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EXPONENTIALLY BOUNDED POSITIVE-DEFINITE FUNCTIONS ON A COMMUTATIVE HYPERGROUP

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Abstract

In this paper we make use of semigroup methods on the space of compactly supported measures to obtain a Bochner representation for α-bounded positive-definite functions on a commutative hypergroup.

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The analysis throughout will be carried out on a (locally compact) hypergroup X admitting a left Haar measure m. (For a definition and properties refer to Jewett [5] whose notation we follow.) This includes those hypergroups that are compact ([5, Theorem 7.2A]), discrete ([5, Theorem 7.1A]) or commutative (Spector [9, Theorem III.4]). We reserve the symbols $M^1(X)$, $M_c^1(X)$ and $M_c(X)$ for the spaces of probability measures, those that have compact support, and the space of measures that have compact support respectively. $L_{loc}^{\infty}(X)$ is just the space of measurable functions that are bounded on every compact subset of X. There is an analogous definition for the space $L_{loc}^1(X)$. We denote the point measure at $x \in X$ by ϵ_x, and the indicator function of a set A by 1_A. The involution on X extends to $M^b(X)$ via $\mu^{\sim}(B) = \overline{\mu(B^-)}$ for all Borel sets $B \subseteq X$.

For each $x, y \in X$ write

$$f(x * y) := \int_X f \, d(\epsilon_x * \epsilon_y), \qquad \mu * f(x) := \int_X f(z^- * x) \, d\mu(z)$$

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and

$$f * g(x) := \int_X f(x * y) g(y^-)\, dm(y) = \int_X f(y) g(y^- * x)\, dm(y).$$

Here f, g are measurable functions on X and $\mu \in M^b(X)$, and the latter equality holds whenever one of f, g is σ-finite (see [5, Theorem 5.1D]). The left x-translate of f is written $f_x(y) = f(x * y)$.

The main objects of interest in this paper are positive-definite functions, that is functions $f \in L^\infty_{loc}(X)$ satisfying

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{a}_j\ f(x_i * x_j^-) \geq 0$$

for each n, and for each choice of complex numbers a_i and points $x_i \in X$. For basic properties of positive-definite functions on hypergroups the reader is referred to [5, Section 11].

DEFINITION 1. We call $\alpha \in L^\infty_{loc}(X)$ with $\alpha \geq 0$ an *absolute value* on X if it satisfies

(i) $\alpha(e) = 1$;
(ii) $\alpha(x * y) \leq \alpha(x)\alpha(y)$;
(iii) $\alpha(x^-) = \alpha(x)$

for all $x, y \in X$.

It should be observed that every continuous absolute value α is positive on X. Indeed if $\alpha(x) = 0$ for some $x \in X$ then $\alpha(x * x^-) \leq \alpha(x)^2 = 0$ shows that $\int_X \alpha\, d(\epsilon_x * \epsilon_{x^-}) = 0$, and hence $\alpha = 0$ on $\{x\} * \{x^-\}$. But this contradicts (i) as $e \in \{x\} * \{x^-\}$.

DEFINITION 2. We say that f on X is *bounded* with respect to an absolute value α, or simply *α-bounded*, if there is a constant K such that $|f(x)| \leq K\alpha(x)$ for all $x \in X$. If there exists an absolute value with respect to which f is bounded then f is called *exponentially bounded.*

PROPOSITION 3. *Every α-bounded positive-definite function f satisfies $|f(x)| \leq f(e)\alpha(x)$ for all $x \in X$.*

PROOF. Write $K = \sup\{\alpha(x)^{-1}\,|f(x)| : x \in X, \alpha(x) \neq 0\}$. Using the positive-definiteness of f we have

$$|f(x)|^2 \leq f(x * x^-) f(e) \leq K\alpha(x * x^-) f(e) \leq K\alpha(x)^2 f(e)$$

so that $|f(x)| \leq (Kf(e))^{1/2}\alpha(x)$ for all $x \in X$. By the choice of K it follows that $K \leq (Kf(e))^{1/2}$ and $K \leq f(e)$.

PROPOSITION 4. *Let α be an absolute value on X. Then A defined by*

$$A(s) := \int_X \alpha \, d\,|s|$$

is submultiplicative on $M_c(X)$.

PROOF. We have, using [5, Lemma 6.1C],

$$\begin{aligned} A(s*t) = \int_X \alpha \, d\,|s*t| &\le \int_X \int_X \alpha(x*y)\, d\,|s|\,(x)\, d\,|t|\,(y) \\ &\le \int_X \int_X \alpha(x)\alpha(y)\, d\,|s|\,(x)\, d\,|t|\,(y) \\ &= A(s)A(t) \end{aligned}$$

and this completes the proof.

Write $L^1_\alpha(X) = \{f \in L^1_{loc}(X) : \int_X |f|\,\alpha \, dm < \infty\}$.

THEOREM 5. *Suppose that $\alpha(x) \ge 1$ for all $x \in X$. With the norm*

$$\|f\|_{1,\alpha} := \int_X |f|\,\alpha \, dm$$

$L^1_\alpha(X)$ is a Banach subalgebra of $L^1(X)$.

PROOF. If $f, g \in L^1_\alpha(X)$ then appealing to [5, Lemma 5.1D]

$$\begin{aligned} \int_X |f*g|\,\alpha \, dm &\le \int_X \int_X |f|\,(y)\,|g|\,(y^- * x)\, dm(y)\,\alpha(x)\, dm(x) \\ &= \int_X \int_X |g|\,(y^- * x)\alpha(x)\, dm(x)\,|f|\,(y)\, dm(y) \\ &= \int_X \int_X |g|\,(x)\alpha(y*x)\, dm(x)\ \ |f|\,(y)\, dm(y) \\ &\le \int_X |f|\,(y)\alpha(y)\, dm(y) \int_X |g|\,(x)\alpha(x)\, dm(x) \\ &= \|f\|_{1,\alpha}\,\|g\|_{1,\alpha} \end{aligned}$$

so that $f*g \in L^1_\alpha(X)$ and $\|f*g\|_{1,\alpha} \le \|f\|_{1,\alpha}\,\|g\|_{1,\alpha}$. Because $\alpha \ge 1$ we must have $L^1_\alpha(X) \subset L^1(X)$, and the rest of the proof is clear.

The assumption $\alpha \geq 1$ is essential to Theorem 5. Indeed (see Ross[8, Section 6]) if X is 2-fold absolutely continuous with trivial centre then $\alpha = 1_{\{e\}}$ is easily seen to define an absolute value on X. However

$$\begin{aligned} L^1_\alpha(X) &= \{f \in L^1_{loc}(X) : \int_X |f|\, 1_{\{e\}}\, dm < \infty\} \\ &= \{f \in L^1_{loc}(X) : |f|\,(e)\, m(\{e\}) < \infty\} \\ &= L^1_{loc}(X) \end{aligned}$$

which in general is not contained in $L^1(X)$, and for which $\|\cdot\|_{1,\alpha}$ does not define a norm when $m(\{e\}) = 0$. An example of such a hypergroup is given in [5, Example 9.5].

In view of the above we assume from now on that $\alpha \geq 1$. We follow the development in Reiter [6, 1.6.1] noting that $L^1_\alpha(X)$ has many of the properties of a Beurling algebra.

LEMMA 6. *$C_c(X)$ is dense in $L^1_\alpha(X)$.*

PROOF. The inclusion $C_c(X) \subset L^1_\alpha(X)$ is clear since $\alpha \in L^\infty_{loc}(X)$. To prove denseness consider $f \in L^1_\alpha(X)$, $\epsilon > 0$ and $k_1 \in C_c(X)$ such that

$$\int_X |f\alpha - k_1|\, dm < \epsilon/2.$$

Let C be a compact set with $\text{supp}(k_1) \subset \text{int}(C)$. Choose a constant K such that $\alpha(x) \leq K$ for all $x \in C$, and then $k \in C_c(X)$ with $\text{supp}(k) \subset C$ and

$$\int_X \left|k_1\alpha^{-1} - k\right|\, dm < \epsilon/(2K).$$

Then

$$\int_X |f - k|\, \alpha\, dm < \epsilon$$

as required.

LEMMA 7. *For all $x \in X$, $\|f_x\|_{1,\alpha} \leq \alpha(x)\, \|f\|_{1,\alpha}$.*

PROOF.

$$\int_X |f_x|\,(y)\alpha(y)\, dm(y) \leq \int_X |f|\,(x * y)\alpha(y^-)\, dm(y)$$

$$\begin{aligned} &= \int_X |f|(y)\alpha(y^- * x)\,dm(y) \\ &\le \alpha(x)\int_X |f|(y)\alpha(y)\,dm(y) \\ &= \alpha(x)\ \|f\|_{1,\alpha}\,. \end{aligned}$$

LEMMA 8. *Given $f \in L^1_\alpha(X)$ and $\epsilon > 0$ there exists a neighbourhood U of e such that $\|f_x - f\|_{1,\alpha} < \epsilon$ for all $x \in U$.*

PROOF. Let V be a compact symmetric neighbourhood of e, and choose a constant K such that $\alpha(x) \le K$ for all $x \in V$. Given $f \in L^1_\alpha(X)$ and $\epsilon > 0$, Lemma 6 gives the existence of $k \in C_c(X)$ such that $\|f - k\|_{1,\alpha} < \epsilon/(3K+1)$.

Now by Bloom and Heyer [1, Corollary 2.7], k is uniformly continuous. Hence, writing $K_1 = \sup\{|\alpha(x)| : x \in V * \operatorname{supp}(k)\}$ there is a neighbourhood $U \subset V$ of e such that $\|k_x - k\|_\infty < \epsilon/(3K_1 m(V * \operatorname{supp}(k)))$ for all $x \in U$. Then making use of [5, Lemma 3.2G], and splitting k into its non-negative parts $k_1 - k_2 + i(k_3 - k_4)$ we see that for $x \in U$

$$\operatorname{supp}(k_x) = \operatorname{supp}(\epsilon_{x^-} * k) \subset \operatorname{supp}(\epsilon_{x^-}) * \operatorname{supp}(k) \subset V * \operatorname{supp}(k)$$

and

$$\begin{aligned} \|k_x - k\|_{1,\alpha} &= \int_X |k(x*y) - k(y)|\,\alpha(y)\,dm(y) \\ &\le (\epsilon/(3K_1 m(V * \operatorname{supp}(k))) \int_X 1_{V*\operatorname{supp}(k)}\alpha\,dm \\ &\le \epsilon/3. \end{aligned}$$

We now have for $x \in U$

$$\begin{aligned} \|f_x - f\|_{1,\alpha} &\le \|f_x - k_x\|_{1,\alpha} + \|k_x - k\|_{1,\alpha} + \|k - f\|_{1,\alpha} \\ &\le \alpha(x)\epsilon/(3K+1) + \epsilon/3 + \epsilon/(3K+1) \\ &< \epsilon \end{aligned}$$

and this completes the proof.

We now show that the algebra $L^1_\alpha(X)$ admits a bounded approximate unit. For the remainder of the paper we assume X to be commutative.

PROPOSITION 9. *Let (V_ι) be a base of relatively compact open neighbourhoods of e, and write $k_\iota = m(V_\iota)^{-1}1_{V_\iota}$. Then $k_\iota m \to \epsilon_e$, and for each $f \in L^1_\alpha(X)$, $k_\iota * f \to f$ in $L^1_\alpha(X)$.*

PROOF. Consider

$$\begin{aligned}\int_X |k_\iota * f - f|\,\alpha\,dm &= \int_X \left|\int_X m(V_\iota)^{-1} 1_{V_\iota}(y) f(x * y^-)\,dm(y) \right.\\ &\qquad \left. - \int_X m(V_\iota)^{-1} 1_{V_\iota}(y) f(x)\,dm(y)\right| \alpha(x)\,dm(x)\\ &\le \int_X \int_X \left|f(x * y^-) - f(x)\right| \alpha(x)\,dm(x) m(V_\iota)^{-1} 1_{V_\iota}(y)\,dm(y).\end{aligned}$$

By Lemma 8, given $\epsilon > 0$ there exists ι_0 such that $\left\|(f^-)_y - f^-\right\|_{1,\alpha} < \epsilon$ for all $y \in V_{\iota_0}$. Thus for $\iota \ge \iota_0$

$$\begin{aligned}\int_X |k_\iota * f - f|\,\alpha\,dm &\le \int_X \int_X \left|f(x^- * y^-) - f(x^-)\right| \alpha(x)\,dm(x) m(V_\iota)^{-1} 1_{V_\iota}(y)\,dm(y)\\ &\le \epsilon\end{aligned}$$

and this gives the result.

DEFINITION 10. A linear functional η on $L^1_\alpha(X)$ is referred to as *multiplicative* and *hermitian* if it is non-trivial, $\eta(f * g) = \eta(f)\eta(g)$ and $\eta(f^-) = \overline{\eta(f)}$ for all $f, g \in L^1_\alpha(X)$.

A *semicharacter* χ is a locally bounded measurable function satisfying $\chi(e) = 1$, $\chi(x * y) = \chi(x)\chi(y)$ and $\chi(x^-) = \overline{\chi(x)}$ for all $x, y \in X$. Observe that every positive semicharacter is automatically an absolute value. It is clear from Proposition 3 that every α-bounded semicharacter χ satisfies $|\chi| \le \alpha$. We denote by $\widehat{X}^\alpha$ the set of α-bounded continuous semicharacters on X. It is easy to see that the Fourier transform $\widehat{f}(\chi)$ is defined for every $\chi \in \widehat{X}^\alpha$, and that $f \to \widehat{f}(\chi)$ is multiplicative and hermitian on $L^1_\alpha(X)$.

THEOREM 11. *Every multiplicative hermitian linear functional on $L^1_\alpha(X)$ is of the form $f \to \widehat{f}(\chi)$ for some $\chi \in \widehat{X}^\alpha$.*

PROOF. Let η be a multiplicative hermitian linear functional on $L^1_\alpha(X)$. Since $L^1_\alpha(X)$ is a commutative Banach algebra, Hewitt and Ross [3, Theorem C.21] gives that η is bounded with norm not exceeding 1.

Consider $k_\iota = m(V_\iota)^{-1} 1_{V_\iota}$. Since α is assumed to be locally bounded we have that $k_\iota \in L^1_\alpha(X)$. Choose $g \in L^1_\alpha(X)$ satisfying $\eta(g) \ne 0$ and consider

$$(1) \qquad (k_\iota)_x * g = \epsilon_{x^-} * k_\iota * g = k_\iota * (\epsilon_{x^-} * g) \to \epsilon_{x^-} * g,$$

the limit holding because of Proposition 9. Also we know from Lemma 7 that $\epsilon_{x^-} * g = g_x \in L^1_\alpha(X)$.

Define $\chi(x) := \eta(\epsilon_{x^-} * g)\eta(g)^{-1}$. From (1) and the continuity of η we have

$$(2) \qquad \eta(\epsilon_{x^-} * g) = \lim_\iota \eta((k_\iota)_x * g) = \lim_\iota \eta\left((k_\iota)_x\right) \eta(g)$$

so that $\chi(x) = \lim_\iota \eta((k_\iota)_x)$. Therefore χ is independent of the choice of $g \in L^1_\alpha(X)$ (with $\eta(g) \neq 0$). Again using the continuity of η we have for any $h \in L^1_\alpha(X)$,

$$\eta(\epsilon_{x^-} * h) = \lim_\iota \eta((k_\iota)_x * h) = \lim_\iota \eta((k_\iota)_x)\eta(h) = \chi(x)\eta(h),$$

and putting $h = \epsilon_{y^-} * g$ gives the third equality in the following:

$$\begin{aligned} \chi(x * y)\eta(g) &= \int_X \eta(\epsilon_{z^-} * g)\, d(\epsilon_x * \epsilon_y)(z) \\ &= \eta(\epsilon_{x^-} * \epsilon_{y^-} * g) = \chi(x)\eta(\epsilon_{y^-} * g) \\ &= \chi(x)\chi(y)\eta(g). \end{aligned}$$

Thus $\chi(x * y) = \chi(x)\chi(y)$. We also have

$$\begin{aligned} \chi(x^-) = \eta(\epsilon_x * g)\eta(g)^{-1} &= \eta((\epsilon_{x^-} * g^-)^-)\eta((g^-)^-)^{-1} \\ &= \overline{\eta(\epsilon_{x^-} * g^-)\eta(g^-)^{-1}} \\ &= \overline{\chi(x)}. \end{aligned}$$

Furthermore, using Lemma 7,

$$|\chi(x)| = |\eta(\epsilon_x * g)|\, |\eta(g)|^{-1} \le \|\epsilon_x * g\|_{1,\alpha}\, |\eta(g)|^{-1} \le \alpha(x)\, \|g\|_{1,\alpha}\, |\eta(g)|^{-1},$$

which shows that χ is α-bounded. That χ is continuous at e follows immediately from Lemma 8, and hence χ is continuous everywhere appealing to Bloom and Ressel [2, Corollary 1.11]. This all shows that χ is a continuous α-bounded character.

Finally, choosing $g \in L^1_\alpha(X)$ with $\eta(g) \neq 0$, we observe that for $f \in L^1_\alpha(X)$,

$$\begin{aligned} \eta(f) = \eta(f * g)\eta(g)^{-1} &= \int_X \eta(\epsilon_x * g)\eta(g)^{-1} f(x)\, dm(x) \\ &= \int_X \chi(x^-) f(x)\, dm(x) \\ &= \widehat{f}(\chi), \end{aligned}$$

as required.

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We use Theorem 11 to show that every α-bounded continuous positive-definite function ϕ on X has a Bochner representation

$$\phi(x) = \int_{\widehat{X}^\alpha} \chi(x)\, d\mu(\chi)$$

where $\mu \in M^+(\widehat{X}^\alpha)$. This should be compared with the special case given in Voit [10, Corollary 2.10] where α is taken to be a positive semicharacter on X.

A hermitian, multiplicative, linear functional ρ on any subalgebra S of $L^1_\alpha(X)$ is called *A-bounded* if there is a positive constant K such that $|\rho(f)| \le K A(f)$ for all $f \in S$, where $A(f) := \|f\|_{1,d}$ coincides with the definition given in Proposition 4 provided f has compact support.

In the following, let H denote the set of all non-trivial hermitian, A-bounded, multiplicative, linear functionals on $L^1_\alpha(X)$. We provide H with the topology of pointwise convergence, and $\widehat{X}^\alpha$ with the topology of uniform convergence on compact subsets of X.

THEOREM 12. *Suppose that the hypergroup X is second countable. Then the canonical mapping $F : \widehat{X}^\alpha \to H$ associating with each $\chi \in \widehat{X}^\alpha$ the functional $f \to \widehat{f}(\chi)$ is a continuous Borel isomorphism.*

PROOF. The remark immediately following Definition 10 shows that F is well-defined, and from Theorem 11 we know that F is onto. If $F(\chi) = F(\gamma)$ then $\widehat{f}(\chi) = \widehat{f}(\gamma)$ for all $f \in C_c(X)$, from which it follows using the continuity of χ, γ that $\chi = \gamma$. So we are left with proving that F is continuous, and that its inverse is Borel measurable.

Let $(\chi_\iota) \subset \widehat{X}^\alpha$ be a net converging to χ. For $f \in L^1_\alpha(X)$ and $\epsilon > 0$ there is a compact set $K \subset X$ such that $\int_{K^c} |f|\alpha\, dm < \epsilon/4$, and then for ι sufficiently large $\max\{|\chi_\iota(x) - \chi(x)| : x \in K\} < \epsilon/(2\|f\|_{1,\alpha} + 1)$. This implies that

$$\begin{aligned} |F(\chi_\iota)(f) - F(\chi)(f)| &= \left| \int_X \overline{(\chi_\iota - \chi)} f\, dm \right| \\ &\le 2\int_{K^c} |f|\alpha\, dm + \int_K |\chi_\iota - \chi|\,|f|\, dm \\ &< \epsilon \end{aligned}$$

which gives the continuity and hence measurability of F.

The space $C(X)$ of all continuous complex-valued functions is, with regard to uniform convergence on compact subsets, a Polish space, hence so is $\widehat{X}^\alpha$ as a closed subspace of $C(X)$. As a continuous one-to-one image of $\widehat{X}^\alpha$ the space H turns out to be a so-called standard or Lusin space, and a deep result from topology (Hoffmann-Jørgensen [4, Ch. III, §7, Theorem 2]) tells us that F^{-1} is measurable, that is, F is a Borel isomorphism.

LEMMA 13. *Let $\rho : L^1_{\alpha,c}(X) \to C$ be a hermitian, A-bounded, multiplicative linear functional. Then ρ extends uniquely to a functional $\widetilde{\rho}$ with the same properties on $L^1_\alpha(X)$. The mapping $\rho \to \widetilde{\rho}$ is a homeomorphism with respect to pointwise convergence.*

PROOF. For $f \in L^1_\alpha(X)$ and $D \subset X$ we put $f_D := f 1_D$. Given $\epsilon > 0$ there is a compact set $C \subset X$ such that $\int_{C^c} |f|\alpha\, dm < \epsilon$. If $D, E \subset X$ are compact sets containing C then $D \triangle E \subset C^c$, and it follows that

$$|\rho(f_D) - \rho(f_E)| = |\rho(f(1_D - 1_E))| \le K \int_X |f|\, 1_{D\triangle E}\alpha\, dm < K\epsilon.$$

Therefore $\widetilde{\rho}(f) := \lim_D \rho(f_D)$ exists in C, and $\widetilde{\rho}$ is easily seen to be linear, multiplicative, hermitian and A-bounded.

The last statement will be clear once we prove that pointwise convergence $\rho_\iota \to \rho$ on $L^1_{\alpha,c}(X)$ implies pointwise convergence $\widetilde{\rho}_\iota \to \widetilde{\rho}$ on $L^1_\alpha(X)$. Let $f \in L^1_\alpha(X)$ and $\epsilon > 0$ be given. There is a compact set $D \subset X$ such that $\int_{D^c} |f|\alpha\, dm < \epsilon/4$ and, for ι large enough, $|\rho_\iota(f_D) - \rho(f_D)| < \epsilon/2$. Appealing to Theorem 11 we have the existence of $\chi_\iota, \chi \in \widehat{X}^\alpha$ such that

$$\begin{aligned}\widetilde{\rho}_\iota(f) - \widetilde{\rho}(f) &= \int_X f(\overline{\chi}_\iota - \overline{\chi})\, dm \\ &= \int_D f(\overline{\chi}_\iota - \overline{\chi})\, dm + \int_{D^c} f(\overline{\chi}_\iota - \overline{\chi})\, dm \\ &= \rho_\iota(f_D) - \rho(f_D) + \int_{D^c} f(\overline{\chi}_\iota - \overline{\chi})\, dm\end{aligned}$$

and for such ι chosen as above $|\widetilde{\rho}_\iota(f) - \widetilde{\rho}(f)| < \epsilon$.

THEOREM 14. *Let X be second countable and let $\phi : X \to C$ be a continuous α-bounded, positive-definite function. Then there is a unique measure $\mu \in M^b_+(\widehat{X}^\alpha)$ such that*

$$\phi(x) = \int_{\widehat{X}^\alpha} \chi(x)\, d\mu(\chi), \qquad x \in X.$$

PROOF. Let $S := M_c(X)$ and extend ϕ to $\Phi : S \to C$ by the natural definition $\Phi(s) := \int_X \phi\, ds$. If $s \in S$ has finite support, say $s = \sum_{i=1}^n a_i \epsilon_{x_i}$, then

$$\begin{aligned}\Phi(s * s^\sim) &= \int_X \int_X \phi(x * y^-)\, ds(x)\, ds^\sim(y) \\ &= \sum\nolimits_{j,k=1}^n a_j \overline{a}_k \phi(x_j * x_k^-) \ge 0.\end{aligned}$$

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Any $s \in S$ may be approximated (setwise on the Borel field) by a net (s_ι) of measures with finite support contained in supp(s). The restriction $\phi\ |_{\text{supp}(s)}$ being bounded and continuous, the above inequality valid for each s_ι extends to s. But then Φ is a positive-definite function on S since

$$\sum_{j,k} c_j \bar{c}_k \Phi(s_j * s_k^{\sim}) = \Phi(s * s^{\sim})$$

with $s := \sum_j c_j s_j$. Because $|\Phi(s)| \leq \phi(e) \int_X \alpha\, d\,|s| = \phi(e)A(s)$, the function Φ is A-bounded, and hence so is its restriction $\Phi' := \Phi\ |_{L^1_{\alpha,c}(X)}$; Theorem 5 in Ressel [7] (in connection with Remark 5 to Theorem 4 in the same reference) gives the representation

$$\Phi'(f * g * h) = \int \rho(f * g * h)\, d\nu(\rho), \qquad f, g, h \in L^1_{\alpha,c}(X),$$

ν being a bounded non-negative Radon measure on the set of all hermitian, A-bounded, multiplicative, linear functionals on $L^1_{\alpha,c}(X)$. By Lemma 13 these may be uniquely extended to such functionals on $L^1_\alpha(X)$, that is, to elements of H, and hence ν may be considered as a Radon measure on H. Let μ' denote the image of ν under F^{-1}, and μ its conjugate (the image of μ' under $\chi \to \overline{\chi}$). Then (recall Theorem 12)

$$\begin{aligned}\Phi'(f * g * h) &= \int_H \rho(f * g * h)\, d\nu(\rho)\\ &= \int_{\widehat{X}^\alpha} \int_X (f * g * h)(x)\chi(x)\, dm(x)\, d\mu(\chi)\\ &= \int_X \left[\int_{\widehat{X}^\alpha} \chi(x)\, d\mu(\chi)\right] (f * g * h)(x)\, dm(x)\end{aligned}$$

for all $f, g, h \in L^1_{\alpha,c}(X)$. Fubini's theorem could be used here, since the function $(\chi, x) \to \chi(x)$ on $\widehat{X}^\alpha \times X$ is (easily seen to be) continuous. Writing $\phi_0(x) := \int_{\widehat{X}^\alpha} \chi(x)\, d\mu(\chi)$ we thus have

$$\int_X \phi_0(f * g * h)\, dm = \int_X \phi(f * g * h)\, dm$$

for all $f, g, h \in L^1_{\alpha,c}(X)$. Standard arguments using a bounded approximate unit in $L^1_{\alpha,c}(X)$ show that $\phi_0(x) = \phi(x)$ for all $x \in X$, and this proves the theorem.

References

[1] W. R. Bloom and H. Heyer, 'Characterisation of potential kernels of transient convolution semigroups on a commutative hypergroup', *Probability measures on groups, IX (Proc. Conf., Oberwolfach, 1988)*, Lecture Notes in Math. 1379 (Springer, Berlin, 1989), pp. 21–35.

[2] W. R. Bloom and P. Ressel, 'Positive definite and related functions on hypergroups', *Canad. J. Math.* **43** (1991), 242–254.
[3] E. Hewitt and K. A. Ross, *Abstract harmonic analysis, vol 1. Structure of topological groups. Integration theory, group representations*, Die Gundlehren der mathematischen Wissenschaften 115 (Springer, Berlin, 1963).
[4] J. Hoffmann-Jørgensen, *The theory of analytic spaces*, Var. Publ. Series 10 (Matematisk Institut, Århus Universitet, 1970).
[5] R. I. Jewett, 'Spaces with an abstract convolution of measures', *Adv. Math.* **18** (1975), 1–101.
[6] H. Reiter, *Classical harmonic analysis and locally compact groups*, Oxford Mathematical Monographs (Clarendon Press, Oxford, 1968).
[7] P. Ressel, 'Integral representations on convex semigroups', *Math. Scand.* **61** (1987), 93–111.
[8] K. A. Ross, 'Centers of hypergroups', *Trans. Amer. Math. Soc.* **243** (1978), 251–269.
[9] R. Spector, 'Mesures invariantes sur les hypergroupes', *Trans. Amer. Math. Soc.* **239** (1978), 147–165.
[10] M. Voit, 'Positive characters on commutative hypergroups and some applications', *Math. Z.* **198** (1988), 405–421.

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