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PRIMITIVE PERMUTATION GROUPS CONTAINING A CYCLE OF PRIME-POWER LENGTH

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An old problem in the theory of permutation groups is the classification of primitive groups which contain an element with a given cycle decomposition. The case that has received the most attention is of an element containing just one cycle of length greater than one. To be specific, let G be a primitive group, not the alternating or symmetric group, of degree $m + k$, containing an element x which is an m-cycle fixing k points. Jordan proved that G is $(k+1)$-transitive ([1]). Then Marggraff showed that $m \geq k$ ([3]). The only further progress on the problem in this generality is the result of Williamson ([8], Theorem 1) that $m \geq k!$. The present knowledge of 4-transitive groups makes it natural to conjecture that $k \leq 2$.

Now suppose that m is a prime power, say $m = p^n$. Then the theorems of Sylow and Witt ensure that x lies in tractable proper subgroups of G, and we expect to be able to prove more. Jordan showed that when $n = 1$ then $k \leq 2$ ([7], Theorem 13.9). In a recent series of papers ([4], [5], [6]) it is proved that for any n, $k \leq 2$. Here we consider what more can be said.

In Chapter 3, we consider the case $p = 2$, $k = 0$. As our hypothesis is not inductive to subgroups, we classify permutation groups of degree 2^n, which contain a 2^n-cycle x and have minimal degree greater than 2^{n-2}. (This includes the primitive groups.) The basis of the proof is the observation that, for such a group G with $n \geq 6$, every

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elementary abelian 2-subgroup of G normalized by x has order at most 8 and any subgroup D of exponent 4, with $\Phi(D) \leq Z(D)$, which is normalized by x is abelian. Hence we have control of the action of elements of odd order on $O_2(G)$. The proof proceeds by choosing a minimal counterexample G and considering the set M of maximal subgroups of G which contain x. Since, by induction, the structure of elements of M is known, we can show that G does not exist. Our conclusion is that if G is primitive, then $G = \mathrm{PGL}_2(q)$, where q is a Mersenne prime with $2^n = q - 1$.

In Chapter 4 we consider the case $k \geq 1$ for all primes. We begin by supposing that $k = 1$, and show that either G has cyclic Sylow p-subgroups, or $p = 2$ and the Sylow 2-subgroups of G are dihedral or semidihedral. It is then easy to show that, if $p = 2$ and $k = 1$, then G is soluble, while if $p = k = 2$, then either G is $\mathrm{PGL}_2(q)$, q a Fermat prime or G is a subgroup of $\mathrm{PGL}_2(9)$. For p odd, we show that G is 3-transitive if $k = 1$ and 5-transitive if $k = 2$. We also prove that $p \neq 3$ and that if $p = 5$, then $k = 1$ and n is even.

We note that some of the results of Chapter 4 have already appeared in [2].

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