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HOMOGENEOUS PBW DEFORMATION FOR ARTIN–SCHELTER REGULAR ALGEBRAS

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Abstract

We introduce a method named *homogeneous* PBW deformation that preserves the regularity and some other homological properties for multigraded algebras. The method is used to produce Artin–Schelter regular algebras without the hypothesis on grading.

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1. Introduction

The project of classifying Artin–Schelter regular algebras began with the work of Artin, Schelter, Tate and Van den Bergh between 1987 and 1991. By their successive work in [1–3], three-dimensional Artin–Schelter regular algebras have been classified completely. Though the whole project is far from finished, much progress has been made on classifying such algebras of higher dimensions (see [7, 13, 15–17, 19–22]). Some effective methods have been presented to produce Artin–Schelter regular algebras, such as the A_∞-algebra method, double Ore extension, Hilbert driven Gröbner basis computations and combinatorics and so on. In this work, the methods usually require a hypothesis of grading on the algebras to endow them with an appropriate $\mathbb{Z}^2$-grading and reduce the computational complexity. A question follows: is it possible to find new Artin–Schelter regular algebras without the hypothesis of grading?

According to an observation in [16, Theorem 2.8], the regularity of algebras can be deduced from their associated $\mathbb{Z}^r$-graded algebras by assigning an appropriate $\mathbb{Z}^r$-grading on the set of generators. We take advantage of this idea to produce new Artin–Schelter regular algebras based on the known ones endowed with $\mathbb{Z}^r$-grading. The main purpose of this paper is to present this method, which we have named *homogeneous PBW deformation*. It partially answers the question in the last paragraph.

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On the other hand, the algebras whose obstructions consist of Lyndon words are studied in [8, 22]. By using the language of Lyndon words, the homological invariants of the algebras can be read off quickly. It is interesting that all the $\mathbb{Z}^2$-graded Artin–Schelter regular algebras of low dimension with two generators fall into this category [21]. If one is able to find an appropriate $\mathbb{Z}^r$-filtration on each such Artin–Schelter regular algebra such that the associated $\mathbb{Z}^r$-graded algebra satisfies some conditions, the homogeneous PBW deformation provides another approach to the classification of Artin–Schelter regular algebras.

The paper is organised as follows. We fix some notation and recall definitions in Section 2. In Section 3, we introduce the concept of the homogeneous PBW deformation and show how to determine it by means of the Gröbner basis. We show that the regularity and some other homological properties are preserved under the deformation. Section 4 is devoted to describing the change of Nakayama automorphisms under the homogeneous PBW deformation. By using the method of the homogeneous PBW deformation, the language of Lyndon words and the χ-condition, we present some new classes of Artin–Schelter regular algebras in the final section.

Throughout the paper, k is a fixed algebraically closed field of characteristic 0. The set of natural numbers is $\mathbb{N} = \{0, 1, 2, \ldots\}$. Unless otherwise stated, graded means $\mathbb{Z}$-graded and the tensor product $\otimes$ means $\otimes_k$. For simplicity, we only consider graded algebras that are generated in degree one. We also set $\mathbb{Z}^r = \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{r}$ with the standard basis $\epsilon_i = (0, \ldots, 1, \ldots, 0)$ for $i = 1, 2, \ldots, r$.

2. Notation and definitions

By the *norm map* on $\mathbb{Z}^r$, we mean the map $\|\cdot\| : \mathbb{Z}^r \to \mathbb{Z}$ which sends $\alpha = (a_1, \ldots, a_r)$ to $\|\alpha\| = \sum_{i=1}^r a_i$. An *admissible ordering* $<$ on $\mathbb{Z}^r$ related to the norm map is a total ordering such that $\|\alpha_1\| < \|\alpha_2\|$ implies $\alpha_1 < \alpha_2$ and $\alpha_1 < \alpha_2$ implies $\alpha_1 + \alpha_3 < \alpha_2 + \alpha_3$ for any $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}^r$.

Let $A = \bigoplus_{\alpha \in \mathbb{Z}^r} A_\alpha$ be a $\mathbb{Z}^r$-graded algebra. For a homogeneous element $a \in A_\alpha$, we call α and $\|\alpha\|$ the *degree* and *total degree* of a, denoted by $\deg_A a$ and $\operatorname{tdeg}_A a$, respectively. We call A *positive* if $A_\alpha = 0$ for all $\alpha \notin \mathbb{N}^r$. A positive $\mathbb{Z}^r$-graded algebra A is called *connected* if $A_0 = k$. A connected $\mathbb{Z}^r$-graded algebra A is called *properly* $\mathbb{Z}^r$-graded if A is generated by $\bigoplus_{i=1}^r A_{\epsilon_i}$ with $A_{\epsilon_i} \neq 0$ for $i = 1, 2, \ldots, r$. For simplicity, we only consider connected properly $\mathbb{Z}^r$-graded algebras.

A $\mathbb{Z}^r$-graded algebra A is called *locally finite* if A_α is finite dimensional for any $\alpha \in \mathbb{Z}^r$. In this case, there exists a map $d : \mathbb{Z}^r \to \mathbb{N}$ defined by $d(\alpha) = \dim_k A_\alpha$, and the *Hilbert series* of A is $H_A(\mathbf{t}) = \sum_{\alpha=(a_1,a_2,\ldots,a_r)} d(\alpha) t_1^{a_1} t_2^{a_2} \cdots t_r^{a_r}$. We denote the Hilbert series of A by $H_A(t)$ if A is just a graded algebra.

For any $\mathbb{Z}^r$-graded left (or right) A-module $M = \bigoplus_{\alpha \in \mathbb{Z}^r} M_\alpha$ and $\beta \in \mathbb{Z}^r$, its shift $M(\beta)$ is also a $\mathbb{Z}^r$-graded left (or right) A-module defined by $M(\beta)_\alpha = M_{\alpha+\beta}$ for any $\alpha \in \mathbb{Z}^r$. Let N be a $\mathbb{Z}^r$-graded left (or right) A-module; we write $\operatorname{Hom}_A(M, N)$ to be the set of A-homomorphisms preserving degrees. Also, write $\underline{\operatorname{Hom}}_A(M, N) = \bigoplus_{\alpha \in \mathbb{Z}^r} \operatorname{Hom}_A(M, N(\alpha))$ and the derived functor $\underline{\operatorname{Ext}}_A^i(M, N) = \bigoplus_{\alpha \in \mathbb{Z}^r} \operatorname{Ext}_A^i(M, N(\alpha))$.

For any $\mathbb{Z}^r$-graded algebra A, we associate it with a graded algebra A^{gr}, where

$$A^{gr} = \bigoplus_n \Big(\bigoplus_{\|\alpha\|=n} A_\alpha \Big).$$

Obviously, A^{gr} is connected and generated by degree one if A is connected and proper.

Definition 2.1. A connected $\mathbb{Z}^r$-graded algebra A is called *Artin–Schelter regular* (AS-regular, for short) of dimension d if the following three conditions hold:

(AS1) A has finite global dimension d;
(AS2) A has finite Gelfand–Kirillov (GK) dimension (GKdim);
(AS3) A is Gorenstein; that is, for some $l \in \mathbb{Z}^r$,

$$\underline{\mathrm{Ext}}^i_A(k, A) = \begin{cases} k(l) & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}$$

where l is called the *Gorenstein parameter*.

This definition is originally due to Artin *et al.* [2] for graded algebras. It makes sense because of the next lemma.

Lemma 2.2 [21, Lemma 1.2]. *Let A be a connected $\mathbb{Z}^r$-graded algebra. Then A is AS-regular if and only if A^{gr} is AS-regular.*

A bridge to link an algebra and a $\mathbb{Z}^r$-graded algebra is $\mathbb{Z}^r$-filtration.

Definition 2.3. Let $<$ be an admissible ordering on $\mathbb{Z}^r$. An algebra B is called a *$\mathbb{Z}^r$-filtered algebra* if there is a family $\{F_\alpha B\}_{\alpha\in\mathbb{Z}^r}$ of k-subspaces of B such that:

(a) $F_\alpha B \subseteq F_{\alpha'} B$ if $\alpha < \alpha' \in \mathbb{Z}^r$, $B = \bigcup_{\alpha\in\mathbb{Z}^r} F_\alpha B$ and $1 \in F_0 B$;
(b) $F_\alpha B \cdot F_{\alpha'} B \subseteq F_{\alpha+\alpha'} B$ for any $\alpha, \alpha' \in \mathbb{Z}^r$.

In the definition above, the family $\{F_\alpha B\}_{\alpha\in\mathbb{Z}^r}$ is called a *$\mathbb{Z}^r$-filtration* of B.

An *associated $\mathbb{Z}^r$-graded algebra* of the $\mathbb{Z}^r$-filtered algebra B is defined by

$$G^r(B) = \bigoplus_{\alpha\in\mathbb{Z}^r} \frac{F_\alpha B}{F_{<\alpha} B},$$

where $F_{<\alpha} B = \bigcup_{\alpha'<\alpha} F_{\alpha'} B$.

The noncommutative Gröbner basis theory plays a pivotal role in this paper. We review it briefly.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and let X^* be the set of words generated by X including 1. We say that u is a *factor* of v if $v = w_1 u w_2$ for some $w_1, w_2 \in X^*$. If $w_1 = 1$ and $w_2 \neq 1$, then u is called a *prefix* of v and, if $w_1 \neq 1$ and $w_2 = 1$, then u is a *suffix* of v. A factor u of v is *proper* if $u \neq v$.

We fix an admissible ordering $\prec$ on X^*. This means a well ordering such that $1 \prec u$ for any $u \in X^*\backslash\{1\}$ and $u \prec v$ imply $wuw' \prec wvw'$ for any $u, v, w, w' \in X^*$. Let f be a polynomial in $k\langle X\rangle$. The *leading monomial* $\mathrm{LM}(f)$ is the largest monomial occurring

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in f, the *leading coefficient* $\mathrm{LC}(f)$ is the coefficient of $\mathrm{LM}(f)$ and the *leading word* $\mathrm{LW}(f)$ is just $\mathrm{LM}(f)/\mathrm{LC}(f)$. For any set $S \subseteq k\langle X\rangle$, set $\mathrm{LW}(S) = \{\mathrm{LW}(f) \mid f \in S\}$. The set S is called *monic* if $\mathrm{LC}(f) = 1$ for any $f \in S$.

Definition 2.4. Let I be an ideal of $k\langle X\rangle$. A set $\mathcal{G} \subseteq I$ is called the *Gröbner basis* of I if any nonzero word in $\mathrm{LW}(I)$ has a factor in the set $\mathrm{LW}(\mathcal{G})$. A monic Gröbner basis $\mathcal{G}$ is called *reduced* if $\mathrm{LW}(f)$ has no factor in the set $\mathrm{LW}(\mathcal{G}\backslash\{f\})$ for any $f \in \mathcal{G}$.

Moreover, if $\mathcal{G}$ is the reduced Gröbner basis of I, we say that $\mathrm{LW}(\mathcal{G})$ is the set of *obstructions* of the algebra $k\langle X\rangle/I$.

3. Homogeneous PBW deformations

In this section, we introduce the concept of homogeneous PBW deformations for the positive $\mathbb{Z}^r$-graded algebra, and some immediate properties are collected. We will discuss the change of Nakayama automorphisms in the process of homogeneous PBW deformations in the next section.

We fix an admissible ordering $<$ on $\mathbb{Z}^r$ related to the norm map.

Let $A = k\langle X\rangle/(S)$ be a positive $\mathbb{Z}^r$-graded algebra, where X is the minimal set of generators. Denote by $k\langle X\rangle_\alpha$ the subspace consisting of all $\mathbb{Z}^r$-homogeneous elements of $k\langle X\rangle$ with degree α for any $\alpha \in \mathbb{Z}^r$. Of course, $k\langle X\rangle_\alpha = 0$ if $\alpha \notin \mathbb{N}^r$.

Instead of a $\mathbb{Z}^r$-homogeneous set of relations S, we may consider a non-$\mathbb{Z}^r$-homogeneous but $\mathbb{Z}$-homogeneous set of relations P and obtain a positive non-$\mathbb{Z}^r$-graded but positive graded algebra $U = k\langle X\rangle/(P)$. As usual for the PBW deformation (see [6]), we assume $p \notin \bigoplus_{\alpha<\deg_A p} k\langle X\rangle_\alpha$ and that the image of p by the canonical projection of $\bigoplus_{\alpha\leq\deg_A p} k\langle X\rangle_\alpha$ to $k\langle X\rangle_{\deg_A p}$ is just an element of S for any $p \in P$. That is to say, we can describe P by

$$P = \{s + \bar{s} \mid s \in S, \bar{s} \in k\langle X\rangle \text{ with } \deg_A \bar{s} < \deg_A s \text{ and } \mathrm{tdeg}_A \bar{s} = \mathrm{tdeg}_A s, \text{ or } \bar{s} = 0\}.$$

Hence, $U = k\langle X\rangle/(P)$ is a graded algebra induced by letting $\deg_U x = \mathrm{tdeg}_A x$ for any $x \in X$. It is not a positive $\mathbb{Z}^r$-graded algebra in general, but there is a natural $\mathbb{Z}^r$-degree on the free algebra $k\langle X\rangle$ induced by $\deg^r_{k\langle X\rangle} x = \deg_A x$ for any $x \in X$. This assigns a $\mathbb{Z}^r$-filtration on U as follows:

$$F_\alpha U = \frac{F_\alpha k\langle X\rangle + (P)}{(P)} \quad \text{for any } \alpha \in \mathbb{Z}^r,$$

where $F_\alpha k\langle X\rangle = \bigoplus_{\alpha'\leq\alpha} k\langle X\rangle_{\alpha'}$ for any $\alpha \in \mathbb{Z}^r$.

Denote by $G^r(U)$ the associated $\mathbb{Z}^r$-graded algebra of U. In the situation described, there is a natural $\mathbb{Z}^r$-graded surjection

$$\varphi : A \to G^r(U).$$

Definition 3.1. We say that the graded algebra U is a *homogeneous PBW deformation* of A if φ is an isomorphism. We say that U is *trivial* if U is isomorphic to A^{gr} as graded algebras.

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Remark 3.2. The homogeneous PBW deformation U is strongly related to the $\mathbb{Z}^r$-grading on A. Suppose that A and B are two nonisomorphic positive $\mathbb{Z}^r$-graded algebras but A^{gr} is isomorphic to B^{gr}. Then the homogeneous PBW deformations of A and of B may be nonisomorphic. It is even possible for one to have only trivial PBW deformations while the other has nontrivial PBW deformations (see Example 5.6 below).

The Gröbner basis is an effective way to describe $G^r(U)$. Choose an arbitrary monomial ordering $\prec$ on X^*. It induces a $\mathbb{Z}^r$-graded admissible ordering $\prec_{\mathbb{Z}^r}$ on X^*: for $u, v \in X^*$, $u \prec_{\mathbb{Z}^r} v$ is defined by:

(a) $\deg^r_{k\langle X\rangle} u < \deg^r_{k\langle X\rangle} v$; or
(b) $\deg^r_{k\langle X\rangle} u = \deg^r_{k\langle X\rangle} v$ and $u \prec v$.

For a nonzero polynomial $f \in k\langle X\rangle$, we can write $f = \sum_{i=1}^{q} f_i$, where each nonzero f_i is $\mathbb{Z}^r$-homogeneous with $\deg^r_{k\langle X\rangle} f_i = \alpha_i$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_q$. We call f_q the *leading homogeneous polynomial* of f, denoted by $\mathrm{LH}(f)$. Let $\mathcal{G}$ be the reduced monic Gröbner basis of (P) with respect to $\prec_{\mathbb{Z}^r}$ and let $\mathrm{LH}(\mathcal{G}) = \{\mathrm{LH}(f) \mid f \in \mathcal{G}\}$. The following proposition says that $\mathrm{LH}(\mathcal{G})$ is a sufficient set for describing $G^r(U)$.

Proposition 3.3. *Suppose that U and $G^r(U)$ are defined as above and $\mathcal{G}$ is the Gröbner basis of (P). Then:*

(a) $\mathrm{LH}(\mathcal{G})$ *is a Gröbner basis of the ideal* $(\mathrm{LH}(\mathcal{G}))$*;*
(b) $G^r(U) \cong k\langle X\rangle/(\mathrm{LH}(\mathcal{G}))$.

Proof. The assertion follows from [12, Ch. 4, Proposition 2.2 and Theorem 2.3]. □

As an immediate consequence, we have a characterisation of homogeneous PBW deformations.

Theorem 3.4. *Let $A = k\langle X\rangle/(S)$ be a positive $\mathbb{Z}^r$-graded algebra and let $U = k\langle X\rangle/(P)$ be defined as above. Suppose that $\mathcal{G}$ is the reduced monic Gröbner basis of (P) with respect to $\prec_{\mathbb{Z}^r}$. Then the following are equivalent.*

(a) *U is a homogeneous PBW deformation of A.*
(b) $(S) = (\mathrm{LH}(\mathcal{G}))$.
(c) $\mathrm{LH}(\mathcal{G})$ *is a Gröbner basis of* (S).
(d) $H_{A^{gr}}(t) = H_U(t)$.

Proof. (a) if and only if (b) if and only if (c) follow by Proposition 3.3. Now we show (a) if and only if (d).

The set $\mathrm{LW}(\mathrm{LH}(\mathcal{G}))$ and the set $\mathrm{LW}(\mathcal{G})$ are the same and that $\mathrm{LH}(\mathcal{G})$ is a Gröbner basis of $(\mathrm{LH}(\mathcal{G}))$ follows from Proposition 3.3(a). Thus, the k-bases of $k\langle X\rangle/(\mathrm{LH}(\mathcal{G}))$ and U have the same form and $H_{G^r(U)^{gr}}(t) = H_U(t)$ by Proposition 3.3(b).

Notice that there exists a natural $\mathbb{Z}^r$-graded surjection $\varphi : A \to G^r(U)$. So, for any $\alpha \in \mathbb{Z}^r$, $\dim_k A_\alpha \geq \dim_k G^r(U)_\alpha$. Hence, $H_{A^{gr}}(t) = H_{G^r(U)^{gr}}(t)$ implies $H_A(\mathbf{t}) = H_U(\mathbf{t})$. Otherwise, there exists at least one $\alpha_0 \in \mathbb{Z}^r$ such that $\dim_k A_{\alpha_0} > \dim_k G^r(U)_{\alpha_0}$.

Therefore, U is a homogeneous PBW deformation of A if and only if $G^r(U) \cong A$ if and only if $H_{A^{gr}}(t) = H_{G^r(U)^{gr}}(t) = H_U(t)$. □

Next, we show some preserved properties under the homogeneous PBW deformation. Before doing so, we give a simple example to motivate the general case. Choose an admissible ordering $<$ on $\mathbb{Z}^r$ related to the norm map. For convenience, we always use lexicographic ordering $<_{lex}$ on $\mathbb{Z}^r$; that is, $\alpha <_{lex} \beta$ if and only if $\|\alpha\| < \|\beta\|$, or $\|\alpha\| = \|\beta\|$ and there exists t $(1 \le t \le r)$ such that $a_i = b_i$ for $i < t$ but $a_t < b_t$, where $\alpha = (a_1, \dots, a_r)$ and $\beta = (b_1, \dots, b_r)$ are two arbitrary elements in $\mathbb{Z}^r$.

It is well known that there are only two classes of two-dimensional AS-regular algebras up to isomorphism, of which one is a positive $\mathbb{Z}^2$-graded algebra called the quantum plane. We show that the other one is just a homogeneous ($\mathbb{Z}^2$-) PBW deformation of it.

Example 3.5. Denote the quantum plane by $\mathfrak{A}(q) = k\langle x, y\rangle/(xy - qyx)$ with $\deg x = (1, 0)$ and $\deg y = (0, 1)$, where $q \in k - \{0\}$. Set

$$P = \{xy - qyx + py^2\}, \quad p \in k.$$

Then we have $U(\mathfrak{A}(q)) = k\langle x, y\rangle/(xy - qyx + py^2)$. An admissible ordering $\prec$ on $\{x, y\}^*$ is provided by $\prec_{lex}$ (see Section 5). It is easy to check that P is the Gröbner basis of (P) with respect to $\prec_{\mathbb{Z}^2}$ and $\mathrm{LH}(P) = \{xy - qyx\}$. Thus, $U(\mathfrak{A}(q))$ is a homogeneous PBW deformation of $\mathfrak{A}(q)$ by Theorem 3.4. Up to graded isomorphism, there are two cases.

(i) $q \neq 1$: $U(\mathfrak{A}(q)) \cong \mathfrak{A}(q)^{gr}$ by the following map:

$$x \mapsto x - \frac{p}{1-q}y, \quad y \mapsto y.$$

(ii) $q = 1$: $U(\mathfrak{A}(1)) \cong k\langle x, y\rangle/(xy - yx - y^2)$ $(p \neq 0)$ is just the other class of AS-regular algebras of dimension two called the Jordan plane. It is also a homogeneous PBW deformation of $\mathfrak{A}(1)$, which is an enveloping algebra of the Lie algebra.

As a consequence, each homogeneous PBW deformation of $\mathfrak{A}(q)$ is an AS-regular algebra. We will show that this is always true in the following proposition.

Proposition 3.6. *Let $A = k\langle X\rangle/(S)$ be a properly $\mathbb{Z}^r$-graded algebra. Suppose that U is the homogeneous PBW deformation of A. Then:*

(a) *$g$ldim $U \le g$ldim A;*
(b) GKdim U = GKdim A;
(c) *if A is AS-regular, then U is AS-regular;*
(d) *if A is strongly noetherian, then so is U;*
(e) *if A is Auslander regular, then so is U. In addition, if A is Cohen–Macaulay, then so is U.*

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Proof. Since $A = k\langle X\rangle/(S)$ is a properly $\mathbb{Z}^r$-graded algebra, there exists a partition of $X = \{X_1, X_2, \ldots, X_r\}$ such that $\deg_A x = (\delta_{1i}, \delta_{2i}, \ldots, \delta_{ri})$ for $x \in X_i$ $(i = 1, 2, \ldots, r)$, where δ_{ij} is the Kronecker symbol. The inherited $\mathbb{Z}^r$-degree $\deg^r_{k\langle X\rangle} x = \deg_A x$ for $x \in X_i$ $(i = 1, 2, \ldots, r)$.

Then (a) and (b) follow from $G^r(U) \cong A$, [16, Lemma 2.6], [9, Theorem 2.8] and [10, Corollary 2.8].

Notice that $G^r(U)$ is the associated $\mathbb{Z}^r$-graded algebra of U for the $\mathbb{Z}^r$-filtration induced by $\deg^r_{k\langle X\rangle}$. It corresponds to a $\mathbb{Z}^r$-filtration on U arising from a partition on the set of generators constructed in [16].

Hence, (c), (d) and the first part of (e) follow from [16, Theorems 2.8 and 2.11]. The second part of (e) is proved in [23]. □

4. Nakayama automorphisms

We describe the change of Nakayama automorphisms in the process of homogeneous PBW deformations. The Nakayama automorphism is a more explicit invariant to characterise the different homological behaviour of algebras. The definition of Nakayama automorphisms of AS-regular algebras is dependent on that of skew Calabi–Yau algebras. We refer the reader to [14] for the details.

In the sequel, we employ the following notation. Let A be an algebra. Denote by A^e the enveloping algebra $A \otimes A^o$, where A^o is the opposite algebra of A. Let M be an A^e-module, which is therefore also an A-bimodule. Suppose that μ, ν are two automorphisms of A. Then the twisting module ${}^{\nu}M^{\mu}$ is defined such that ${}^{\nu}M^{\mu} = M$ as k-spaces, and the A^e-module structure becomes $a_1 * m * a_2 = \nu(a_1)m\mu(a_2)$ for any $a_1, a_2 \in A$ and $m \in {}^{\nu}M^{\mu}$.

Lemma 4.1 [14, Lemma 1.2]. *Let A be a connected $\mathbb{Z}^r$-graded algebra. Then A is AS-regular of global dimension d if and only if A satisfies:*

(a) *A as a $\mathbb{Z}^r$-graded A^e-module has a projective resolution that has finite length and such that each term in the projective resolution is finitely generated;*
(b) *there exists a $\mathbb{Z}^r$-graded automorphism μ of A such that*

$$\underline{\mathrm{Ext}}^i_{A^e}(A, A^e) = \begin{cases} {}^1A^{\mu} & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}$$

as A^e-modules, where 1 denotes the identity map of A.

In this case, we say that A is an AS-regular algebra with Nakayama automorphism μ.

Let σ be an algebra endomorphism of a $\mathbb{Z}^r$-filtered algebra B. We say that σ is *$\mathbb{Z}^r$-filtered* if $\sigma(F_\alpha B) \subseteq \operatorname{Im}\sigma \cap F_\alpha B$ for any $\alpha \in \mathbb{Z}^r$. Also, there is an induced $\mathbb{Z}^r$-graded algebra endomorphism $G^r(\sigma)$ of $G^r(B)$. If the inclusion is the strict equality, then we say that σ is *strict*. There are many examples of AS-regular algebras with $\mathbb{Z}^r$-filtered graded automorphisms.

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REMARK 4.2. Let U be a homogeneous PBW deformation of a $\mathbb{Z}^r$-graded algebra $A = k\langle X\rangle/(S)$. We adopt the $\mathbb{Z}^r$-filtration on U as the one induced by $\mathbb{Z}^r$-grading on $k\langle X\rangle$. Hence, each $\mathbb{Z}^r$-filtered graded endomorphism of U is $\mathbb{Z}^r$-filtered with respect to this particular $\mathbb{Z}^r$-filtration on U.

THEOREM 4.3. *Let $A = k\langle X\rangle/(S)$ be a $\mathbb{Z}^r$-graded AS-regular algebra with Nakayama automorphism μ. Suppose that U is a homogeneous PBW deformation of A with a $\mathbb{Z}^r$-filtered Nakayama automorphism σ. Then $G^r(\sigma) = \mu$.*

The proof of this theorem appears later in this section. For the sake of completeness, we first give some definitions and their consequences from [16].

DEFINITION 4.4. Let B be a $\mathbb{Z}^r$-filtered algebra and M a B-module. The module M is called *$\mathbb{Z}^r$-filtered* if there exists a $\mathbb{Z}^r$-filtration on it; that is, a family $\{F_\alpha M\}_{\alpha\in\mathbb{Z}^r}$ of k-subspaces of M such that:

(a) $F_\alpha M \subseteq F_{\alpha'} M$ if $\alpha < \alpha'$, $M = \bigcup_{\alpha\in\mathbb{N}^r} F_\alpha M$;
(b) $F_\alpha B.F_{\alpha'} M \subseteq F_{\alpha+\alpha'} M$ for any $\alpha, \alpha' \in \mathbb{N}^r$.

There is an *associated $\mathbb{Z}^r$-graded module* of M; that is, $G^r(M) = \bigoplus_{\alpha\in\mathbb{Z}^r} F_\alpha M/F_{<\alpha} M$, where $F_{<\alpha} M = \bigcup_{\alpha'<\alpha} F_{\alpha'} M$. Notice that $G^r(M)$ is a $\mathbb{Z}^r$-graded $G^r(B)$-module.

Let M be a $\mathbb{Z}^r$-filtered B-module. If L is a submodule of M, there is an *induced $\mathbb{Z}^r$-filtration* $\{F_\alpha L\}_{\alpha\in\mathbb{Z}^r}$ on L, where $F_\alpha L = L \bigcap F_\alpha M$. An *induced $\mathbb{Z}^r$-filtration* $\{F_\alpha(M/L)\}_{\alpha\in\mathbb{Z}^r}$ on M/L is defined by $F_\alpha(M/L) = (F_\alpha M + L)/L$. In the following, by referring to a $\mathbb{Z}^r$-filtration on a submodule (a quotient module, respectively), we always mean the induced one. For the direct sum of a set of $\mathbb{Z}^r$-filtered modules $\{M_i\}_{i\in I}$, there exists a $\mathbb{Z}^r$-filtration on $\bigoplus_{i\in I} M_i$ such that $F_\alpha(\bigoplus_{i\in I} M_i) = \bigoplus_{i\in I} F_\alpha M_i$ for any $\alpha \in \mathbb{Z}^r$.

Suppose that B is also graded. We next define the *$\mathbb{Z}^r$-filtered shift* of the $\mathbb{Z}^r$-filtered graded B-module M. Fix $\beta \in \mathbb{Z}^r$ and define the $\mathbb{Z}^r$-filtered shift $M(-\beta)$ as follows: as a graded B-module it is $M(-\|\beta\|)$ and the $\mathbb{Z}^r$-filtration on $M(-\beta)$ becomes $F_\alpha(M(-\beta)) = F_{\alpha-\beta} M$ for all $\alpha \in \mathbb{Z}^r$. In this case, $G^r(M(-\beta)) \cong G^r(M)(-\beta)$.

DEFINITION 4.5. For two $\mathbb{Z}^r$-filtered graded B-modules M and N, let ϕ be a B-homomorphism from M to N. We say that ϕ is *$\mathbb{Z}^r$-filtered* if $\phi(F_\alpha M) \subseteq F_\alpha N$ for any $\alpha \in \mathbb{Z}^r$. In addition, if $\phi(F_\alpha M) = \phi(M) \bigcap F_\alpha N$ for all $\alpha \in \mathbb{N}^r$, we say that ϕ is *strict*.

Moreover, ϕ induces a $G^r(B)$-homomorphism from $G^r(M)$ to $G^r(N)$ denoted by $G^r(\phi)$.

Now let $B = k\langle X\rangle/I$ be a graded algebra. We define a $\mathbb{Z}^r$-filtration on B based on the $\mathbb{Z}^r$-grading on $k\langle X\rangle$. That is, given a partition on $X = \{X_1, \ldots, X_r\}$ endowed with $\mathbb{Z}^r$-degree $\deg^r x = (\delta_{1i}, \ldots, \delta_{ri})$ if $x \in X_i$, the $\mathbb{Z}^r$-filtration on B is defined by

$$F_\alpha B = \frac{F_\alpha k\langle X\rangle + I}{I},$$

where $F_\alpha k\langle X\rangle = \mathrm{Span}_k\{u \in X^* \mid \deg^r u \leq \alpha\}$ for any $\alpha \in \mathbb{Z}^r$. Its associated $\mathbb{Z}^r$-graded algebra is still denoted by $G^r(B)$.

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The $\mathbb{Z}^r$-filtration on B induces a $\mathbb{Z}^r$-filtration on B^e by defining

$$F_\alpha(B^e) = \sum_{\alpha'+\alpha''=\alpha} F_{\alpha'}B \otimes F_{\alpha''}B^o,$$

and it is trivial to check that $G^r(B^e) \cong G^r(B) \otimes G^r(B^o)$ as $\mathbb{Z}^r$-graded algebras.

We quote three elementary lemmas, which are special cases of ones in [16]. The analogous results and proofs can be found in [9, 12].

Lemma 4.6. *Suppose that we have a $\mathbb{Z}^r$-filtered sequence of graded B^e-modules*

$$\bigoplus_{i=1}^{s_1} B(-\alpha_i) \xrightarrow{\varphi_1} \bigoplus_{i=1}^{s_2} B(-\beta_i) \xrightarrow{\varphi_2} \bigoplus_{i=1}^{s_3} B(-\gamma_i), \tag{$\natural$}$$

where s_1, s_2, s_3 are finite positive integers, and $\alpha_i, \beta_j, \gamma_k \in \mathbb{Z}^r$. The associated $\mathbb{Z}^r$-graded sequence is

$$\bigoplus_{i=1}^{s_1} G^r(B)(-\alpha_i) \xrightarrow{G^r(\varphi_1)} \bigoplus_{i=1}^{s_2} G^r(B)(-\beta_i) \xrightarrow{G^r(\varphi_2)} \bigoplus_{i=1}^{s_3} G^r(B)(-\gamma_i) \tag{$G^r(\natural)$}$$

and the sequence ($G^r(\natural)$) is exact if and only if the sequence ($\natural$) is exact and φ_1, φ_2 are strict.

Lemma 4.7. *Let M_1 and M_2 be $\mathbb{Z}^r$-filtered graded B^e-modules.*

(a) *Let $\phi : M_1 \to M_2$ be a strict $\mathbb{Z}^r$-filtered graded B^e-module homomorphism. Then* $\operatorname{Im} G^r(\phi) \cong G^r(\operatorname{Im} \phi)$ *and* $\operatorname{Ker} G^r(\phi) \cong G^r(\operatorname{Ker} \phi)$.

(b) *If M_2 is a submodule of M_1, then $G^r(M_1/M_2) \cong G^r(M_1)/G^r(M_2)$.*

Lemma 4.8. *Suppose that $G^r(B)$ as a graded $G^r(B^e)$-module has a finite free resolution:*

$$0 \longrightarrow \underline{P}_m \xrightarrow{\underline{\partial}_m} \underline{P}_{m-1} \xrightarrow{\underline{\partial}_{m-1}} \cdots \xrightarrow{\underline{\partial}_2} \underline{P}_1 \xrightarrow{\underline{\partial}_1} \underline{P}_0 \xrightarrow{\underline{\partial}_0} G^r(B) \longrightarrow 0, \tag{$\underline{P}.$}$$

where $\underline{P}_j = \bigoplus_{i=1}^{s_j} G^r(B^e)(-\alpha_i^j)$ with $\alpha_i^j \in \mathbb{Z}^r$ for $0 \le j \le m$. Then

(i) *there exists a finite free resolution of B as graded B^e-module:*

$$0 \longrightarrow P_m \xrightarrow{\partial_m} P_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} B \longrightarrow 0, \tag{$P.$}$$

where $P_j = \bigoplus_{i=1}^{s_j} B^e(-\alpha_i^j)$ is such that $G^r(P_j) \cong \underline{P}_j$ and $G^r(\partial_j) = \underline{\partial}_j$ for all $0 \le j \le m$.

(ii) *applying functors* $\operatorname{Hom}_{G^r(B^e)}(-, G^r(B^e))$ *and* $\operatorname{Hom}_{B^e}(-, B^e)$ *to ($\underline{P}.$) and ($P.$), respectively, yields two complexes:*

$$0 \longleftarrow \underline{P}'_m \xleftarrow{\underline{\partial}'_m} \underline{P}'_{m-1} \xleftarrow{\underline{\partial}'_{m-1}} \cdots\cdots \xleftarrow{\underline{\partial}'_0} \underline{P}'_0 \longleftarrow 0,$$

where $\underline{P}'_j = \bigoplus_{i=1}^{s_j} G^r(B^e)(\alpha_i^j)$ with $\alpha_i^j \in \mathbb{Z}^r$ for $0 \le j \le m$, and

$$0 \longleftarrow P'_m \xleftarrow{\partial'_m} P'_{m-1} \xleftarrow{\partial'_{m-1}} \cdots\cdots \xleftarrow{\partial'_0} P'_0 \longleftarrow 0,$$

where $P'_j = \bigoplus_{i=1}^{s_j} B^e(\alpha_i^j)$ for $0 \le j \le d$ such that $G^r(P'_j) \cong \underline{P}'_j$ and $G^r(\partial'_j) = \underline{\partial}'_j$ for all $0 \le j \le m$.

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LEMMA 4.9. *Let μ be a $\mathbb{Z}^r$-filtered graded algebra automorphism of B. Then the following conditions hold.*

(a) *μ is strict and $G^r(\mu)$ is a $\mathbb{Z}^r$-graded algebra automorphism of $G^r(B)$.*
(b) *If M is a $\mathbb{Z}^r$-filtered graded right B-module, and M^μ is endowed with the same $\mathbb{Z}^r$-filtration, then $G^r(M^\mu) \cong G^r(M)^{G^r(\mu)}$ as graded right $G^r(B)$-modules.*

PROOF. (a) μ is strict if and only if $\mu(F_\alpha B) = F_\alpha B$ for any $\alpha \in \mathbb{Z}^r$. Obviously, μ is injective when it is restricted on any $F_\alpha B$. However, each $F_\alpha B$ is finite dimensional, by the construction. Thus, the strictness of μ is clear. The statement for $G^r(\mu)$ is an immediate consequence.

(b) Since the $\mathbb{Z}^r$-filtrations on M and M^μ are the same, there exists a graded right $G^r(B)$-module homomorphism between $G^r(M^\mu)$ and $G^r(M)^{G^r(\mu)}$, which is just the identity when it is restricted to vector spaces. □

PROOF OF THEOREM 4.3. From Lemma 4.1, A has a finite projective resolution as an A^e-module and U is a homogeneous PBW deformation of A. That is, $G^r(U) \cong A$ for the $\mathbb{Z}^r$-filtration induced by the $\mathbb{Z}^r$-grading of A. By Lemma 4.8(b), $\underline{\mathrm{Ext}}^*_{A^e}(A, A^e)$ and $\underline{\mathrm{Ext}}^*_{U^e}(U, U^e)$ can be computed respectively by the complexes

$$0 \longleftarrow \underline{P}_d \xleftarrow{\underline{\partial}_d} \underline{P}_{d-1} \xleftarrow{\underline{\partial}_{d-1}} \cdots\cdots \xleftarrow{\underline{\partial}_0} \underline{P}_0 \longleftarrow 0,$$

where $\underline{P}_j = \bigoplus_{i=1}^{s_j} A^e(\alpha_i^j)$ with $\alpha_i^j \in \mathbb{Z}^r$ for $0 \le j \le d$, and

$$0 \longleftarrow P_d \xleftarrow{\partial_d} P_{d-1} \xleftarrow{\partial_{d-1}} \cdots\cdots \xleftarrow{\partial_0} P_0 \longleftarrow 0,$$

where $P_j = \bigoplus_{i=1}^{s_j} U^e(\alpha_i^j)$ for $0 \le j \le d$ such that $G^r(\partial_j) = \underline{\partial}_j$ for all $0 \le j \le d$.

Using Lemma 4.1 again, we find $\underline{\mathrm{Ext}}^i_{A^e}(A, A^e) = 0$ for $i < d$. This implies that ∂_d is strict by Lemma 4.6. Notice that $\underline{\mathrm{Ext}}^d_{U^e}(U, U^e)$ is a quotient of P_d and the $\mathbb{Z}^r$-filtration on it is induced from P_d. Thus, as A^e-modules,

$$\begin{aligned} G^r(\underline{\mathrm{Ext}}^d_{U^e}(U, U^e)) &= G^r(P_d/\operatorname{Im}\partial_d) \cong G^r(P_d)/G^r(\operatorname{Im}\partial_d) \cong \underline{P}_d/\operatorname{Im}\underline{\partial}_d \\ &= \underline{\mathrm{Ext}}^d_{A^e}(A, A^e) \cong {}^1A^\mu. \end{aligned}$$

On the other hand, U is an AS-regular algebra, so $\underline{\mathrm{Ext}}^d_{U^e}(U, U^e) \cong {}^1U^\sigma$. Now the result follows from Lemma 4.9. □

5. Artin–Schelter regular algebras

We expect to find new AS-regular algebras by means of those that are $\mathbb{Z}^r$-graded. In this section, we first give a sufficient condition of AS-regularity by using the χ-condition on $G^r(A)$ and then present some new AS-regular algebras by the method of homogeneous PBW deformations.

A recent study on algebras with obstructions consisting of Lyndon words shows that those algebras are closely related to AS-regular algebras. The invariants of the

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algebras are controlled strictly by their relations. We review the definition of Lyndon words briefly and refer the reader to [8, 22] for the details.

We fix an ordering on $X = \{x_1, x_2, \ldots, x_n\}$ such as $x_1 < x_2 < \cdots < x_n$. The lex order on X^* is given by: for $u, v \in X^*$, $u \prec_{lex} v$ if and only if:

- either there are factorisations $u = rx_i s$, $v = rx_j t$ with $x_i < x_j$;
- or v is a proper prefix of u.

Definition 5.1. A word $u \in X^*$ is called a *Lyndon word* if $u \neq 1$ and $wv \prec_{lex} u$ for every $u = vw$ with $v, w \neq 1$.

When X is endowed with $\mathbb{Z}^r$-grading $\deg^r$, we need accordingly a $\mathbb{Z}^r$-grading admissible ordering $\prec_{\mathbb{Z}^r-lex}$ on X^* in the sequel. We say $u \prec_{\mathbb{Z}^r-lex} v$ for $u, v \in X^*$ if and only if $\deg^r u <_{lex} \deg^r v$ or $\deg^r u = \deg^r v$ and $u \prec_{lex} v$.

It is well known that noetherian AS-regular algebras satisfy an important condition called χ (see [4, Theorem 8.1]). Let A be a noetherian connected $\mathbb{Z}^r$-graded algebra and M be a left (or right) $\mathbb{Z}^r$-graded module. In our case, $\chi(M)$ holds if and only if $\underline{\mathrm{Ext}}^i_A({}_A k, M)$ (or $\underline{\mathrm{Ext}}^i_A(k_A, M)$) is finite dimensional for any $i \geq 0$. It is also equivalent that the local cohomological modules of M are all right bounded. We say that A (respectively A^o) satisfies χ if $\chi(M)$ holds for any finitely generated left (respectively right) A-module M. This condition is strongly related to the Gorenstein condition.

Theorem 5.2. *Let $A = k\langle X\rangle/I$ be a connected graded algebra with finite global dimension and GK dimension. Suppose that $G^r(A)$ is noetherian and $\chi(G^r(A))$ holds as right and left $G^r(A)$-module for some appropriate $\mathbb{Z}^r$-grading on $k\langle X\rangle$. Let $\mathcal{G}$ be the Gröbner basis of I with respect to $\prec_{\mathbb{Z}^r-lex}$. Suppose that the leading words of $\mathcal{G}$ are Lyndon words. Then $G^r(A)$ is AS-regular.*

In this case, A is an AS-regular algebra.

Proof. Set $B = G^r(A)$. By [10, Theorem 2.8], $\mathrm{GKdim}\, B = \mathrm{GKdim}\, A$ is finite.

We know that $B \cong k\langle X\rangle/(\mathrm{LH}(\mathcal{G}))$ by Proposition 3.3 and that $\mathrm{LH}(\mathcal{G})$ is the Gröbner basis of $(\mathrm{LH}(\mathcal{G}))$. From this, the leading words of the Gröbner basis of the relations of B are Lyndon words. Since $\mathrm{GKdim}\, B$ is finite, $gl\dim B = gl\dim A$ is finite by [22, Theorem 2.11].

By the definition, both $\underline{\mathrm{Ext}}^i_B({}_B k, B)$ and $\underline{\mathrm{Ext}}^i_B(k_B, B)$ are finite dimensional for every i, since $\chi(B)$ holds for both sides. Then B satisfies the Gorenstein condition by [18, Theorem 0.3]. □

The homogeneous PBW deformation is an additional method for the classification of all AS-regular algebras whose obstructions consist of Lyndon words. The second and third authors proved that the obstructions of low-dimensional $\mathbb{Z}^2$-graded AS-regular algebras with two generators are all Lyndon words under $\prec_{\mathbb{Z}^2-lex}$ [21, Theorem 8.1]. In the sequel, we obtain new classes of AS-regular algebras which are homogeneous PBW deformations of such AS-regular algebras. As before, we use the admissible ordering $\prec_{\mathbb{Z}^r-lex}$.

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Example 5.3. We start from the down–up algebra, which was introduced in [5]. It is an associative algebra $A(\alpha,\beta,\gamma) = k\langle d,u\rangle/(S)$, where

$$S = \{d^2u - \alpha dud - \beta ud^2 - \gamma d, du^2 - \alpha udu - \beta u^2d - \gamma u\}.$$

It is known that $A(\alpha,\beta,0)$ is a noetherian AS-regular algebra of dimension three if $\beta \neq 0$ (see [11]). In particular, $A(2,-1,0)$ is the enveloping algebra of a Lie algebra of dimension three. Now we apply homogeneous ($\mathbb{Z}^2$-) PBW deformation to $A = A(\alpha,\beta,0)$ with $\beta \neq 0$.

Set the $\mathbb{Z}^2$-grading on A by $\deg d = (1,0)$ and $\deg u = (0,1)$ and fix $d > u > 1$. Then the Gröbner basis $\mathcal{G}(A)$ of (S) is just S.

Let $U(A) = k\langle d,u\rangle/(P)$, where

$$P = \left\{\begin{matrix} d^2u - \alpha dud - \beta ud^2 - \lambda_1 du^2 - \lambda_2 udu - \lambda_3 u^2d - \eta u^3, \\ du^2 - \alpha udu - \beta u^2d - \rho u^3 \end{matrix} \;\middle|\; \begin{matrix} \lambda_i,\eta,\rho \in k \\ i = 1,2,3 \end{matrix}\right\}.$$

Without loss of generality, we assume $\lambda_1 = 0$. By Theorem 3.4, we find the solutions for $\{\lambda_2,\lambda_3,\eta,\rho\}$ such that the leading homogeneous polynomials of the Gröbner basis of (P) equal $\mathcal{G}(A)$. In this case, it is equivalent to resolving the overlap in $\mathrm{LW}(P)$. After a few steps of computation, there are five solutions:

(i)	$\alpha = 2,\ \beta = -1$	$\lambda_2 = \lambda_2$	$\lambda_3 = -\lambda_2$	$\eta = \eta$	$\rho = 0$
(ii)	$\alpha = 2,\ \beta = -1$	$\lambda_2 = 3\rho$	$\lambda_3 = 0$	$\eta = \eta$	$\rho \neq 0$
(iii)	$\alpha = 0,\ \beta = 1$	$\lambda_2 = \lambda_2$	$\lambda_3 = -\lambda_2$	$\eta = \eta$	$\rho = 0$
(iv)	$\alpha + \beta \neq 1$	$\lambda_2 = \frac{(\alpha^2-\alpha+\beta-1)\rho}{\alpha+\beta-1}$	$\lambda_3 = \frac{\alpha(\beta+1)\rho}{\alpha+\beta-1}$	$\eta = \frac{\alpha\rho^2}{\alpha+\beta-1}$	$\rho = \rho$
(v)	$\alpha + \beta = 1, \beta^2 \neq 1$	$\lambda_2 = \lambda_2$	$\lambda_3 = -\lambda_2$	$\eta = 0$	$\rho = 0$

As a graded automorphism, the homogeneous PBW deformation is trivial in the case (v). Thus, we obtain four nontrivial classes of homogeneous PBW deformations of A up to graded isomorphism which are AS-regular algebras:

(i) $U_1(A) = k\langle d,u\rangle/(f_{11}, f_{12})$, where

$$f_{11} = d^2u - 2dud + ud^2 + pudu - pu^2d + qu^3,$$
$$f_{12} = du^2 - 2udu + u^2d.$$

(ii) $U_2(A) = k\langle d,u\rangle/(f_{21}, f_{22})$, where

$$f_{21} = d^2u - 2dud + ud^2 + 3udu,$$
$$f_{22} = du^2 - 2udu + u^2d + u^3.$$

(iii) $U_3(A) = k\langle d,u\rangle/(f_{31}, f_{32})$, where

$$f_{31} = d^2u - ud^2 + pudu - pu^2d + qu^3,$$
$$f_{32} = du^2 - u^2d.$$

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(iv) $U_4(A) = k\langle d, u\rangle/(f_{41}, f_{42})$, where $\alpha + \beta \neq 1$ and

$$f_{41} = d^2u - \alpha dud - \beta ud^2 + qu^3,$$
$$f_{42} = du^2 - \alpha udu - \beta u^2 d.$$

EXAMPLE 5.4. Secondly, we consider the homogeneous ($\mathbb{Z}^2$-) PBW deformation of $\mathfrak{C} = D(-2, -1)$ (see [13]). It is an enveloping algebra of a Lie algebra of global dimension four. Here $\mathfrak{C} = k\langle x, y\rangle/(g_1, g_2)$, where

$$g_1 = xy^2 - 2yxy + y^2x,$$
$$g_2 = x^3y - 3x^2yx + 3xyx^2 - yx^3.$$

Construct the positive $\mathbb{Z}^2$-graded algebra $\mathfrak{C}$ with $\deg x = (1, 0)$ and $\deg y = (0, 1)$ and fix $x \succ y \succ 1$. Then the Gröbner basis $\mathcal{G}(\mathfrak{C})$ of (g_1, g_2) is $\{g_1, g_2, g_3\}$, where

$$g_3 = x^2yxy - 3xyx^2y + 2xyxyx + 3yx^2yx - 5yxyx^2 + 2y^2x^3.$$

Since $\deg g_1 = (1, 2)$ and $\deg g_2 = (3, 1)$, we suppose $U(\mathfrak{C}) = k\langle x, y\rangle/(P)$, where P is the reduced form

$$\left\{\begin{matrix} xy^2 - 2yxy + y^2x + ay^3, \\ x^3y - 3x^2yx + 3xyx^2 - yx^3 + b_1xyxy + b_2yx^2y \\ \quad +b_3yxyx + b_4y^2x^2 + c_1y^2xy + c_2y^3x + dy^4 \end{matrix} \;\middle|\; \begin{matrix} a, b_i, c_j, d \in k \\ i = 1, 2, 3, 4 \\ j = 1, 2 \end{matrix}\right\}.$$

With the help of Maple, we found all the possible solutions as follows, such that $\mathrm{LH}(\mathcal{G}(U(\mathfrak{C})))$ is just $\mathcal{G}(\mathfrak{C})$, where $\mathcal{G}(U(\mathfrak{C}))$ is a Gröbner basis of (P).

Case 1. $a = 0, b_1 = -b_2 - b_3 - b_4$, where $b_2, b_3, b_4, c_1, c_2, d$ are free parameters.

Case 2. $a \neq 0, b_1 = 6a + b_3 + 2b_4,\ b_2 = -6a - 2b_3 - 3b_4,\ c_1 = 3ab_4 - c_2,$ where a, b_3, b_4, c_2, d are free parameters.

This yields two classes of AS-regular algebras with all the coefficients occurring below in k.

(i) $U_1(\mathfrak{C}) = k\langle x, y\rangle/(g'_{11}, g'_{12})$, where

$$g'_{11} = xy^2 - 2yxy + y^2x,$$
$$\begin{aligned} g'_{12} = x^3y - 3x^2yx + 3xyx^2 - yx^3 \\ - (b_1 + b_2 + b_3)xyxy + b_1yx^2y + b_2yxyx + b_3y^2x^2 \\ + c_1y^2xy + c_2y^3x + dy^4. \end{aligned}$$

(ii) $U_2(\mathfrak{C}) = k\langle x, y\rangle/(g'_{21}, g'_{22})$, where

$$g'_{21} = xy^2 - 2yxy + y^2x + y^3,$$
$$\begin{aligned} g'_{22} = x^3y - 3x^2yx + 3xyx^2 - yx^3 \\ + (p + q + 2m + 2n)xyxy - (p + 2q + 3m + 3n)yx^2y \\ + qyxyx + (m + n)y^2x^2 + 3my^2xy + 3ny^3x + ty^4. \end{aligned}$$

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REMARK 5.5. The paper [16] shows that there exists only one class of AS-regular algebras $\mathcal{J}$ whose Frobenius data (see [13, 16]) is of Jordan type. It is just the case in $U_1(\mathfrak{C})$ when taking $b_1 = u, b_2 = u - 3, b_3 = 2 - u, c_1 = v, c_2 = -v, d = w$. In other cases, the Frobenius data of $U_1(\mathfrak{C})$ falls within diagonal type.

EXAMPLE 5.6. There are two enveloping algebras of graded Lie algebras of dimension five with two generators. They are as follows.

(i) $\mathfrak{D} = k\langle x, y\rangle/(h_1, h_2, h_3)$, where

$$\begin{aligned} h_1 &= x^3y - 3x^2yx + 3xyx^2 - yx^3, \\ h_2 &= x^2y^2 - 2xyxy + 2yxyx - y^2x^2, \\ h_3 &= xy^3 - 3yxy^2 + 3y^2xy - y^3x. \end{aligned}$$

(ii) $\mathfrak{F} = k\langle x, y\rangle/(p_1, p_2, p_3)$, where

$$\begin{aligned} p_1 &= x^2y - 2xyx + yx^2, \\ p_2 &= xyxy^2 - 3xy^2xy + 2xy^3x + 2yxyxy - 3yxy^2x + y^2xyx, \\ p_3 &= xy^4 - 4yxy^3 + 6y^2xy^2 - 4y^3xy + y^4x. \end{aligned}$$

We still define the $\mathbb{Z}^2$-degree by $\deg x = (1, 0)$ and $\deg y = (0, 1)$ and $x \succ y \succ 1$. After a straightforward computation, we know that $\mathfrak{D}$ and $\mathfrak{F}$ have only the trivial homogeneous ($\mathbb{Z}^2$-) PBW deformation.

However, there exists a new class of AS-regular algebras coming from $\mathfrak{F}$ if we consider another $\mathbb{Z}^2$-grading on $\mathfrak{F}$. Let $\deg x = (0, 1)$ and $\deg y = (1, 0)$ and the ordering on $\{x, y\}^*$ is still $x \succ y \succ 1$. Under those conditions, we get the new class of AS-regular algebras $U(\mathfrak{F}) = k\langle x, y\rangle/(q_1, q_2, q_3)$, where

$$\begin{aligned} q_1 &= x^2y - 2xyx + yx^2, \\ q_2 &= xyxy^2 - 3xy^2xy + 2xy^3x + 2yxyxy - 3yxy^2x + y^2xyx, \\ q_3 &= xy^4 - 4yxy^3 + 6y^2xy^2 - 4y^3xy + y^4x - (a + 2b)y^3x^2 + (3a + 4b)y^2xyx \\ &\quad - (3a + b)yxy^2x - 2byxyxy + axy^3x + bxy^2xy + (c + 2d)y^2x^3 \\ &\quad - (2c + 3d)yxyx^2 + cxy^2x^2 + dxyxyx + txyx^3 - tyx^4. \end{aligned}$$

The last two examples mentioned above are both homogeneous PBW deformations of enveloping algebras of Lie algebras. It is well known that enveloping algebras of finite-dimensional positively graded Lie algebras are AS-regular. Such algebras are always endowed with a well-defined $\mathbb{Z}^r$-grading for some $r > 1$. Then one can find new classes of AS-regular algebras via homogeneous PBW deformation.

The polynomial algebra $\mathfrak{P} = k[x_1, x_2, \ldots, x_n]$, which is the enveloping algebra of an abelian Lie algebra, is the only commutative AS-regular algebra of dimension n. It is obviously $\mathbb{Z}^n$-graded. One could consider homogeneous ($\mathbb{Z}^n$-) PBW deformation U of $\mathfrak{P}$. Set $\deg x_i = (\delta_{ni}, \ldots, \delta_{1i})$ and $x_1 \prec x_2 \cdots \prec x_n$. Without loss of generality,

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the substitutive relations of U by the definition should be

$$P = \Big\{x_j x_i - x_i x_j + \sum_{1\le s<i} a_{jisj} x_s x_j + \sum_{1\le s\le t<j} a_{jist} x_s x_t \mid 1 \le i < j \le n\Big\}.$$

Since the Gröbner basis of relations for $\mathfrak{P}$ is $\{x_j x_i - x_i x_j \mid 1 \le i < j \le n\}$, the question turns on finding solutions of coefficients $\{a_{jist}\}$ such that all overlaps in P are resolvable. Furthermore, considering $\mathfrak{P}$ as a $\mathbb{Z}^r$-graded algebra, where $1 \le r < n$, then homogeneous ($\mathbb{Z}^r$-) PBW deformation of $\mathfrak{P}$ would give more new classes of AS-regular algebras.

Remark 5.7. There are other deformations of Lie algebras based on generalising the Lie bracket, such as Lie superalgebras and colour Lie superalgebras. When restricted to finite-dimensional positively graded algebras, the enveloping algebras are $\mathbb{Z}^r$-graded AS-regular algebras. Recently, in [22], the last two authors defined a generalised Lie bracket using Lyndon words to produce $\mathbb{Z}^r$-graded AS-regular algebras. We can also construct new examples of AS-regular algebras by applying homogeneous PBW deformation to those kinds of algebras.

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