\frac{\varrho(h(u))}{h(\varphi(2\psi(u)))} = \frac{4\varphi\big(2\psi(u)\big) - 4\varphi\big(3\psi(u)\big) + \varphi\big(4\psi(u)\big)}{2\varphi\big(2\psi(u)\big) - \varphi\big(4\psi(u)\big)} = 2 + \frac{3\varphi\big(4\psi(u)\big) - 4\varphi\big(3\psi(u)\big)}{2\varphi\big(2\psi(u)\big) - \varphi\big(4\psi(u)\big)}.$$

Since $\psi(u)$ is decreasing and $\frac{3\varphi(4t)-4\varphi(3t)}{2\varphi(2t)-\varphi(4t)}$ is increasing, we conclude that $\frac{3\varphi(4\psi(u))-4\varphi(3\psi(u))}{2\varphi(2\psi(u))-\varphi(4\psi(u))}$ is decreasing and then $\frac{h(\varphi(2\psi(u)))}{\varrho(h(u))}$ is increasing. Now, based on Theorem 5.1, we reach $T_c \geq_{\mathrm{rh}} T_s$.

(ii) Based on (5.5) and (5.6), we have

$$h'\big(\varphi(2\psi(u))\big)\varphi'\big(2\psi(u)\big)\psi'(u) = \big[h\big(\varphi(2\psi(u))\big)\big]'/2 = 2\psi'(u)\big[\varphi'\big(2\psi(u)\big) - \varphi'\big(4\psi(u)\big)\big]',$$

$$\varrho'\big(h(u)\big)h'(u) = \big[\varrho\big(h(u)\big)\big]' = 4\psi'(u)\big[2\varphi'\big(2\psi(u)\big) - 3\varphi'\big(3\psi(u)\big) + \varphi'\big(4\psi(u)\big)\big].$$

Hence, it holds that

$$\frac{\varrho'\big(h(u)\big)h'(u)}{h'\big(\varphi(2\psi(u))\big)\varphi'\big(2\psi(u)\big)\psi'(u)} = 4 + 6\frac{\varphi'\big(4\psi(u)\big) - \varphi'\big(3\psi(u)\big)}{\varphi'\big(2\psi(u)\big) - \varphi'\big(4\psi(u)\big)}.$$

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Since $\psi(u)$ is decreasing and $\frac{\varphi'(4t)-\varphi'(3t)}{\varphi'(2t)-\varphi'(4t)}$ is increasing, $\frac{\varphi'(4\psi(u))-\varphi'(3\psi(u))}{\varphi(2\psi(u))-\varphi'(4\psi(u))}$ is decreasing and hence $\frac{h'(\varphi(2\psi(u)))\varphi'(2\psi(u))\psi'(u)}{\varrho'(h(u))h'(u)}$ is increasing. Thus, we reach $T_c \geq_{\rm lr} T_s$ owing to Theorem 5.1(ii). □

For a two-component series system with Archimedean copula of component and redundancy lifetimes, Corollary 5.2 presents sufficient conditions for BP principle in the sense of the likelihood ratio order and reversed hazard rate order, respectively. It is routine to check that $\left(\frac{e^t-1}{1-e^{2t}}\right)' = e^t\frac{(e^t-1)^2}{(1-e^{2t})^2} \geq 0$ for all $t \geqslant 0$. Therefore, corresponding to the independent copula the generator $\varphi(t) = e^{-t}$ is such that $\frac{\varphi'(4t)-\varphi'(3t)}{\varphi'(2t)-\varphi'(4t)} = \frac{e^t-1}{1-e^{2t}}$ increases in $t \geqslant 0$. This invokes the condition (ii) of Theorem 5.2. Therefore, the redundancy at component level outfits the redundancy at system level in the sense of the likelihood ratio order. Besides, as is illustrated by Example 5.3, Clayton copula also fulfills such a sufficient condition.

Example 5.3. Suppose homogeneous lifetimes X_1, X_2, Y_1, Y_2 are linked by Clayton copula with generator $\varphi(t) = (1+t)^{-1}$. Since $\varphi'(t) = -(1+t)^{-2}$ and hence $\frac{\varphi'(4t)-\varphi'(3t)}{\varphi'(2t)-\varphi'(4t)} = -\frac{1}{4}l(t)$, where $l(t) = (2+15t+36t^2+28t^3)/(1+3t)^3$ is such that $l'(t) = -3\frac{1+6t+8t^2}{(1+3t)^4} \leq 0$, $l(t)$ is decreasing and hence $\frac{\varphi'(4t)-\varphi'(3t)}{\varphi'(2t)-\varphi'(4t)}$ is increasing. Thus, the condition of Theorem 5.2(ii) is fulfilled.

Denote $\Delta_1(t) = 3\varphi(4t) - 4\varphi(3t)$ and $\Delta_2(t) = 2\varphi(2t) - \varphi(4t)$. If $\frac{\varphi'(4t)-\varphi'(3t)}{\varphi'(2t)-\varphi'(4t)}$ is increasing, then, the ratio $\frac{\Delta_1'(t)}{\Delta_2'(t)} = 3\frac{\varphi'(4t)-\varphi'(3t)}{\varphi'(2t)-\varphi'(4t)}$ is increasing. That is, $\frac{\Delta_1'(x)}{\Delta_2'(x)} \leq \frac{\Delta_1'(y)}{\Delta_2'(y)}$ for $y > x \geqslant 0$. Since φ' is increasing, it holds that $\Delta_2'(x)\Delta_2'(y) > 0$ for $y > x \geqslant 0$ and hence $\Delta_1'(x)\Delta_2'(y) \leq \Delta_1'(y)\Delta_2'(x)$ for $y > x \geqslant 0$. In view of $\lim_{t\to+\infty}\varphi(t) = 0$, we have

$$-\Delta_1'(x)\Delta_2(x) = \Delta_1'(x)\int_x^{+\infty}\Delta_2'(y)\mathrm{d}y \leq \int_x^{+\infty}\Delta_1'(y)\mathrm{d}y\Delta_2'(x) = -\Delta_2'(x)\Delta_1(x),$$

which implies that $\left(\frac{\Delta_1(x)}{\Delta_2(x)}\right)' = \frac{\Delta_1'(x)\Delta_2(x)-\Delta_2'(x)\Delta_1(x)}{\Delta_2^2(x)} \geq 0$ for all $x \geqslant 0$. Since $\frac{\varphi'(4t)-\varphi'(3t)}{\varphi'(2t)-\varphi'(4t)}$ is increasing in $\geqslant 0$, we conclude that $\frac{3\varphi(4t)-4\varphi(3t)}{2\varphi(2t)-\varphi(4t)}$ is also increasing in $t \geqslant 0$.

Lemma 2.2 ensures that $\boldsymbol{X} \vee \boldsymbol{Y}$ inherits the dependence structure of $\boldsymbol{X}$ whenever $(\boldsymbol{X}, \boldsymbol{Y})$ are of an Archimedean copula, and this plays a critically important role in developing the proof of the main results in this section. As for more general copulas, such a nice result is not necessarily true any more (see Example 3.1), and thus we encounter here the difficulty in extending the preceding results to other kinds of copulas.

6. Systems with i.i.d. component and redundancy lifetimes

As is remarked in Section 1, most of the research on BP principle in the literature are performed for i.i.d. component and redundancy lifetimes. Since the independence copula is one typical member of the Archimedean family, in this section, we present several corollaries of Theorem 5.1 to justify those typical results on the BP principle in related references.

Note that the system distortion transform $\bar{h}(u) = 1 - h(1-u)$ on $(0,1)$, the generator $\varphi(t) = e^{-t}$ for the independence copula has general inverse $\psi(u) = -\ln u$ such that $\varphi(2\psi(u)) = u^2$ on $(0,1)$, and due to the independence the dual distortion function $\varrho(u) = u^2$ on $(0,1)$.

Corollary 6.1. *Suppose that lifetimes* $(\boldsymbol{X}, \boldsymbol{Y})$ *are i.i.d.. Then,* $T(\boldsymbol{X} \vee \boldsymbol{Y}) \geq_{\rm rh} T(\boldsymbol{X}) \vee T(\boldsymbol{Y})$ *whenever* $(1-u)\bar{h}'(u)/[1-\bar{h}(u)]$ *is increasing on* $(0,1)$.

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Proof. Note that $\varrho(u) = u^2$, $\varphi(t) = e^{-t}$, $\psi(u) = -\ln u$ and $\varphi(2\psi(u)) = u^2$. It holds that

$$\eta(u) = \frac{h\big(\varphi(2\psi(u))\big)}{\varrho\big(h(u)\big)} = \frac{1-\bar{h}(1-u^2)}{[1-\bar{h}(1-u)]^2}, \qquad \text{for all } u \in (0,1).$$

Since $1-u^2 \geqslant 1-u$ for $u \in (0,1)$, the increasing property of $(1-u)\bar{h}'(u)/[1-\bar{h}(u)]$ implies that

$$\begin{aligned} \eta'(u) &\propto 2u\bar{h}'(1-u^2)[1-\bar{h}(1-u)]^2 - 2[1-\bar{h}(1-u^2)][1-\bar{h}(1-u)]\bar{h}'(1-u) \\ &\propto u\frac{\bar{h}'(1-u^2)}{1-\bar{h}(1-u^2)} - \frac{\bar{h}'(1-u)}{1-\bar{h}(1-u)} \\ &\propto [1-(1-u^2)]\frac{\bar{h}'(1-u^2)}{1-\bar{h}(1-u^2)} - [1-(1-u)]\frac{\bar{h}'(1-u)}{1-\bar{h}(1-u)} \geqslant 0, \qquad \text{for all } u \in (0,1), \end{aligned}$$

and hence $\eta(u)$ is increasing in $u \in (0,1)$. Consequently, $T\big(\boldsymbol{X} \vee \boldsymbol{Y}\big) \geq_{\text{rh}} T(\boldsymbol{X}) \vee T(\boldsymbol{Y})$ follows from Theorem 5.1(i) immediately. □

It should be remarked that Theorem 3.2 of [15] built the BP principle in the sense of the reversed hazard rate order in the context that $(1-u)\bar{h}'(u)/[1-\bar{h}(u)]$ is increasing and $u \geqslant \bar{h}(u)$ on $(0,1)$. As per Corollary 6.1, $u \geqslant \bar{h}(u)$ on $(0,1)$ is superfluous.

Corollary 6.2. *Suppose that* $(\boldsymbol{X},\boldsymbol{Y})$ *are i.i.d.. Then,* $T\big(\boldsymbol{X} \vee \boldsymbol{Y}\big) \geq_{\text{hr}} T(\boldsymbol{X}) \vee T(\boldsymbol{Y})$ *if* $\bar{h}(u) = 1-h(1-u)$ *is such that (i)* $u\bar{h}'(u)/\bar{h}(u)$ *is decreasing and (ii)* $u \geqslant \bar{h}(u)$ *for* $u \in (0,1)$.

Proof. Similar to Corollary 6.1, it holds that $\ell(u) = \frac{1-h(\varphi(2\psi(u)))}{1-\varrho(h(u))} = \frac{\bar{h}(1-u^2)}{\bar{h}(1-u)[2-\bar{h}(1-u)]}$ and hence

$$\begin{aligned} \ell'(u) &\propto -2u\bar{h}'(1-u^2)[2-\bar{h}(1-u)]\bar{h}(1-u) - 2\bar{h}(1-u^2)\bar{h}'(1-u)[-1+\bar{h}(1-u)] \\ &\propto (1-u)\frac{\bar{h}'(1-u)}{\bar{h}(1-u)}\frac{1-\bar{h}(1-u)}{2-\bar{h}(1-u)}\frac{1+u}{u} - (1-u^2)\frac{\bar{h}'(1-u^2)}{\bar{h}(1-u^2)} = \gamma(u). \end{aligned}$$

Since $\bar{h}(1-u) \leqslant 1-u$ implies $u \leqslant 1-\bar{h}(1-u)$ on $(0,1)$ and $\frac{x}{1+x}$ is increasing, it holds that $\frac{1-\bar{h}(1-u)}{1+(1-\bar{h}(1-u))} \geqslant \frac{u}{1+u}$ for all $u \in (0,1)$. Owing to the decreasing $u\bar{h}'(u)/\bar{h}(u)$, we conclude that

$$\ell'(u) \propto \gamma(u) \geqslant (1-u)\frac{\bar{h}'(1-u)}{\bar{h}(1-u)} - (1-u^2)\frac{\bar{h}'(1-u^2)}{\bar{h}(1-u^2)} \geqslant 0, \qquad \text{for all } u \in (0,1),$$

that is, $\ell(u)$ is increasing. Thus, $T\big(\boldsymbol{X} \vee \boldsymbol{Y}\big) \geq_{\text{hr}} T(\boldsymbol{X}) \vee T(\boldsymbol{Y})$ follows from Theorem 5.1(ii). □

It should be mentioned here that Theorem 2 of [5] developed the sufficient condition of Corollary 6.2, which gives rise to the characterization of Theorem 5.1(ii).

Corollary 6.3. *Suppose that* $(\boldsymbol{X},\boldsymbol{Y})$ *are i.i.d.. Then,* $T\big(\boldsymbol{X} \vee \boldsymbol{Y}\big) \geq_{\text{lr}} T(\boldsymbol{X}) \vee T(\boldsymbol{Y})$ *if and only if* $\frac{uh'((u^2)}{h'(u)h(u)}$ *is increasing on* $(0,1)$.

Proof. In view of $\varrho(u) = u^2$, $\varphi(t) = e^{-t}$, $\psi(u) = -\ln u$, and $\varphi(2\psi(u)) = u^2$, we have

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$$\frac{h'\big(\varphi(2\psi(u))\big)\varphi'(2\psi(u))\psi'(u)}{\varrho'\big(h(u)\big)h'(u)} = \frac{uh'(u^2)}{2h'(u)h(u)}, \qquad \text{for all } u \in (0,1).$$

Therefore, $T\big(\boldsymbol{X} \vee \boldsymbol{Y}\big) \geq_{\rm lr} T(\boldsymbol{X}) \vee T(\boldsymbol{Y})$ follows directly from Theorem 5.1(iii). □

According to Theorem 3.2 of [24], $T\big(\boldsymbol{X} \vee \boldsymbol{Y}\big) \geq_{\rm lr} T(\boldsymbol{X}) \vee T(\boldsymbol{Y})$ if and only if $\frac{1-\bar{h}(u)}{1-u}\frac{\bar{h}'(u)}{\bar{h}'(u(2-u))}$ is increasing in $u \in (0,1)$. It is not difficult to check that such a characterization result is equivalent to the necessary and sufficient condition of Corollary 6.3. As thus, Theorem 5.1(iii) serves as one substantial generalization of Theorem 3.2 of [24].

Denote $T_{k,n}(\boldsymbol{X}) = X_{n-k+1:n}$ the lifetime of k-out-of-n:G system, $k = 1, \ldots, n$.

Corollary 6.4. *If* $(\boldsymbol{X}, \boldsymbol{Y})$ *are i.i.d., then,* $T_{k,n}\big(\boldsymbol{X} \vee \boldsymbol{Y}\big) \geq_{\rm lr} T_{k,n}(\boldsymbol{X}) \vee T_{k,n}(\boldsymbol{Y})$ *for* $k = 1, \ldots, n$.

Proof. For $n=1$, $T_{1,n}\big(\boldsymbol{X} \vee \boldsymbol{Y}\big) = \max_{1\leqslant i\leqslant n} X_i \vee Y_i = \max\{T_{1,n}(\boldsymbol{X}), T_{1,n}(\boldsymbol{Y})\}$, the likelihood ratio order is trivially true. Let us assume $2 \leqslant k \leqslant n$. Denote F and f the cdf and pdf of X_1, respectively. In according to [8], $T_{k,n}(\boldsymbol{X})$ has the cdf $K(t) = h\big(F(t)\big)$, where

$$h(u) = \int_0^u \frac{n!}{(n-k)!(k-1)!} v^{n-k}(1-v)^{k-1}\mathrm{d}v, \quad \text{for any } v \in (0,1).$$

Since $(\boldsymbol{X}, \boldsymbol{Y})$ are i.i.d., the system lifetime $T_s = T_{k,n}(\boldsymbol{X}) \vee T_{k,n}(\boldsymbol{Y})$ attains cdf $H_s(t) = \mathrm{P}\left(T_{k,n}(\boldsymbol{X}) \vee T_{k,n}(\boldsymbol{Y}) \leqslant t\right) = K^2(t) = [h(F(t))]^2 = \varrho\big(h(F(t))\big)$, where $\varrho(u) = u^2$.

In the setting of independent component and redundancy lifetimes, we have $\varphi(t) = e^{-t}$ and $\psi(u) = -\ln u$. Since $\varrho'(u) = 2u$ and $h'(u) = \frac{n!}{(n-k)!(k-1)!}u^{n-k}(1-u)^{k-1}$, it is easy to check that

$$\frac{h'\big(\varphi(2\psi(u))\big)\varphi'(2\psi(u))\psi'(u)}{\varrho'\big(h(u)\big)h'(u)} = u^{n-k+1}(1+u)^{k-1}\Big/2\int_0^u v^{n-k}(1-v)^{k-1}\mathrm{d}v.$$

According to the proof of the technical lemma in [34], this ratio is increasing in $u \in (0,1)$. Thus, the desired order follows as a direct consequence of Theorem 5.1(ii). □

For k-out-of-n:G systems with i.i.d. component and redundancy lifetimes, [34] independently proved the BP principle in terms of the likelihood ratio order. As thus, Theorem 5.1(ii) forms as an essential extension of the result in the setting of component and redundancy lifetimes linked by an Archimedean copula.

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