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RODRIGUES FORMULA AND LINEAR INDEPENDENCE FOR VALUES OF HYPERGEOMETRIC FUNCTIONS WITH VARYING PARAMETERS

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Abstract

In this article, we prove a generalized Rodrigues formula for a wide class of holonomic Laurent series, which yields a new linear independence criterion concerning their values at algebraic points. This generalization yields a new construction of Padé approximations including those for Gauss hypergeometric functions. In particular, we obtain a linear independence criterion over a number field concerning values of Gauss hypergeometric functions, allowing *the parameters of Gauss hypergeometric functions to vary.*

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1. Introduction

We give here a linear independence criterion for values over number fields, by using the Padé approximation, for a certain class of holonomic Laurent series with algebraic coefficients.

As a consequence, over a number field we show a linear independence criterion of values of Gauss hypergeometric functions, where we let the parameters vary, which is the novel part.

The Padé approximation has appeared as one of the major methods in Diophantine problems since the works of Hermite and Padé [24, 25]. To solve a number theoretical program by the Padé approximation, we usually need to construct a system of Padé approximants in an explicit form. Padé approximants can be constructed by

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linear algebra with estimates using Siegel's lemma via Dirichlet's box principle. However, it is not always enough to establish arithmetic applications such as the linear independence criterion. Indeed, we are obliged to explicitly construct Padé approximants to provide sufficiently sharp estimates instead. In general, it is known that this step can be performed for specific functions only.

In this article, we succeed in proving a generalized Rodrigues formula, which gives an explicit construction of Padé approximations for a new and wide class of holonomic Laurent series. We introduce a linear map φ_f (see Equation (2-1)) with respect to a given holonomic Laurent series $f(z)$, which describes a necessary and sufficient condition to explicitly construct Padé approximants by studying $\ker \varphi_f$. We state necessary properties of $\ker \varphi_f$ by looking at related differential operators.

The construction of Padé approximants for Laurent series dates back to the classical works of Legendre and Rodrigues. In 1782, Legendre discovered a system of orthogonal polynomials the so-called Legendre polynomials. In 1816, Rodrigues established a simple expression for Legendre polynomials, called the Rodrigues formula by Hermite. See [5], where Askey described a short history of the Rodrigues formula. It is known that Legendre polynomials provide Padé approximants of the logarithmic function. After Legendre and Rodrigues, various kinds of Padé approximants of Laurent series have been developed by Rasala [26], Aptekarev *et al.* [4], Rivoal [28] and Sorokin [31–33]. We note that Alladi and Robinson [1], also Beukers [6–8], applied the Legendre polynomials to solve central irrationality questions, and many results are shown in the following papers by Rhin and Toffin [27], Hata [17–19] and Marcovecchio [21]. The author together with David and Hirata-Kohno [10–13] also proved the linear independence criterion concerning certain specific functions in a different setting.

By trying a new approach, distinct from those in [13], the author shows how to construct new generalized Padé approximants of Laurent series. This method allows us to provide a linear independence criterion for Gauss hypergeometric functions, letting the parameters vary. The case has not been previously been considered among the known results, although the Gauss hypergeometric function is a well-known classical function.

The approach relies on the linear map φ_f (see Equation (2-1)) to construct the Padé approximants in *an explicit but formal manner*. This idea has been partly used but in a different expression in [10–13], as well as in [20] by Poëls and the author.

The main point in this article is that we re-describe the Rodrigues formula itself from a formal point of view to find suitable differential operators that enable us to construct Padé approximants themselves, *instead of Padé-type approximants*. This part is done for the functions whose Padé approximants have never been explicitly given before.

Consequently, our corollary provides arithmetic applications, for example, the linear independence of the concerned values at different points for a wider class of functions, which was not achieved in [4].

In the first part of this article, we discuss an explicit construction of Padé approximants. Our final aim is to find a general method to explicitly obtain Padé approximants for given Laurent series. Here, we partly succeed in giving a solution to this fundamental question on the Rodrigues formula for specific Laurent series that can be transformed to polynomials by the differential operator of order 1. Precisely speaking, we indeed generalize the Rodrigues formula to a new class of holonomic series (see Theorem 4.2).

In the second part, we apply our explicit Padé approximants of holonomic Laurent series for the linear independence problems of their values. As a corollary, we show below a new linear independence criterion for values of the Gauss hypergeometric function, letting the parameters vary. We recall the Gauss hypergeometric function. For a rational number x and a nonnegative integer k, we denote the kth Pochhammer symbol: $(x)_0 = 1$, $(x)_k = x(x+1)\cdots(x+k-1)$. For $a, b, c \in \mathbb{Q}$ that are nonnegative integers, we define

$$ {}_2F_1(a, b, c\,|z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k. $$

We can now state the following theorem.

THEOREM 1.1. *Let u, α be integers with $u \geq 2$ and $|\alpha| \geq 2$. Assume*

$$ V(\alpha) := \log|\alpha| - \log 2 - \Big(2 - \frac{1}{u}\Big)\Big(\log u + \sum_{\substack{q:\text{prime} \\ q|u}} \frac{\log q}{q-1}\Big) - \frac{u-1}{\varphi(u)} > 0, $$

where φ is Euler's totient function. Then the real numbers:

$$ 1, {}_2F_1\Big(\frac{1+l}{u}, 1, \frac{u+l}{u} \,\Big|\, \frac{1}{\alpha^u}\Big) \quad (0 \leq l \leq u-2) $$

are linearly independent over $\mathbb{Q}$.

The following table gives suitable data for u and α so as to ensure $V(\alpha) > 0$:

u	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\alpha \geq$	$e^{3.78}$	$e^{4.44}$	$e^{5.84}$	$e^{5.32}$	$e^{8.76}$	$e^{5.91}$	$e^{7.65}$	$e^{7.22}$	$e^{9.40}$	$e^{6.73}$	$e^{10.59}$	$e^{7.04}$	$e^{9.92}$	$e^{9.52}$

The present article is organized as follows. In Section 2, we collect basic notions and recall the Padé-type approximants of Laurent series. To achieve an explicit construction of Padé approximants, which is of particular interest, we introduce a morphism φ_f associated with a Laurent series $f(z)$. To analyse the structure of $\ker \varphi_f$ is a crucial point for our program (see Proposition 2.3). Indeed, we provide a proper subspace, in some cases this is the whole space, of $\ker \varphi_f$ derived from the differential operator that annihilates f (see Corollary 2.6). This is the key ingredient required to generalize the Rodrigues formula.

In Section 3, we introduce the weighted Rodrigues operator, which is first defined in [4] as well as basic properties that are going to be used in the course of the proof.

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In Section 4, we give a generalization of the Rodrigues formula to Padé approximants of certain holonomic series by using the weighted Rodrigues operators (see Theorem 4.2). In Section 5, we introduce the determinants associated with the Padé approximants obtained in Theorem 4.2. To prove the nonvanishing of these determinants is one of the most crucial steps to obtain irrationality as well as linear independence results. We discuss some examples of Theorem 4.2 and Proposition 5.2 in Section 6. Example 6.1 is the particular example concerning Theorem 1.1. In Section 7, we state a more precise theorem than Theorem 1.1 (see Theorem 7.1). This section is devoted to the proof of Theorem 7.1. The Appendix is devoted to describing a result due to Fischler and Rivoal in [15]. They gave a condition on the differential operator of order 1 with polynomial coefficients so as to be a G-operator. Indeed, this result is crucial to apply Theorem 4.2 to G-functions. More precisely, whenever the operator is a G-operator, then the Laurent series considered in Theorem 7.1 turn out to be G-functions.

2. Padé-type approximants of Laurent series

Throughout this section, we fix a field K of characteristic 0. We denote the formal power series ring of variable $1/z$ with coefficients K by $K[[1/z]]$ and the field of fractions by $K((1/z))$. We say an element of $K((1/z))$ is a formal Laurent series. We define the order function at $z = \infty$ by

$$\mathrm{ord}_\infty : K((1/z)) \longrightarrow \mathbb{Z} \cup \{\infty\};\ \sum_k \frac{a_k}{z^k} \mapsto \min\{k \in \mathbb{Z} \cup \{\infty\} \mid a_k \neq 0\}.$$

Note that for $f \in K((1/z))$, $\mathrm{ord}_\infty f = \infty$ if and only if $f = 0$. We recall without proof the following elementary lemma.

LEMMA 2.1. *Let m be a nonnegative integer, $f_1(z), \dots, f_m(z) \in (1/z) \cdot K[[1/z]]$ and $\boldsymbol{n} = (n_1, \dots, n_m) \in \mathbb{N}^m$. Put $N = \sum_{j=1}^m n_j$. For a nonnegative integer M with $M \geq N$, there exist polynomials $(P, Q_1, \dots, Q_m) \in K[z]^{m+1} \setminus \{\mathbf{0}\}$ satisfying the following conditions:*

$$(i)\ \deg P \leq M;$$
$$(ii)\ \mathrm{ord}_\infty(P(z)f_j(z) - Q_j(z)) \geq n_j + 1 \quad \textit{for } 1 \leq j \leq m.$$

DEFINITION 2.2. We say that a vector of polynomials $(P, Q_1, \dots, Q_m) \in K[z]^{m+1}$ satisfying properties (i) and (ii) is a weight $\boldsymbol{n}$ and degree M Padé-type approximant of $(f_1, \dots, f_m)$. For such approximants $(P, Q_1, \dots, Q_m)$ of $(f_1, \dots, f_m)$, we call the formal Laurent series $(P(z)f_j(z) - Q_j(z))_{1 \leq j \leq m}$, that is to say remainders, as weight $\boldsymbol{n}$ degree M Padé-type approximations of $(f_1, \dots, f_m)$.

Let $f(z) = \sum_{k=0}^\infty f_k/z^{k+1} \in (1/z) \cdot K[[1/z]]$. We define a K-linear map $\varphi_f \in \mathrm{Hom}_K(K[t], K)$ by

$$\varphi_f : K[t] \longrightarrow K; \quad t^k \mapsto f_k \quad (k \geq 0). \tag{2-1}$$

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The above linear map extends naturally to a $K[z]$-linear map $\varphi_f : K[z,t] \to K[z]$, and then to a $K[z][[1/z]]$-linear map $\varphi_f : K[z,t][[1/z]] \to K[z][[1/z]]$. With this notation, the formal Laurent series $f(z)$ satisfies the following crucial identities (see [23, Equation (6.2) page 60 and Equation (5.7) page 52]):

$$f(z) = \varphi_f\Big(\frac{1}{z-t}\Big), \quad P(z)f(z) - \varphi_f\Big(\frac{P(z)-P(t)}{z-t}\Big) \in (1/z)\cdot K[[1/z]] \quad \text{for any } P(z) \in K[z].$$

LEMMA 2.3. *Let m be a nonnegative integer, $f_1(z),\dots,f_m(z) \in (1/z)\cdot K[[1/z]]$ and $\boldsymbol{n} = (n_1,\dots,n_m) \in \mathbb{N}^m$. Let M be a positive integer and $P(z) \in K[z]$ a nonzero polynomial with $M \geq \sum_{j=1}^m n_j$ and $\deg P \leq M$. Put $Q_j(z) = \varphi_{f_j}((P(z)-P(t))/(z-t)) \in K[z]$ for $1 \leq j \leq m$.*

Then the following are equivalent.

(i) *The vector of polynomials $(P, Q_1,\dots,Q_m)$ is a weight $\boldsymbol{n}$ Padé-type approximant of $(f_1,\dots,f_m)$.*
(ii) *We have $t^kP(t) \in \ker\varphi_{f_j}$ for $1 \leq j \leq m$, $0 \leq k \leq n_j - 1$.*

PROOF. By the definition of $Q_j(z)$,

$$P(z)f_j(z) - Q_j(z) = \varphi_{f_j}\Big(\frac{P(t)}{z-t}\Big) \in (1/z)\cdot K[[1/z]].$$

The above equality yields that the vector of polynomials $(P, Q_1,\dots,Q_m)$ being a weight $\boldsymbol{n}$ Padé-type approximant of $(f_1,\dots,f_m)$ is equivalent to the order of the Laurent series

$$\varphi_{f_j}\Big(\frac{P(t)}{z-t}\Big) = \sum_{k=0}^{\infty} \frac{\varphi_{f_j}(t^kP(t))}{z^{k+1}}$$

being greater than or equal to $n_j + 1$ for $1 \leq j \leq m$. This shows the equivalence of items (i) and (ii). □

Lemma 2.3 indicates that it is useful to study $\ker\varphi_f$ for the explicit construction of Padé-type approximants of Laurent series. We are now going to investigate $\ker\varphi_f$ for a holonomic Laurent series $f \in (1/z)\cdot K[[1/z]]$. We denote the differential operator d/dz (respectively d/dt) by ∂_z (respectively ∂_t). We describe the action of a differential operator D on a function f by $D \cdot f$ and denote $\partial_z \cdot f$ by f'.

To begin with, let us introduce a map

$$\iota : K(z)[\partial_z] \longrightarrow K(t)[\partial_t]; \quad \sum_j P_j(z)\partial_z^j \mapsto \sum_j (-1)^j \partial_t^j P_j(t).$$

Note, for $D \in K(z)[\partial_z]$, $\iota(D)$ is called the adjoint of D and relates to the dual of differential module $K(z)[\partial_z]/K(z)[\partial_z]D$ (see [2, Exercise III(3)]). For $D \in K(z)[\partial_z]$, we denote $\iota(D)$ by D^*. Notice that we have $(DE)^* = E^*D^*$ for any $D, E \in K(z)[\partial_z]$.

LEMMA 2.4. *For $D \in K[z,\partial_z]$, there exists a polynomial $P(t,z) \in K[t,z]$ satisfying*

$$D \cdot \frac{1}{z-t} = P(t,z) + D^* \cdot \frac{1}{z-t}.$$

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PROOF. Let m, n be nonnegative integers. It suffices to prove the case $D = z^m \partial_z^n$. Then,

$$D \cdot \frac{1}{z-t} = \frac{(-1)^n n!\, z^m}{(z-t)^{n+1}} = (-1)^n \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} \frac{t^k}{z^{k+1+n-m}}. \tag{2-2}$$

We define a polynomial $P(t, z)$ by 0 if $m \leq n$ and

$$P(t, z) = (-1)^n \sum_{k=0}^{m-n-1} \frac{(n+k)!}{k!} t^k z^{m-n-k-1}$$

for $m > n$. Equation (2-2) implies

$$\begin{aligned} D \cdot \frac{1}{z-t} - P(t, z) &= (-1)^n \sum_{k=\max\{m-n,0\}}^{\infty} \frac{(n+k)!}{k!} \frac{t^k}{z^{k+1+n-m}} \\ &= (-1)^n \sum_{k=0}^{\infty} (k+1+m-n) \cdots (m+k) \frac{t^{k+m-n}}{z^{k+1}}. \end{aligned}$$

However,

$$\begin{aligned} D^* \cdot \frac{1}{z-t} = (-1)^n \partial_t^n \cdot \frac{t^m}{z-t} &= (-1)^n \sum_{k=0}^{\infty} \partial_t^n \cdot \frac{t^{m+k}}{z^{k+1}} \\ &= (-1)^n \sum_{k=0}^{\infty} (k+1+m-n) \cdots (m+k) \frac{t^{k+m-n}}{z^{k+1}}. \end{aligned}$$

The above equalities yield

$$D \cdot \frac{1}{z-t} - P(t, z) = D^* \cdot \frac{1}{z-t}.$$

This completes the proof of Lemma 2.4. □

We introduce the projection morphism π by

$$\pi : K[z][[1/z]] \longrightarrow K[z][[1/z]]/K[z] \cong (1/z) \cdot K[[1/z]]; \quad f(z) = P(z) + \tilde{f}(z) \mapsto \tilde{f}(z),$$

where $P(z) \in K[z]$ and $\tilde{f}(z) \in (1/z) \cdot K[[1/z]]$. Lemma 2.4 allows us to show the following key proposition.

PROPOSITION 2.5. *Let $D \in K[z, \partial_z]$ and $f(z) \in (1/z) \cdot K[[1/z]]$. We have $\varphi_{\pi(D \cdot f)} = \varphi_f \circ D^*$.*

PROOF. First, since φ_f acts only on the parameter t,

$$D \cdot f = D \circ \varphi_f\Big(\frac{1}{z-t}\Big) = \varphi_f\Big(D \cdot \frac{1}{z-t}\Big).$$

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Lemma 2.4 implies that there exists a polynomial $P(z)$ with

$$D\cdot f = P(z) + \varphi_f\Big(D^*\cdot \frac{1}{z-t}\Big) = P(z) + \sum_{k=0}^{\infty}\frac{\varphi_f(D^*\cdot t^k)}{z^{k+1}}.$$

Note that $P(z) = \varphi_f(P(t,z))$, where $P(t,z)\in K[t,z]$ is defined in Lemma 2.4. This shows that $\pi(D\cdot f) = \sum_{k=0}^{\infty}\varphi_f(D^*\cdot t^k)/z^{k+1}$ and therefore

$$\varphi_{\pi(D\cdot f)}(t^k) = \varphi_f\circ D^*(t^k) \quad \text{for all } k\geq 0.$$

This concludes the proof of Proposition 2.5. □

As a corollary of Proposition 2.5, the following crucial equivalence relations hold.

COROLLARY 2.6. *Let $f(z)\in(1/z)\cdot K[[1/z]]$ and $D\in K[z,\partial_z]$.*
The following are equivalent:

(i) $D\cdot f\in K[z]$;
(ii) $D^*(K[t])\subseteq \ker\varphi_f$.

PROOF. Conditions (i) and (ii) are equivalent to $\pi(D\cdot f)=0$ and $\varphi_f\circ D^*=0$, respectively. Therefore, by Proposition 2.5, we obtain the assertion. □

3. Weighted Rodrigues operators

Let K be a field of characteristic 0. Let us introduce the weighted Rodrigues operator, which is first defined by A. I. Aptekarev, A. Branquinho and W. Van Assche in [4].

DEFINITION 3.1 See [4, Equation (2.5)]. Let $l\in\mathbb{N}$, $a_1(z),\ldots,a_l(z)\in K[z]\setminus\{0\}$, $b(z)\in K[z]$. Put $a(z)=a_1(z)\cdots a_l(z)$, $D=-a(z)\partial_z+b(z)$. For $n\in\mathbb{N}$ and a weight $\vec{r}=(r_1,\ldots,r_l)\in\mathbb{Z}^l$ with $r_i\geq 0$, we define the weighted Rodrigues operator associated with D by

$$R_{D,n,\vec{r}} = \frac{1}{n!}\Big(\partial_z+\frac{b(z)}{a(z)}\Big)^n a(z)^n\prod_{v=1}^{l}a_v(z)^{-r_v}\in K(z)[\partial_z].$$

In the case of $\vec{r}=(0,\ldots,0)$, we denote $R_{D,n,\vec{r}}=R_{D,n}$ and call this operator the nth Rodrigues operator associated with D.

We denote the generalized Rodrigues operator associated with D with respect to the parameter t by

$$\mathcal{R}_{D,n,\vec{r}} = \frac{1}{n!}\Big(\partial_t+\frac{b(t)}{a(t)}\Big)^n a(t)^n\prod_{v=1}^{l}a_v(t)^{-r_v}\in K(t)[\partial_t],$$

and $\mathcal{R}_{D,n,\vec{r}}=\mathcal{R}_{D,n}$ in the case of $\vec{r}=(0,\ldots,0)$.

Let us show some basic properties of the weighted Rodrigues operator in order to obtain a generalization of Rodrigues formula of Padé approximants of holonomic

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Laurent series. In the following, for $a(z) \in K[z]$ (respectively $a(t) \in K[t]$), we denote the ideal of $K[z]$ (respectively $K[t]$), generated by $a(z)$ (respectively $a(t)$) by $(a(z))$ (respectively $(a(t))$).

PROPOSITION 3.2. *Let $a(t), b(t) \in K[t]$ with $a(t) \neq 0$. Put $\mathcal{E}_{a,b} = \partial_t + b(t)/a(t) \in K(t)[\partial_t]$.*

(i) Let n, k be nonnegative integers. Then there exist integers $(c_{n,k,l})_{0 \le l \le \min\{n,k\}}$ with

$$c_{n,k,\min\{n,k\}} = (-1)^n k(k-1)\cdots(k-n+1),$$

$$t^k \mathcal{E}_{a,b}^n = \sum_{l=0}^{\min\{n,k\}} c_{n,k,l} \mathcal{E}_{a,b}^{n-l} t^{k-l} \in K(t)[\partial_t].$$

(ii) Assume there exist polynomials $a_1(t), \ldots, a_l(t) \in K[t]$ with $a(t) = a_1(t)\cdots a_l(t)$. For an l-tuple of nonnegative integers $\boldsymbol{s} := (s_1, \ldots, s_l)$, we denote by $\mathrm{I}(\boldsymbol{s})$ the ideal of $K[t]$ generated by $\prod_{v=1}^{l} a_v(t)^{s_v}$. Then for $n \ge 1$ and $F(t) \in \mathrm{I}(\boldsymbol{s})$,

$$\mathcal{E}_{a,b}^n a(t)^n \cdot F(t) \in \mathrm{I}(\boldsymbol{s}). \tag{3-1}$$

PROOF. (i) We prove the assertion by induction on $(n,k) \in \mathbb{Z}^2$ with $n, k \ge 0$. In the case of $n = 0$ and any $k \ge 0$, the statement is trivial. Let n, k be nonnegative integers with $n \ge 1$ or $k \ge 1$. We assume that the assertion holds for any elements of the set $\{(\tilde{n}, \tilde{k}) \in \mathbb{Z}^2 \mid 0 \le \tilde{n}, \tilde{k} \text{ and } \tilde{n} < n \text{ and } \tilde{k} \le k\}$. The equality $t^k \mathcal{E}_{a,b} = \mathcal{E}_{a,b} t^k - k t^{k-1}$ in $K[t, \partial_t]$ implies that we have

$$\begin{aligned} t^k \mathcal{E}_{a,b}^n &= (\mathcal{E}_{a,b} t^k - k t^{k-1}) \mathcal{E}_{a,b}^{n-1} \\ &= \mathcal{E}_{a,b} \sum_{l=0}^{\min\{n-1,k\}} c_{n-1,k,l} \mathcal{E}_{a,b}^{n-1-l} t^{k-l} - k \sum_{l=0}^{\min\{n-1,k-1\}} c_{n,k-1,l} \mathcal{E}_{a,b}^{n-1-l} t^{k-1-l} \qquad (3\text{-}2) \\ &= \sum_{l=0}^{\min\{n-1,k\}} c_{n-1,k,l} \mathcal{E}_{a,b}^{n-l} t^{k-l} - \sum_{l=0}^{\min\{n-1,k-1\}} k c_{n-1,k-1,l} \mathcal{E}_{a,b}^{n-1-l} t^{k-1-l}. \end{aligned}$$

Note that we use the induction hypothesis in line (3-2). This concludes the assertion for (n,k).

(ii) Let us prove the statement by induction on n. In the case of $n = 1$, since

$$\mathcal{E}_{a,b} a(t) \cdot F(t) = (\partial_t a(t) + b(t)) \cdot F(t) = a'(t)F(t) + a(t)F'(t) + b(t)F(t),$$

using the Leibniz formula, we obtain Equation (3-1). We assume Equation (3-1) holds for $n \ge 1$. In the case of $n + 1$,

$$\mathcal{E}_{a,b}^{n+1} a(t)^{n+1} \cdot F(t) = \mathcal{E}_{a,b} \mathcal{E}_{a,b}^n a(t)^n \cdot a(t)F(t). \tag{3-3}$$

Note that we have $a(t)F(t) \in \mathrm{I}(\boldsymbol{s} + \mathbf{1})$, where $\boldsymbol{s} + \mathbf{1} := (s_1 + 1, \ldots, s_d + 1) \in \mathbb{N}^d$. Relying on the induction hypothesis, we deduce $\mathcal{E}_{a,b}^n a(t)^n \cdot a(t)F(t) \in \mathrm{I}(\boldsymbol{s} + \mathbf{1})$. Thus, there exists

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a polynomial $\tilde{F}(t) \in \mathrm{I}(\boldsymbol{s})$ with $\mathcal{E}_{a,b}^{n} a(t)^n \cdot a(t)F(t) = a(t)\tilde{F}(t)$. Substituting this equality into Equation (3-3), by using a similar argument to the case of $n = 1$, we conclude $\mathcal{E}_{a,b}^{n+1} a(t)^{n+1} \cdot F(t) \in \mathrm{I}(\boldsymbol{s})$. □

COROLLARY 3.3. *(i) Let* $a(z) \in K[z] \setminus \{0\}$ *and* $b(z) \in K[z]$. *We put* $D = -a(z)\partial_z + b(z)$. *Let* $f(z) \in (1/z) \cdot K[[1/z]] \setminus \{0\}$ *with* $D \cdot f(z) \in K[z]$. *Put* $\mathcal{E}_{a,b} = \partial_t + b(t)/a(t) \in K(t)[\partial_t]$. *Then, for* $n, k \in \mathbb{Z}$ *with* $0 \leq k < n$,

$$t^k \mathcal{E}_{a,b}^{n} \cdot (a(t)^n) \subseteq \ker \varphi_f.$$

(ii) Let $d, l \in \mathbb{N}$, $(n_1, \dots, n_d) \in \mathbb{N}^d$ *and* $a_1(t), \dots, a_l(t) \in K[t] \setminus \{0\}$. *Put* $a(t) = a_1(t) \cdots a_l(t)$. *For* $b_1(t), \dots, b_d(t) \in K[t]$ *and* l*-tuple of nonnegative integers* $\vec{r}_j = (r_{j,1}, \dots, r_{j,l})(1 \leq j \leq d)$, *we put* $D_j = -a(z)\partial_z + b_j(z)$ *and*

$$\mathcal{R}_{j,n_j} = \mathcal{R}_{D_j,n_j,\vec{r}_j} = \frac{1}{n_j!} \mathcal{E}_{a,b_j}^{n_j} a(t)^{n_j} \prod_{v=1}^{l} a_v(t)^{-r_{j,v}} \in K(t)[\partial_t].$$

Let $s_1, \dots, s_d$ *be nonnegative integers and* $F(t) \in (\prod_{v=1}^{l} a_v(t)^{s_v + \sum_{j=1}^{d} r_{j,v}})$. *Then,*

$$\prod_{j=1}^{d} \mathcal{R}_{j,n_j} \cdot F(t) \in \left(\prod_{v=1}^{l} a_v(t)^{s_v}\right)$$

(The statement in the first term holds for any order of product of operators $(\mathcal{R}_{j,n_j})_j$.)

PROOF. (i) By the definition of D, we have $D^* = \mathcal{E}_{a,b} a(t)$. Since we have $\mathcal{E}_{a,b} \cdot (a(t)) \subseteq \ker \varphi_f$, by Corollary 2.6, it suffices to show $t^k \mathcal{E}_{a,b}^{n} \cdot (a(t)^n) \subset \mathcal{E}_{a,b} \cdot (a(t))$. Relying on Proposition 3.2 (i), there are $\{c_{n,k,l}\}_{0 \leq l \leq k} \subset \mathbb{Z}$ with

$$t^k \mathcal{E}_{a,b}^{n} = \sum_{l=0}^{k} c_{n,k,l} \mathcal{E}_{a,b}^{n-l} t^{k-l}. \tag{3-4}$$

For an integer l with $0 \leq l \leq k$,

$$\mathcal{E}_{a,b}^{n-l} t^{k-l} \cdot (a(t)^n) \subset \mathcal{E}_{a,b} \mathcal{E}_{a,b}^{n-l-1} \cdot (a(t)^n).$$

The Leibniz formula allows us to get $\mathcal{E}_{a,b}^{n-l-1} \cdot (a(t)^n) \subset (a(t))$. Combining Equation (3-4) and the above relation gives

$$t^k \mathcal{E}_{a,b}^{n} \cdot (a(t)^n) \subset \mathcal{E}_{a,b} \cdot (a(t)).$$

This completes the proof of item (*i*).

(ii) It suffices to prove the assertion in the case of $d = 1$. By the definition of $\mathcal{R}_{1,n_1}$,

$$\mathcal{R}_{1,n_1} \cdot F(t) = \frac{1}{n_1!} \mathcal{E}_{a,b_1}^{n_1} a(t)^{n_1} \prod_{v=1}^{l} a_v(t)^{-r_{1,v}} \cdot F(t) \in \mathcal{E}_{a,b_1}^{n_1} a(t)^{n_1} \cdot \left(\prod_{v=1}^{l} a_v(t)^{s_v}\right).$$

Using Proposition 3.2(ii), we conclude that $\mathcal{E}_{a,b_1}^{n_1} a(t)^{n_1} \cdot (\prod_{v=1}^{l} a_v(t)^{s_v}) \subset (\prod_{v=1}^{l} a_v(t)^{s_v})$. This completes the proof of item (ii). □

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4. Rodrigues formula of Padé approximants

LEMMA 4.1. *Let* $a(z), b(z) \in K[z]$ *with* $a(z) \neq 0$, $\deg a = u$ *and* $\deg b = v$. *Put*

$$D = -a(z)\partial_z + b(z) \in K[z, \partial_z], \quad a(z) = \sum_{i=0}^{u} a_i z^i, \quad b(z) = \sum_{j=0}^{v} b_j z^j,$$

and $w = \max\{u-2, v-1\}$. *Assume* $w \geq 0$ *and*

$$a_u(k+u) + b_v \neq 0 \quad \text{for all } k \geq 0 \quad \text{if } u-2 = v-1. \tag{4-1}$$

Then there exist $f_0(z), \dots, f_w(z) \in (1/z) \cdot K[[1/z]]$ *that are linearly independent over* K *and satisfy* $D \cdot f_l(z) \in K[z]$ *for* $0 \leq l \leq w$.

PROOF. Let $f(z) = \sum_{k=0}^{\infty} f_k/z^{k+1} \in (1/z) \cdot K[[1/z]]$ be a Laurent series. There exists a polynomial $A(z) \in K[z]$ that depends on the operator D and f with $\deg A \leq w$, satisfying

$$D \cdot f(z) = A(z) + \sum_{k=0}^{\infty} \frac{\sum_{i=0}^{u} a_i(k+i) f_{k+i-1} + \sum_{j=0}^{v} b_j f_{k+j}}{z^{k+1}}.$$

Put

$$\sum_{i=0}^{u} a_i(k+i) f_{k+i-1} + \sum_{j=0}^{v} b_j f_{k+j} = c_{k,0} f_{k-1} + \cdots + c_{k,w} f_{k+w} + c_{k,w+1} f_{k+w+1} \quad \text{for } k \geq 0,$$

with $c_{0,0} = 0$. We remark that $c_{k,l}$ depends only on $a(z), b(z)$. Notice that $c_{k,w+1}$ is $a_u(k+u)$ if $u-2 > v-1$, b_v if $u-2 < v-1$ and $a_u(k+u) + b_v$ if $u-2 = v-1$. Then by Equation (4-1), we have $\min\{k \geq 0 \mid c_{k',w+1} \neq 0 \text{ for all } k' \geq k\} = 0$ and thus the K-linear map:

$$K^{w+1} \longrightarrow \{f \in (1/z) \cdot K[[1/z]] \mid D \cdot f \in K[z]\}; \quad (f_0, \dots, f_w) \mapsto \sum_{k=0}^{\infty} \frac{f_k}{z^{k+1}},$$

where, for $k \geq w+1$, f_k is determined inductively by

$$\sum_{i=0}^{u} a_i(k+i) f_{l,k+i-1} + \sum_{j=0}^{v} b_j f_{l,k+j} = 0 \quad \text{for } k \geq 0$$

is an isomorphism. This completes the proof of Lemma 4.1. □

Let us state a generalization of the Rodrigues formula for Legendre polynomials to Padé approximants of certain holonomic Laurent series, which gives a generalization of [4, Theorem 1]. In the following theorem, we construct Padé approximants of the family of Laurent series considered in Lemma 4.1.

THEOREM 4.2. *Let* $l, d \in \mathbb{N}$, $(a_1(z), \dots, a_l(z)) \in (K[z] \setminus \{0\})^l$ *and* $(b_1(z), \dots, b_d(z)) \in K[z]^d$. *Put* $a(z) = a_1(z) \cdots a_l(z)$. *Put* $D_j = -a(z)\partial_z + b_j(z) \in K[z, \partial_z]$ *and* $w_j = \max\{\deg a - 2, \deg b_j - 1\}$. *Assume* $w_j \geq 0$ *for* $1 \leq j \leq d$ *and Equation (4-1) for* D_j.

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Let $f_{j,0}(z), \dots, f_{j,w_j}(z) \in (1/z)\cdot K[[1/z]]$ be formal Laurent series that are linearly independent over K satisfying

$$D_j \cdot f_{j,u_j}(z) \in K[z] \quad \text{for } 0 \le u_j \le w_j.$$

(The existence of such series is ensured by Lemma 4.1.) Let $(n_1, \dots, n_d) \in \mathbb{N}^d$. For an l-tuple of nonnegative integers $\vec{r}_j = (r_{j,1}, \dots, r_{j,l})$ $(1 \le j \le d)$, we denote by R_{j,n_j} the weighted Rodrigues operator $R_{D_j,n_j,\vec{r}_j}$ associated with D_j. Assume

$$R_{j_1,n_{j_1}} R_{j_2,n_{j_2}} = R_{j_2,n_{j_2}} R_{j_1,n_{j_1}} \quad \text{for } 1 \le j_1, j_2 \le d.$$

Take a nonzero polynomial $F(z)$ that is contained in the ideal $(\prod_{v=1}^{l} a_v(z)^{\sum_{j=1}^{d} r_{j,v}})$ and put

$$P(z) = \prod_{j=1}^{d} R_{j,n_j} \cdot F(z),$$

$$Q_{j,u_j}(z) = \varphi_{f_{j,u_j}}\Big(\frac{P(z)-P(t)}{z-t}\Big) \quad \text{for } 1 \le j \le d,\ 0 \le u_j \le w_j.$$

Assume $P(z) \neq 0$. (We need to assume $P(z) \neq 0$. For example, in the case of $d = 1$, $D = -\partial_z z^2 = -z^2\partial - 2z$ and $n = 1$, we have $P(z) = (\partial_z - 2/z)z^2 \cdot 1 = 0$.) Then the vector of polynomials $(P(z), Q_{j,u_j}(z))_{\substack{1\le j\le d\\ 0\le u_j\le w_j}}$ is a weight $(\boldsymbol{n}_1, \dots, \boldsymbol{n}_m) \in \mathbb{N}^{\sum_{j=1}^{d}(w_j+1)}$ Padé-type approximants of $(f_{j,u_j}(z))_{\substack{1\le j\le d\\ 0\le u_j\le w_j}}$, where $\boldsymbol{n}_j = (n_j, \dots, n_j) \in \mathbb{N}^{w_j+1}$ for $1 \le j \le d$.

PROOF. By Lemma 2.3, it suffices to prove that any triple (j, u_j, k) with $1 \le j \le d$, $0 \le u_j \le w_j$, $0 \le k \le n_j - 1$ satisfies $t^k P(t) \in \ker \varphi_{f_{j,u_j}}$. Put $\mathcal{R}_{j,n_j} = \mathcal{R}_{D_j,n_j,\vec{r}_j}$. Then we have $P(t) = \prod_{j=1}^{d} \mathcal{R}_{j,n_j} \cdot F(t)$ and thus

$$t^k P(t) = t^k \mathcal{R}_{j,n_j} \prod_{j' \neq j} \mathcal{R}_{j',n_{j'}} \cdot F(t). \tag{4-2}$$

Since $F(t) \in (\prod_{v=1}^{l} a_v(z)^{r_{j,v} + \sum_{j'\neq j}^{d} r_{j',v}})$, using Corollary 3.3(ii),

$$\prod_{j' \neq j} \mathcal{R}_{j',n_{j'}} \cdot F(t) \in \Big(\prod_{v=1}^{l} a_v(t)^{r_{j,v}}\Big).$$

Combining Equation (4-2) and the above relation yields

$$t^k P(t) \in t^k \mathcal{R}_{j,n_j} \cdot \Big(\prod_{v=1}^{l} a_v(t)^{r_{j,v}}\Big) \subseteq t^k \mathcal{E}_{a,b_j}^{n_j} \cdot (a(t)^{n_j}) \subseteq \ker \varphi_{f_{j,u_j}}.$$

Note that the last inclusion is obtained from Corollary 3.3(i) for $D_j \cdot f_{j,u_j}(z) \in K[z]$. □

4.1. Commutativity of differential operators. In this subsection, we give a sufficient condition under which weighted Rodrigues operators commute. We denote $\partial_z \cdot c(z)$ by $c'(z)$ for any rational function $c(z) \in K(z)$.

LEMMA 4.3. *Let $a(z), b(z) \in K[z]$ and $c(z) \in K(z)$ with $a(z)c(z) \neq 0$. Let $w(z)$ be a nonzero solution of $-a(z)\partial_z + b(z)$ in some differential extension $\mathcal{K}$ of $K(z)$ and n a nonnegative integer. Put*

$$R_n = \frac{1}{n!}\Big(\partial_z + \frac{b(z)}{a(z)}\Big)^n c(z)^n \in K(z)[\partial_z].$$

Then, in the ring $\mathcal{K}[\partial_z]$, we have the following equality:

$$R_n = \frac{1}{n!}w(z)^{-1}\partial_z^n w(z)c(z)^n = \frac{1}{n!}R_1(R_1 + c'(z))\cdots(R_1 + (n-1)c'(z)).$$

PROOF. The first equality is readily obtained using the identity

$$\partial_z w(z) = w(z)\Big(\partial_z + \frac{b(z)}{a(z)}\Big).$$

The second equality is proved using the identity

$$\begin{aligned}\Big(\partial_z + \frac{b(z)}{a(z)}\Big)c(z)^n &= \Big[c(z)^{n-1}\Big(\partial_z + \frac{b(z)}{a(z)}\Big) + (n-1)c'(z)c(z)^{n-2}\Big]c(z)\\ &= c(z)^{n-1}(R_1 + (n-1)c'(z)).\end{aligned}$$

This completes the proof of Lemma 4.3. □

LEMMA 4.4. *Let $a(z), b_1(z), b_2(z), c(z) \in K[z]$ with $a(z)c(z) \neq 0$. For a nonnegative integer n and $j = 1, 2$,*

$$R_{j,n} = \frac{1}{n!}\Big(\partial_z + \frac{b_j(z)}{a(z)}\Big)^n c(z)^n.$$

Assume $\deg c \leq 1$. *Then the following are equivalent.*

(i) *For any $n_1, n_2 \in \mathbb{N}$, we have $R_{1,n_1}R_{2,n_2} = R_{2,n_2}R_{1,n_1}$.*
(ii) *We have $(b_2(z) - b_1(z))/a(z)c(z) \in K$.*

PROOF. Since $\deg c \leq 1$ and therefore $c'(z) \in K$, using Lemma 4.3, we see that item (i) is equivalent to $R_{1,1}R_{2,1} = R_{2,1}R_{1,1}$. Let us show that the commutativity of $R_{j,1} (j = 1, 2)$ is equivalent to item (ii). According to the identity,

$$R_{1,1}R_{2,1} = R_{2,1}R_{1,1} + (R_{2,1} - R_{1,1})c'(z) + \Big(\frac{b_2(z) - b_1(z)}{a(z)}\Big)' c(z)^2,$$

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the identity $R_{1,1}R_{2,1} = R_{2,1}R_{1,1}$ is equivalent to

$$\begin{aligned}(R_{2,1} - R_{1,1})c'(z) &+ \Big(\frac{b_2(z) - b_1(z)}{a(z)}\Big)' c(z)^2 \\ &= \Big(\frac{b_2(z) - b_1(z)}{a(z)} c'(z) + \Big(\frac{b_2(z) - b_1(z)}{a(z)}\Big)' c(z)\Big) c(z) \\ &= \Big(\frac{b_2(z) - b_1(z)}{a(z)} c(z)\Big)' c(z) = 0,\end{aligned}$$

which means item (ii) holds. This completes the proof of Lemma 4.4. □

5. Determinants associated with Padé approximants

Let $f_{j,u_j}(z)$ be the Laurent series in Theorem 4.2. To consider the linear independence results on the values of $f_{j,u_j}(z)$ á la the method of Siegel (see [30]), we need to study the nonvanishing of determinants of certain matrices. In this section, we compute the determinants of specific matrices whose entries are given by the Padé approximants of $f_{j,u_j}(z)$ obtained in Theorem 4.2.

First, let d be a nonnegative integer and $a_1(z), a_2(z), b_1(z), \ldots, b_d(z) \in K[z]$. Put $a(z) = a_1(z)a_2(z)$, $w_j = \max\{\deg a - 2, \deg b_j - 1\}$ and $W = w_1 + \cdots + w_d + d$.

Assume $w_j \geq 0$, $\deg a_1 \leq 1$, a_1 is a monic polynomial and

$$\gamma_{j_1,j_2} = \frac{b_{j_1}(z) - b_{j_2}(z)}{a_2(z)} \in K \setminus \{0\} \quad \text{for } 1 \leq j_1 < j_2 \leq d.$$

Denote $D_j = -a(z)\partial_z + b_j(z) \in K[z, \partial_z]$ and assume Equation (4-1) for D_j. Lemma 4.1 implies that there exist Laurent series $f_{j,0}(z), \ldots, f_{j,w_j}(z) \in (1/z) \cdot K[[1/z]]$ that are linearly independent over K and satisfy

$$D_j \cdot f_{j,u_j}(z) \in K[z] \quad \text{for } 1 \leq j \leq d, \ \ 0 \leq u_j \leq w_j.$$

We now fix these series. For $n \in \mathbb{N}$, we denote the weighted Rodrigues operator associated with D_j by

$$R_{j,n} = \frac{1}{n!}\Big(\partial_z + \frac{b_j(z)}{a(z)}\Big)^n a_1(z)^n \quad \text{for } 1 \leq j \leq d.$$

Lemma 4.4 to the case of $a(z) = a_1(z)a_2(z)$ and $c(z) = a_1(z)$ asserts the commutativity of the differential operators $R_{j,n}$, namely

$$R_{j_1,n}R_{j_2,n} = R_{j_2,n}R_{j_1,n} \quad \text{for } 1 \leq j_1, j_2 \leq d.$$

Put $\varphi_{j,u_j} = \varphi_{f_{j,u_j}}$. For $0 \leq h \leq W$, we define

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$$P_{n,h}(z) = P_h(z) = \prod_{j=1}^{d} R_{j,n} \cdot [z^h a_2(z)^{dn}],$$

$$Q_{n,j,u_j,h}(z) = Q_{j,u_j,h}(z) = \varphi_{j,u_j}\Big(\frac{P_h(z) - P_h(t)}{z-t}\Big) \quad \text{for } 1 \leq j \leq d,\ 0 \leq u_j \leq w_j,$$

$$\mathfrak{R}_{n,j,u_j,h}(z) = \mathfrak{R}_{j,u_j,h}(z) = P_h(z) f_{j,u_j}(z) - Q_{j,u_j,h}(z) \quad \text{for } 1 \leq j \leq d,\ 0 \leq u_j \leq w_j.$$

Assume $P_h(z) \neq 0$. Theorem 4.2 yields that the vector of polynomials $(P_h, Q_{j,u_j,h})_{\substack{1\leq j\leq d\\ 0\leq u_j\leq w_j}}$ is a weight $(n, \dots, n) \in \mathbb{N}^W$ Padé-type approximant of $(f_{j,u_j})_{\substack{1\leq j\leq d\\ 0\leq u_j\leq w_j}}$.

First we compute the coefficients of $1/z^{n+1}$ of $\mathfrak{R}_{j,u_j,h}(z)$.

Lemma 5.1. *Let notation be as above. For* $1 \leq j \leq d$, $0 \leq u_j \leq w_j$ *and* $0 \leq h \leq W$,

$$\mathfrak{R}_{j,u_j,h}(z) = \sum_{k=n}^{\infty} \frac{\varphi_{j,u_j}(t^k P_h(t))}{z^{k+1}}$$

and

$$\varphi_{j,u_j}(t^n P_h(t)) = \frac{(-1)^n}{(n!)^{d-1}} \prod_{\substack{1\leq j'\leq d\\ j'\neq j}} \Big[\prod_{k=1}^{n} (\gamma_{j',j} - k\varepsilon_{a_1})\Big] \varphi_{j,u_j}(t^h a_1(t)^n \cdot a_2(t)^{dn}),$$

where $\varepsilon_{a_1} = 1$ *if* $\deg a_1 = 1$ *and* $\varepsilon_{a_1} = 0$ *if* $\deg a_1 = 0$.

Proof. Since $(\mathfrak{R}_{j,u_j,h})_{j,u_j}$ is a weight $(n, \dots, n) \in \mathbb{N}^W$ Padé-type approximation of $(f_{j,u_j})_{j,u_j}$, we have $\mathrm{ord}_\infty \mathfrak{R}_{j,u_j,h} \geq n+1$ and the first equality is obtained by

$$\mathfrak{R}_{j,u_j,h}(z) = \varphi_{j,u_j}\Big(\frac{P_h(t)}{z-t}\Big) = \sum_{k=n}^{\infty} \frac{\varphi_{j,u_j}(t^k P_h(t))}{z^{k+1}}.$$

We prove the second equality. Fix j and put $\mathcal{E}_{a,b_{j'}} = \partial_t + b_{j'}(t)/a(t)$ for $1 \leq j' \leq d$. Then,

$$\mathcal{E}_{a,b_{j'}} = \mathcal{E}_{a,b_j} + \frac{\gamma_{j',j}}{a_1(t)} \tag{5-1}$$

and $\mathcal{R}_{j',n} = (1/n!)\mathcal{E}^n_{a,b_{j'}} a_1(t)^n$. By Proposition 3.2(i), there is a set $\{c_{j,l} \mid l = 0, 1, \dots, n\}$ of integers with $c_{j,n} = (-1)^n n!$ and

$$t^n \mathcal{R}_{j,n} = \sum_{l=0}^{n} \frac{c_{j,l}}{n!} \mathcal{E}^{n-l}_{a,b_j} t^{n-l} a_1(t)^n.$$

Note, by the Leibniz formula, the polynomial $\prod_{j'\neq j} \mathcal{R}_{j',n} \cdot [t^h a_2(t)^{dn}]$ is contained in the ideal $(a_2(t)^n)$. By Corollary 3.3(i),

$$\mathcal{E}^{n-l}_{a,b_j} a_1(t)^n \cdot (a_2(t)^n) \subseteq \ker \varphi_{j,u_j} \quad \text{for } 0 \leq l \leq n-1$$

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and thus,

$$\begin{aligned} t^n P_h(t) &= t^n \mathcal{R}_{j,n} \prod_{j' \neq j} \mathcal{R}_{j',n} \cdot [t^h a_2(t)^{dn}] \\ &= \sum_{l=0}^{n} \frac{c_{j,l}}{n!} \mathcal{E}_{a,b_j}^{n-l} t^{n-l} a_1(t)^n \prod_{j' \neq j} \mathcal{R}_{j',n} \cdot [t^h a_2(t)^{dn}] \\ &\equiv (-1)^n a_1(t)^n \prod_{j' \neq j} \mathcal{R}_{j',n} \cdot [t^h a_2(t)^{dn}] \bmod \ker \varphi_{j,u_j}. \end{aligned} \tag{5-2}$$

Equation (5-1) yields

$$\begin{aligned} a_1(t)^n \mathcal{R}_{j',n} &= \frac{a_1(t)^n}{n!} \Big(\mathcal{E}_{a,b_j} + \frac{\gamma_{j',j}}{a_1(t)} \Big)^n a_1(t)^n \\ &= \frac{1}{n!} \Big(\mathcal{E}_{a,b_j} a_1(t)^n + (\gamma_{j',j} - n\varepsilon_{a_1}) a_1(t)^{n-1} \Big) \Big(\mathcal{E}_{a,b_j} + \frac{\gamma_{j',j}}{a_1(t)} \Big)^{n-1} a_1(t)^n \\ &\equiv \frac{1}{n!} (\gamma_{j',j} - n\varepsilon_{a_1}) a_1(t)^{n-1} \Big(\mathcal{E}_{a,b_j} + \frac{\gamma_{j',j}}{a_1(t)} \Big)^{n-1} a_1(t)^n \bmod \{\mathcal{E}_{a,b_j} a_1(t) \cdot K[t, \partial_t]\} \\ &\equiv \frac{1}{n!} \prod_{k=1}^{n} (\gamma_{j',j} - k\varepsilon_{a_1}) a_1(t)^n \bmod \{\mathcal{E}_{a,b_j} a_1(t) \cdot K[t, \partial_t]\}. \end{aligned} \tag{5-3}$$

Note that we use the assumption $\deg a_1 \leq 1$ and the equality $\partial_t \cdot a(t) = \varepsilon_{a_1}$ in Equation (5-3). Combining the above equality and Equation (5-2) yields

$$\varphi_{j,u_j}(t^n P_h(t)) = \frac{(-1)^n}{(n!)^{d-1}} \prod_{\substack{1 \leq j' \leq d \\ j' \neq j}} \Big[\prod_{k=1}^{n} (\gamma_{j',j} - k\varepsilon_{a_1}) \Big] \varphi_{j,u_j}(t^h a_1(t)^n \cdot a_2(t)^{dn}).$$

This completes the proof of Lemma 5.1. □

For a nonnegative integer n, we now consider the determinant of the following $(W+1) \times (W+1)$ matrix:

$$\Delta_n(z) = \Delta(z) = \det \begin{pmatrix} P_0(z) & P_1(z) & \dots & P_W(z) \\ Q_{1,0,0}(z) & Q_{1,0,1}(z) & \dots & Q_{1,0,W}(z) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{1,w_1,0}(z) & Q_{1,w_1,1}(z) & \dots & Q_{1,w_1,W}(z) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{d,0,0}(z) & Q_{d,0,1}(z) & \dots & Q_{d,0,W}(z) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{d,w_d,0}(z) & Q_{d,w_d,1}(z) & \dots & Q_{d,w_d,W}(z) \end{pmatrix}.$$

Notice that the determinant $\Delta(z)$ is a polynomial.

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To compute $\Delta(z)$, we define the determinant of following $W \times W$ matrix:

$$\Theta_n = \Theta = \det \begin{pmatrix} \varphi_{1,0}(a_1(t)^n a_2(t)^{dn}) & \varphi_{1,0}(ta_1(t)^n a_2(t)^{dn}) & \dots & \varphi_{1,0}(t^{W-1}a_1(t)^n a_2(t)^{dn}) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{1,w_1}(a_1(t)^n a_2(t)^{dn}) & \varphi_{1,w_1}(ta_1(t)^n a_2(t)^{dn}) & \dots & \varphi_{1,w_1}(t^{W-1}a_1(t)^n a_2(t)^{dn}) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{d,0}(a_1(t)^n a_2(t)^{dn}) & \varphi_{d,0}(ta_1(t)^n a_2(t)^{dn}) & \dots & \varphi_{d,0}(t^{W-1}a_1(t)^n a_2(t)^{dn}) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{d,w_d}(a_1(t)^n a_2(t)^{dn}) & \varphi_{d,w_d}(ta_1(t)^n a_2(t)^{dn}) & \dots & \varphi_{d,w_d}(t^{W-1}a_1(t)^n a_2(t)^{dn}) \end{pmatrix}.$$

Notice that $\Theta \in K$. Replace the coefficient of $z^{(n+1)W}$ of the polynomial P_W by p_W, that is,

$$p_W = \frac{1}{[(n+1)W]!}\partial_z^{(n+1)W} \cdot P_W(z).$$

Then we have the following proposition.

PROPOSITION 5.2. $\Delta(z) \in K$. *More precisely,*

$$\Delta(z) = \left(\frac{-1}{(n!)^{d-1}}\right)^W p_W \cdot \prod_{j=1}^{d}\Bigg[\prod_{\substack{1\le j'\le d \\ j'\neq j}}\prod_{k=1}^{n}(\gamma_{j',j} - k\varepsilon_{a_1})\Bigg]^{w_j+1} \cdot \Theta,$$

where ε_{a_1} is the real number defined in Lemma 5.1.

PROOF. First, by the definition of $P_l(z)$,

$$\deg P_l \le nW + l. \tag{5-4}$$

For the matrix in the definition of $\Delta(z)$, for $1 \le j \le d, 0 \le u_j \le w_j$, adding $-f_{j,u_j}(z)$ times the first row to the $(w_1 + \cdots + w_{j-1}) + u_j + 1$ th row,

$$\Delta(z) = (-1)^W \det \begin{pmatrix} P_0(z) & \dots & P_W(z) \\ \mathfrak{R}_{1,0,0}(z) & \dots & \mathfrak{R}_{1,0,W}(z) \\ \vdots & \ddots & \vdots \\ \mathfrak{R}_{1,w_1,0}(z) & \dots & \mathfrak{R}_{1,w_1,W}(z) \\ \vdots & \ddots & \vdots \\ \mathfrak{R}_{d,0,0}(z) & \dots & \mathfrak{R}_{d,0,W}(z) \\ \vdots & \ddots & \vdots \\ \mathfrak{R}_{d,w_d,0}(z) & \dots & \mathfrak{R}_{d,w_d,W}(z) \end{pmatrix}.$$

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We denote the (s,t) th cofactor of the matrix in the right-hand side of the above equality by $\Delta_{s,t}(z)$. Then we have, developing along the first row,

$$\Delta(z) = (-1)^W\Big(\sum_{l=0}^{W} P_l(z)\Delta_{1,l+1}(z)\Big). \tag{5-5}$$

Since

$$\mathrm{ord}_\infty\, \mathfrak{R}_{l,h}(z) \geq n+1 \quad \text{for } 1 \leq j \leq d, 0 \leq u_j \leq w_j, 0 \leq h \leq W,$$

$$\mathrm{ord}_\infty\, \Delta_{1,l+1}(z) \geq (n+1)W \quad \text{for } 0 \leq l \leq W.$$

Combining Equation (5-4) and the above inequality yields

$$P_l(z)\Delta_{1,l+1}(z) \in (1/z)\cdot K[[1/z]] \quad \text{for } 0 \leq l \leq W-1,$$

and

$$P_W(z)\Delta_{1,W+1}(z) \in K[[1/z]].$$

Note that in the above relation, the constant term of $P_W(z)\Delta_{1,W+1}(z)$ is

$$p_W \cdot \text{‘Coefficient of } 1/z^{(n+1)W} \text{ of } \Delta_{1,W+1}(z)\text{’}. \tag{5-6}$$

Equation (5-5) implies $\Delta(z)$ is a polynomial in z with nonpositive valuation with respect to ord_∞. Thus, it has to be a constant. At last, by Lemma 5.1, the coefficient of $1/z^{(n+1)W}$ of $\Delta_{1,W+1}(z)$ is

$$\det\begin{pmatrix} (-1)^n\varphi_{1,0}(t^nP_0(t)) & \dots & (-1)^n\varphi_{1,0}(t^nP_{W-1}(t)) \\ \vdots & \ddots & \vdots \\ (-1)^n\varphi_{1,w_1}(t^nP_0(t)) & \dots & (-1)^n\varphi_{1,w_1}(t^nP_{W-1}(t)) \\ \vdots & \ddots & \vdots \\ (-1)^n\varphi_{d,0}(t^nP_0(t)) & \dots & (-1)^n\varphi_{d,0}(t^nP_{W-1}(t)) \\ \vdots & \ddots & \vdots \\ (-1)^n\varphi_{d,w_d}(t^nP_0(t)) & \dots & (-1)^n\varphi_{d,w_d}(t^nP_{W-1}(t)) \end{pmatrix}$$

$$= \Big(\frac{1}{(n!)^{d-1}}\Big)^W \prod_{j=1}^{d}\Big[\prod_{\substack{1\leq j'\leq d \\ j'\neq j}}\prod_{k=1}^{n}(\gamma_{j',j} - k\varepsilon_{a_1})\Big]^{w_j+1}\cdot\Theta.$$

Combining Equations (5-5), (5-6) and the above equality yields the assertion. This completes the proof of Proposition 5.2. □

6. Examples

In this section, let us describe some examples of the application of Theorem 4.2 and Proposition 5.2.

EXAMPLE 6.1. Let us give a generalization of the Chevyshev polynomials (see [3, Section 5.1]). Let $u \geq 2$ be an integer. Put $D = -(z^u-1)\partial_z - z^{u-1} \in K[z,\partial_z]$. The Laurent series

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$$f_l(z) = \sum_{k=0}^{\infty} \frac{(\frac{1+l}{u})_k}{(\frac{u+l}{u})_k} \frac{1}{z^{uk+l+1}} = \frac{1}{z^{l+1}} \cdot {}_2F_1\Big(\frac{1+l}{u}, 1, \frac{u+l}{u} \Big| \frac{1}{z^u}\Big) \quad \text{for } 0 \le l \le u-2$$

are linearly independent over K and satisfy $D \cdot f_l(z) \in K[z]$. Note that $f_0(z) = (z^u - 1)^{-1/u}$. We denote $\varphi_{f_l} = \varphi_l$. For $h, n \in \mathbb{N}$ with $0 \le h \le u-1$, we define

$$P_{n,h}(z) = P_h(z) = \frac{1}{n!}\Big(\partial_z - \frac{z^{u-1}}{z^u - 1}\Big)^n (z^u - 1)^n \cdot z^h,$$
$$Q_{n,l,h}(z) = Q_{l,h}(z) = \varphi_l\Big(\frac{P_h(z) - P_h(t)}{z - t}\Big) \quad \text{for } 0 \le l \le u-2.$$

Theorem 4.2 yields that the vector of polynomials $(P_h, Q_{j,h})_{0 \le j \le u-2}$ is a weight $(n, \dots, n) \in \mathbb{N}^{u-1}$ Padé-type approximant of $(f_0, \dots, f_{u-2})$. Define

$$\Delta_n(z) = \det \begin{pmatrix} P_0(z) & \cdots & P_{u-1}(z) \\ Q_{0,0}(z) & \cdots & Q_{0,u-1}(z) \\ \vdots & \ddots & \vdots \\ Q_{u-2,0}(z) & \cdots & Q_{u-2,u-1}(z) \end{pmatrix}.$$

The determinant $\Delta_n(z)$ is computed in Lemma 7.2.

EXAMPLE 6.2. In this example, we give a generalization of the Bessel polynomials (see [16]). Let d, n be nonnegative integers and $\gamma_1, \dots, \gamma_d \in K$ that are not integers less than -1 with

$$\gamma_{j_2} - \gamma_{j_1} \notin \mathbb{Z} \quad \text{for } 1 \le j_1 < j_2 \le d.$$

Put $D_j = -z^2\partial_z + \gamma_j z - 1$,

$$f_j(z) = \sum_{k=0}^{\infty} \frac{1}{(2+\gamma_j)_k} \frac{1}{z^{k+1}}$$

and $\varphi_{f_j} = \varphi_j$. A straightforward computation yields $D_j \cdot f_j(z) \in K$. Put

$$R_{j,n} = \frac{1}{n!}\Big(\partial_z + \frac{\gamma_j z - 1}{z^2}\Big)^n z^n.$$

Lemma 4.4 yields

$$R_{j_1,n_1} R_{j_2,n_2} = R_{j_2,n_2} R_{j_1,n_1} \quad \text{for } 1 \le j_1, j_2 \le d \text{ and } n_{j_1}, n_{j_2} \in \mathbb{N}.$$

For $h \in \mathbb{Z}$ with $0 \le h \le d$, we define

$$P_{n,h}(z) = P_h(z) = \prod_{j=1}^{d} R_{j,n} \cdot z^{dn+h},$$
$$Q_{n,j}(z) = Q_j(z) = \varphi_j\Big(\frac{P_h(z) - P_h(t)}{z - t}\Big) \quad \text{for } 1 \le j \le d.$$

Then Theorem 4.2 yields that the vector of polynomials $(P_h, Q_{j,h})_{1\le j\le d}$ is a weight $(n,\dots,n)\in\mathbb{N}^d$ Padé approximant of $(f_1,\dots,f_d)$. By the definition of $P_d(z)$,

$$P_d(z)=\frac{\prod_{j=1}^{d}(d(n+1)+\gamma_j+1)_n}{(n!)^d}z^{d(n+1)}+(\text{lower degree terms}). \tag{6-1}$$

Define

$$\Delta_n(z)=\det\begin{pmatrix} P_0(z) & P_1(z) & \dots & P_d(z)\\ Q_{1,0}(z) & Q_{1,1}(z) & \dots & Q_{1,d}(z)\\ \vdots & \vdots & \ddots & \vdots\\ Q_{d,0}(z) & Q_{d,1}(z) & \dots & Q_{d,d}(z)\end{pmatrix},$$

$$\Theta_n=\det\begin{pmatrix}\varphi_1(t^{(d+1)n}) & \dots & \varphi_1(t^{(d+1)n+d-1})\\ \vdots & \ddots & \vdots\\ \varphi_d(t^{(d+1)n}) & \dots & \varphi_d(t^{(d+1)n+d-1})\end{pmatrix}.$$

Let us compute Θ_n. By the definition of φ_j and the properties of determinants,

$$\begin{aligned}\Theta_n&=\det\begin{pmatrix}\frac{1}{(2+\gamma_1)_{(d+1)n}} & \dots & \frac{1}{(2+\gamma_1)_{(d+1)n+d-1}}\\ \vdots & \ddots & \vdots\\ \frac{1}{(2+\gamma_d)_{(d+1)n}} & \dots & \frac{1}{(2+\gamma_d)_{(d+1)n+d-1}}\end{pmatrix}\\ &=\prod_{j=1}^{d}\frac{1}{(2+\gamma_j)_{(d+1)n+d-1}}\cdot\det\begin{pmatrix}(2+\gamma_1+(d+1)n)_{d-1} & \dots & (2+\gamma_1+(d+1)n)_0\\ \vdots & \ddots & \vdots\\ (2+\gamma_d+(d+1)n)_{d-1} & \dots & (2+\gamma_d+(d+1)n)_0\end{pmatrix}.\end{aligned}$$

Here, by using the properties of the determinant again,

$$\begin{aligned}&\det\begin{pmatrix}(2+\gamma_1+(d+1)n)_{d-1} & \dots & (2+\gamma_1+(d+1)n)_0\\ \vdots & \ddots & \vdots\\ (2+\gamma_d+(d+1)n)_{d-1} & \dots & (2+\gamma_d+(d+1)n)_0\end{pmatrix}\\ &=(-1)^{(d-1)d/2}\det\begin{pmatrix}(2+\gamma_1+(d+1)n)_0 & \dots & (2+\gamma_1+(d+1)n)_{d-1}\\ \vdots & \ddots & \vdots\\ (2+\gamma_d+(d+1)n)_0 & \dots & (2+\gamma_d+(d+1)n)_{d-1}\end{pmatrix}\\ &=(-1)^{(d-1)d/2}\det\begin{pmatrix}1 & \gamma_1 & \dots & \gamma_1^{d-1}\\ \vdots & \ddots & \vdots & \\ 1 & \gamma_d & \dots & \gamma_d^{d-1}\end{pmatrix}.\end{aligned}$$

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Since the last determinant is a Vandermonde determinant,

$$\Theta_n = \prod_{j=1}^{d} \frac{1}{(2+\gamma_j)_{(d+1)n+d-1}} \cdot (-1)^{(d-1)d/2} \prod_{1\le j_1<j_2\le d} (\gamma_{j_2}-\gamma_{j_1}).$$

Proposition 5.2 and Equation (6-1) imply that

$$\Delta_n(z) = (-1)^{(d-1)d/2}\Big(\frac{-1}{(n!)^d}\Big)^d \cdot \prod_{j=1}^{d}\Big[\prod_{\substack{1\le j'\le d\\ j'\ne j}}\prod_{k=1}^{n}(\gamma_{j'}-\gamma_j-k)\Big]$$
$$\times \prod_{j=1}^{d} \frac{(d(n+1)+\gamma_j+1)_n}{(2+\gamma_j)_{(d+1)n+d-1}} \cdot \prod_{1\le j_1<j_2\le d}(\gamma_{j_2}-\gamma_{j_1}).$$

Especially, we have $\Delta_n(z) \in K \setminus \{0\}$.

EXAMPLE 6.3. In this example, we give a generalization of the Laguerre polynomials (see [3, Section 6.2]). Let $d, n \in \mathbb{N}$, $\gamma_1, \dots, \gamma_d \in K \setminus \{0\}$ be pairwise distinct and $\delta \in K$ be a nonnegative integer. Put $D_j = -z\partial_z - \gamma_j z + \delta$,

$$f_j(z) = \sum_{k=0}^{\infty} (1+\delta)_k \Big(\frac{1}{\gamma_j z}\Big)^{k+1}$$

and $\varphi_{f_j} = \varphi_j$. A straightforward computation shows $D_j \cdot f_j(z) \in K$. Put

$$R_{j,n} = \frac{1}{n!}\Big(\partial_z - \frac{\gamma_j z - \delta}{z}\Big)^n.$$

By Lemma 4.4,

$$R_{j_1,n_{j_1}} R_{j_2,n_{j_2}} = R_{j_2,n_{j_2}} R_{j_1,n_{j_1}} \quad \text{for } 1 \le j_1, j_2 \le d, \ \ n_{j_1}, n_{j_2} \in \mathbb{N}.$$

For $h \in \mathbb{Z}$ with $0 \le h \le d$, we define

$$P_{n,h}(z) = P_h(z) = \prod_{j=1}^{d} R_{j,n} \cdot z^{dn+h},$$
$$Q_{n,j}(z) = Q_j(z) = \varphi_j\Big(\frac{P_h(z) - P_h(t)}{z-t}\Big) \quad \text{for } 1 \le j \le d.$$

Then Theorem 4.2 yields that the vector of polynomials $(P_h, Q_{j,h})_{1\le j\le d}$ is a weight $(n, \dots, n) \in \mathbb{N}^d$ Padé-type approximant of $(f_j)_{1\le j\le d}$. By the definition of $P_d(z)$,

$$P_d(z) = \frac{\prod_{j=1}^{d} \gamma_j^n}{(n!)^d} z^{d(n+1)} + \text{(lower degree terms)}. \tag{6-2}$$

Define

$$\Delta_n(z) = \det\begin{pmatrix} P_0(z) & P_1(z) & \dots & P_d(z) \\ Q_{1,0}(z) & Q_{1,1}(z) & \dots & Q_{1,d}(z) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{d,0}(z) & Q_{d,1}(z) & \dots & Q_{d,d}(z) \end{pmatrix}, \quad \Theta_n = \det\begin{pmatrix} \varphi_1(t^{dn}) & \dots & \varphi_1(t^{d(n+1)-1}) \\ \vdots & \ddots & \vdots \\ \varphi_d(t^{dn}) & \dots & \varphi_d(t^{d(n+1)-1}) \end{pmatrix}.$$

We now compute Θ_n. By the definition of φ_j and the properties of the determinant,

$$\Theta_n = \det\begin{pmatrix} \frac{(1+\delta)_{dn}}{\gamma_1^{dn+1}} & \dots & \frac{(1+\delta)_{d(n+1)-1}}{\gamma_1^{d(n+1)}} \\ \vdots & \ddots & \vdots \\ \frac{(1+\delta)_{dn}}{\gamma_d^{dn+1}} & \dots & \frac{(1+\delta)_{d(n+1)-1}}{\gamma_d^{d(n+1)}} \end{pmatrix}$$

$$= \prod_{j=1}^{d} \frac{(1+\delta)_{dn+j-1}}{\gamma_j^{d(n+1)}} \cdot \det\begin{pmatrix} 1 & \gamma_1 & \dots & \gamma_1^{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \gamma_d & \dots & \gamma_d^{d-1} \end{pmatrix}.$$

Since the last determinant is nothing but a Vandermonde determinant,

$$\Theta_n = \prod_{j=1}^{d} \frac{(1+\delta)_{dn+j-1}}{\gamma_j^{d(n+1)}} \cdot \prod_{1\le j_1<j_2\le d} (\gamma_{j_2}-\gamma_{j_1}).$$

Proposition 5.2 and Equation (6-2) imply that

$$\Delta_n(z) = \left(\frac{-1}{(n!)^d}\right)^d \cdot \prod_{j=1}^{d}\Big[\prod_{\substack{1\le j'\le d\\ j'\neq j}} (\gamma_{j'}-\gamma_j)^n\Big] \cdot \prod_{j=1}^{d} \frac{(1+\delta)_{dn+j-1}}{\gamma_j^{(d-1)n+d}} \cdot \prod_{1\le j_1<j_2\le d} (\gamma_{j_2}-\gamma_{j_1}) \in K\setminus\{0\}.$$

EXAMPLE 6.4. Let us give an alternative generalization of the Laguerre polynomials. Let $d, n \in \mathbb{N}$, $\gamma \in K\setminus\{0\}$, and $\delta_1, \dots, \delta_d \in K$ be nonnegative integers with

$$\delta_{j_1} - \delta_{j_2} \notin \mathbb{Z} \quad \text{for } 1 \le j_1 < j_2 \le d.$$

Put $D_j = -z\partial_z - \gamma z + \delta_j$,

$$f_j(z) = \sum_{k=0}^{\infty} (1+\delta_j)_k \Big(\frac{1}{\gamma z}\Big)^{k+1},$$

and $\varphi_{f_j} = \varphi_j$. Then we have $D_j \cdot f_j(z) \in K$. Put

$$R_{j,n} = \frac{1}{n!}\Big(\partial_z - \frac{\gamma z - \delta_j}{z}\Big)^n z^n.$$

By Lemma 4.4,

$$R_{j_1,n_{j_1}} R_{j_2,n_{j_2}} = R_{j_2,n_{j_2}} R_{j_1,n_{j_1}} \quad \text{for } 1 \le j_1, j_2 \le d, \ \ n_{j_1}, n_{j_2} \in \mathbb{N}.$$

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For $h \in \mathbb{Z}$ with $0 \leq h \leq d$, we define

$$P_{n,h}(z) = P_h(z) = \prod_{j=1}^{d} R_{j,n} \cdot z^h,$$
$$Q_{n,j}(z) = Q_j(z) = \varphi_j\Big(\frac{P_h(z) - P_h(t)}{z - t}\Big) \quad \text{for } 1 \leq j \leq d.$$

Then Theorem 4.2 yields that the vector of polynomials $(P_h, Q_{j,h})_{1 \leq j \leq d}$ is a weight $(n, \dots, n) \in \mathbb{N}^d$ Padé-type approximant of $(f_j)_{1 \leq j \leq d}$. By the definition of $P_d(z)$,

$$P_d(z) = \frac{\gamma^{dn}}{(n!)^d} z^{d(n+1)} + (\text{lower degree terms}). \tag{6-3}$$

Define

$$\Delta_n(z) = \det \begin{pmatrix} P_0(z) & P_1(z) & \dots & P_d(z) \\ Q_{1,0}(z) & Q_{1,1}(z) & \dots & Q_{1,d}(z) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{d,0}(z) & Q_{d,1}(z) & \dots & Q_{d,d}(z) \end{pmatrix}, \quad \Theta_n = \det \begin{pmatrix} \varphi_1(t^n) & \dots & \varphi_1(t^{d+n-1}) \\ \vdots & \ddots & \vdots \\ \varphi_d(t^n) & \dots & \varphi_d(t^{d+n-1}) \end{pmatrix}.$$

Let us compute Θ_n. By the definition of φ_j and the properties of the determinant,

$$\Theta_n = \det \begin{pmatrix} \frac{(1+\delta_1)_n}{\gamma^{n+1}} & \dots & \frac{(1+\delta_1)_{d+n-1}}{\gamma^{d+n}} \\ \vdots & \ddots & \vdots \\ \frac{(1+\delta_d)_n}{\gamma^{n+1}} & \dots & \frac{(1+\delta_d)_{d+n-1}}{\gamma^{d+n}} \end{pmatrix}$$
$$= \prod_{j=1}^{d} \frac{(1+\delta_j)_n}{\gamma^{n+j}} \cdot \begin{pmatrix} (n+\delta_1)_0 & \dots & (n+\delta_1)_{d-1} \\ \vdots & \ddots & \vdots \\ (n+\delta_1)_0 & \dots & (n+\delta_d)_{d-1} \end{pmatrix}.$$

A similar computation as in Example 6.2 leads us to get

$$\Theta_n = \prod_{j=1}^{d} \frac{(1+\delta_j)_n}{\gamma^{n+j}} \cdot \prod_{1 \leq j_1 < j_2 \leq d} (\delta_{j_2} - \delta_{j_1}).$$

Proposition 5.2 and Equation (6-3) imply that

$$\Delta_n(z) = \Big(\frac{-1}{(n!)^d}\Big)^d \cdot \prod_{j=1}^{d} \Big[\prod_{\substack{1 \leq j' \leq d \\ j' \neq j}} \prod_{k=1}^{n} (\delta_{j'} - \delta_j - k)\Big]$$
$$\times \prod_{j=1}^{d} \frac{(1+\delta_j)_n}{\gamma^j} \cdot \prod_{1 \leq j_1 < j_2 \leq d} (\delta_{j_2} - \delta_{j_1}) \in K \setminus \{0\}.$$

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EXAMPLE 6.5. In this example, we give a generalization of the Hermite polynomials (see [3, Section 6.1]). Let $d, n \in \mathbb{N}$, $\gamma \in K \setminus \{0\}$ and $\delta_1, \ldots, \delta_d \in K$ be pairwise distinct. Put $D_j = -\partial_z + \gamma z + \delta_j$,

$$f_j(z) = \sum_{k=0}^{\infty} \frac{f_{j,k}}{z^{k+1}},$$

where $f_{j,0} = 1$, $f_{j,1} = -\delta_j/\gamma$ and

$$f_{j,k+2} = -\frac{\delta_j f_{j,k+1} + (k+1) f_{j,k}}{\gamma} \quad \text{for } k \geq 0, \tag{6-4}$$

and $\varphi_{f_j} = \varphi_j$. Then we have $D_j \cdot f_j(z) \in K$. Put

$$R_{j,n} = \frac{1}{n!}(\partial_z + \gamma z + \delta_j)^n.$$

By Lemma 4.4,

$$R_{j_1,n_{j_1}} R_{j_2,n_{j_2}} = R_{j_2,n_{j_2}} R_{j_1,n_{j_1}} \quad \text{for } 1 \leq j_1, j_2 \leq d, n_{j_1}, n_{j_2} \in \mathbb{N}.$$

For $h \in \mathbb{Z}$ with $0 \leq h \leq d$, we define

$$P_{n,h}(z) = P_h(z) = \prod_{j=1}^{d} R_{j,n} \cdot z^h,$$

$$Q_{n,j,h}(z) = Q_{j,h}(z) = \varphi_j\Big(\frac{P_h(z) - P_h(t)}{z - t}\Big) \quad \text{for } 1 \leq j \leq d.$$

Then Theorem 4.2 yields that the vector of polynomials $(P_h, Q_{j,h})_{1\leq j\leq d}$ is a weight $(n, \ldots, n) \in \mathbb{N}^d$ Padé-type approximant of $(f_j)_{1\leq j\leq d}$. By the definition of $P_d(z)$,

$$P_d(z) = \frac{\gamma^{dn}}{(n!)^d} z^{d(n+1)} + (\text{lower degree terms}). \tag{6-5}$$

Define

$$\Delta_n(z) = \det\begin{pmatrix} P_0(z) & P_1(z) & \ldots & P_d(z) \\ Q_{1,0}(z) & Q_{1,1}(z) & \ldots & Q_{1,d}(z) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{d,0}(z) & Q_{d,1}(z) & \ldots & Q_{d,d}(z) \end{pmatrix}, \quad \Theta_n = \det\begin{pmatrix} \varphi_1(1) & \ldots & \varphi_1(t^{d-1}) \\ \vdots & \ddots & \vdots \\ \varphi_d(1) & \ldots & \varphi_d(t^{d-1}) \end{pmatrix}.$$

Let us compute Θ_n. By the definition of φ_j,

$$\Theta_n = \det\begin{pmatrix} f_{1,0} & f_{1,1} & \ldots & f_{1,d-1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{d,0} & f_{d,1} & \ldots & f_{d,d-1} \end{pmatrix}. \tag{6-6}$$

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Here, using Equation (6-4) and the properties of the determinant repeatedly,

$$\det\begin{pmatrix} f_{1,0} & f_{1,1} & \dots & f_{1,d-1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{d,0} & f_{d,1} & \dots & f_{d,d-1} \end{pmatrix} = \det\begin{pmatrix} 1 & \frac{-\delta_1}{\gamma} & \dots & \left(\frac{-\delta_1}{\gamma}\right)^{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{-\delta_d}{\gamma} & \dots & \left(\frac{-\delta_d}{\gamma}\right)^{d-1} \end{pmatrix}. \tag{6-7}$$

Combining Equations (6-6) and (6-7) implies

$$\Theta_n = \left(\frac{-1}{\gamma}\right)^{1+2+\cdots+(d-1)} \cdot \prod_{1\le j_1<j_2\le d} (\delta_{j_2} - \delta_{j_1}).$$

Proposition 5.2 and Equation (6-5) imply that

$$\begin{aligned} \Delta_n(z) = \left(\frac{-1}{(n!)^d}\right)^d \cdot \prod_{j=1}^{d} \Bigg[\prod_{\substack{1\le j'\le d \\ j'\neq j}} (\delta_{j'} - \delta_j)^n \Bigg] \\ \times (-1)^{(d-1)d/2} \gamma^{dn-(d-1)d/2} \cdot \prod_{1\le j_1<j_2\le d} (\delta_{j_2} - \delta_{j_1}) \in K \setminus \{0\}. \end{aligned}$$

EXAMPLE 6.6. In this example, we consider a generalization of the Legendre polynomials (see [3, Remark 5.3.1]). Let $d, m, n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_m \in K \setminus \{0\}$ be pairwise distinct and $\gamma_1, \dots, \gamma_d \in K$ be nonnegative integers, satisfying $\gamma_{j_1} - \gamma_{j_2} \notin \mathbb{Z}$ for $1 \le j_1 < j_2 \le d$. Put $a_2(z) = \prod_{i=1}^{m}(z - \alpha_i)$, $D_j = -za_2(z)\partial_z + \gamma_j a_2(z)$,

$$f_{i,j}(z) = \sum_{k=0}^{\infty} \frac{1}{k+1+\gamma_j}\left(\frac{\alpha_i}{z}\right)^{k+1} \quad \text{for } 1 \le i \le m, 1 \le j \le d,$$

and $\varphi_{f_{i,j}} = \varphi_{i,j}$. Then we have $D_j \cdot f_{i,j}(z) \in K[z]$. Put

$$R_{j,n} = \frac{1}{n!}\left(\partial_z + \frac{\gamma_j}{z}\right)^n z^n.$$

By Lemma 4.4, we have

$$R_{j_1,n_{j_1}} R_{j_2,n_{j_2}} = R_{j_2,n_{j_2}} R_{j_1,n_{j_1}} \quad \text{for } 1 \le j_1, j_2 \le d, n_{j_1}, n_{j_2} \in \mathbb{N}.$$

For $h \in \mathbb{Z}$ with $0 \le h \le dm$, we define

$$\begin{aligned} P_{n,h}(z) &= P_h(z) = \prod_{j=1}^{d} R_{j,n} \cdot [z^h a_2(z)^{dn}], \\ Q_{n,i,j,h}(z) &= Q_{i,j,h}(z) = \varphi_{i,j}\left(\frac{P_h(z) - P_h(t)}{z - t}\right) \quad \text{for } 1 \le i \le m, 1 \le j \le d. \end{aligned}$$

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Then Theorem 4.2 yields that the vector of polynomials $(P_h, Q_{i,j,h})_{\substack{1\le i\le m\\ 1\le j\le d}}$ is a weight $(n,\dots,n)\in\mathbb{N}^{dm}$ Padé-type approximant of $(f_{i,j})_{\substack{1\le i\le m\\ 1\le j\le d}}$. Define

$$\Delta_n(z)=\det\begin{pmatrix} P_0(z) & P_1(z) & \dots & P_{dm}(z)\\ Q_{1,1,0}(z) & Q_{1,1,1}(z) & \dots & Q_{1,1,dm}(z)\\ \vdots & \vdots & \ddots & \vdots\\ Q_{m,1,0}(z) & Q_{m,1,1}(z) & \dots & Q_{m,1,dm}(z)\\ \vdots & \vdots & \ddots & \vdots\\ Q_{1,d,0}(z) & Q_{1,d,1}(z) & \dots & Q_{1,d,dm}(z)\\ \vdots & \vdots & \ddots & \vdots\\ Q_{m,d,0}(z) & Q_{m,d,1}(z) & \dots & Q_{m,d,dm}(z) \end{pmatrix}.$$

The nonvanishing of $\Delta_n(z)$ has been proven in [12, Proposition 4.1].

REMARK 6.7. We mention that Examples 6.2, 6.3, 6.4 and 6.6 can be applicable to prove the linear independence of the values of the series which are considered in each example. However, such results have been obtained as follows.

In Example 6.2, for $\gamma_1,\dots,\gamma_d\in\mathbb{Q}$, the series $f_j(z)$ become E-functions in the sense of Siegel (see [30]). The linear independence result for the values of these E-functions has been studied by Väänänen in [35]. In Example 6.3, for $\delta\in\mathbb{Q}$ and $\gamma_1,\dots,\gamma_d\in K$ for an algebraic number field K, the series $f_j(z)$ are Euler-type series. In the case of $\delta=0$, the global relations among the values of these Euler-type series have been studied by Matala-aho and Zudilin for $d=1$ in [22] and L. Seppälä for general d in [29]. Likewise, Example 6.4, for $\delta_1,\dots,\delta_d\in\mathbb{Q}$ and $\gamma=1$, treats Euler-type series. In [34], Väänänen studied the global relations among the values of these Euler-type series. In Example 6.6, for $\gamma_1,\dots,\gamma_d\in\mathbb{Q}$ and $\alpha_1,\dots,\alpha_m\in K$ for an algebraic number field K, the series $f_{i,j}(z)$ become G-functions in the sense of Siegel (see [30]) called the first Lerch functions. The linear independence of values of these functions has been studied by David, Hirata-Kohno and the author in [12, Theorem 2.1].

7. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We prove the more precise theorem that we state below. To state the theorem, we prepare the notation.

Let K be an algebraic number field. We denote the set of places of K by $\mathfrak{M}_K$. For $v\in\mathfrak{M}_K$, we denote the completion of K with respect to v by K_v and define the normalized absolute value $|\cdot|_v$ as follows:

$$|p|_v=p^{-[K_v:\mathbb{Q}_p]/[K:\mathbb{Q}]}\ \text{if } v\mid p,\qquad |x|_v=|\iota_v x|^{[K_v:\mathbb{R}]/[K:\mathbb{Q}]}\ \text{if } v\mid\infty,$$

where p is a prime number and ι_v the embedding $K\hookrightarrow\mathbb{C}$ corresponding to v.

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Let $\beta \in K$. We define the absolute Weil height of β as

$$\mathrm{H}(\beta) = \prod_{v \in \mathfrak{M}_K} \max\{1, |\beta|_v\}.$$

Let m be a positive integer and $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_m) \in \mathbb{P}_m(K)$. We define the absolute Weil height of $\boldsymbol{\beta}$ by

$$\mathrm{H}(\boldsymbol{\beta}) = \prod_{v \in \mathfrak{M}_K} \max\{|\beta_0|_v, \ldots, |\beta_m|_v\},$$

and the logarithmic absolute Weil height by $\mathrm{h}(\boldsymbol{\beta}) = \log \mathrm{H}(\boldsymbol{\beta})$. Let $v \in \mathfrak{M}_K$, then $\mathrm{h}_v(\boldsymbol{\beta}) = \log \|\boldsymbol{\beta}\|_v$ where $\|\cdot\|_v$ is the sup v-adic norm. Then we have $\mathrm{h}(\boldsymbol{\beta}) = \sum_{v \in \mathfrak{M}_K} \mathrm{h}_v(\boldsymbol{\beta})$ and for $\beta \in K$, $\mathrm{h}(\beta)$ is the height of the point $(1, \beta) \in \mathbb{P}_1(K)$.

Let u be an integer with $u \geq 2$. We put $\nu(u) = u \prod_{q:\text{prime}, q|u} q^{1/(q-1)}$. Let v_0 be a place of K, $\alpha \in K$ with $|\alpha|_{v_0} > 2$. In the case where v_0 is a nonarchimedean place, we denote the prime number under v_0 by p_{v_0} and put $\varepsilon_{v_0}(u) = 1$ if u is coprime with p_{v_0} and $\varepsilon_{v_0}(u) = 0$ if u is divisible by p_{v_0}. We denote Euler's totient function by φ.

We define real numbers

$$\mathbb{A}_{v_0}(\alpha) = \mathrm{h}_{v_0}(\alpha) - \begin{cases} \mathrm{h}_{v_0}(2) & \text{if } v_0 \mid \infty \\ \dfrac{\varepsilon_{v_0}(u) \log |p_{v_0}|_{v_0}}{p_{v_0} - 1} & \text{if } v_0 \nmid \infty, \end{cases}$$

$$\mathbb{B}_{v_0}(\alpha) = (u-1)\mathrm{h}(\alpha) + (u+1)\mathrm{h}(2) + \frac{(2u-1)\log \nu(u)}{u}$$
$$+ \frac{u-1}{\varphi(u)} - (u-1)\mathrm{h}_{v_0}(\alpha) - \begin{cases} (u+1)\mathrm{h}_{v_0}(2) & \text{if } v_0 \mid \infty \\ \log |\nu(u)|_{v_0}^{-1} & \text{if } v_0 \nmid \infty, \end{cases}$$

$$U_{v_0}(\alpha) = (u-1)\mathrm{h}_{v_0}(\alpha) + \begin{cases} (u+1)\mathrm{h}_{v_0}(2) & \text{if } v_0 \mid \infty \\ \log |\nu(u)|_{v_0}^{-1} & \text{if } v_0 \nmid \infty, \end{cases}$$

$$V_{v_0}(\alpha) = \mathbb{A}_{v_0}(\alpha) - \mathbb{B}_{v_0}(\alpha).$$

We can now state the following theorem.

THEOREM 7.1. *Assume $V_{v_0}(\alpha) > 0$. Then, for any positive number ε with $\varepsilon < V_{v_0}(\alpha)$, there exists an effectively computable positive number H_0 depending on ε and the given data such that the following property holds. For any $\boldsymbol{\lambda} = (\lambda, \lambda_l)_{0 \leq l \leq u-2} \in K^u \setminus \{\mathbf{0}\}$ satisfying $H_0 \leq \mathrm{H}(\boldsymbol{\lambda})$, then*

$$\left| \lambda + \sum_{l=0}^{u-2} \lambda_l \cdot \frac{1}{\alpha^{l+1}} {}_2F_1\left(\frac{1+l}{u}, 1, \frac{u+l}{u} \,\middle|\, \frac{1}{\alpha^u} \right) \right|_{v_0} > C(\alpha, \varepsilon) \mathrm{H}_{v_0}(\boldsymbol{\lambda}) \mathrm{H}(\boldsymbol{\lambda})^{-\mu(\alpha, \varepsilon)},$$

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where

$$\mu(\alpha,\varepsilon)=\frac{\mathbb{A}_{v_0}(\alpha)+U_{v_0}(\alpha)}{V_{v_0}(\alpha)-\varepsilon}\quad \textit{and}\quad C(\alpha,\varepsilon)$$
$$=\exp\Big(-\Big(\frac{\log(2)}{V_{v_0}(\alpha)-\varepsilon}+1\Big)(\mathbb{A}_{v_0}(\alpha)+U_{v_0}(\alpha))\Big).$$

We derive Theorem 1.1 from Theorem 7.1.

PROOF OF THEOREM 1.1. Let us consider the case of $K=\mathbb{Q}$, $v_0=\infty$ and $\alpha\in\mathbb{Z}\setminus\{0,\pm1\}$. Then we see that $V_\infty(\alpha)=V(\alpha)$ where $V(\alpha)$ is defined in Theorem 1.1. Assume $V(\alpha)>0$. Choose some $\boldsymbol{\lambda}=(\lambda,\lambda_0\dots,\lambda_{u-2})\in\mathbb{Q}^u\setminus\{\mathbf{0}\}$ such that

$$\lambda_0+\sum_{l=0}^{u-2}\lambda_l\cdot\frac{1}{\alpha^{l+1}}{}_2F_1\Big(\frac{1+l}{u},1,\frac{u+l}{u}\,\Big|\,\frac{1}{\alpha^u}\Big)=0.$$

If $H(\boldsymbol{\lambda})\geq H_0$ (where H_0 is as in Theorem 7.1), there is nothing more to prove. Otherwise, let $m>0$ be a rational integer such that $H(m\boldsymbol{\lambda})\geq H_0$. Then Theorem 7.1 ensures that

$$m\Big(\lambda_0+\sum_{l=0}^{u-2}\lambda_l\cdot\frac{1}{\alpha^{l+1}}{}_2F_1\Big(\frac{1+l}{u},1,\frac{u+l}{u}\,\Big|\,\frac{1}{\alpha^u}\Big)\Big)\neq 0.$$

This is a contradiction and completes the proof of Theorem 1.1. □

Now we start the proof of Theorem 7.1. The proof is relying on the Padé approximants obtained in Example 6.1. In the following, we use the same notation as in Example 6.1.

7.1. Computation of determinants

LEMMA 7.2. *Let n be a positive integer. Put $n=uN+s$ for nonnegative integers N,s with $0\leq s\leq u-1$. Then,*

$$\Delta_n(z)=(-1)^{(uN+s+1)(u-1)}\frac{((uN+s+1)u-1-uN)_{uN+s}}{(uN+s)!}\prod_{l=0}^{u-2}\frac{(\frac{u-1}{u})_{uN+s}}{(\frac{u+l}{u})_{uN+s}}\in K\setminus\{0\}.$$

PROOF. Put

$$\Theta_n=\det\begin{pmatrix}\varphi_0((t^u-1)^n) & \dots & \varphi_0(t^{u-2}(t^u-1)^n)\\ \vdots & \ddots & \vdots\\ \varphi_{u-2}((t^u-1)^n) & \dots & \varphi_{u-2}(t^{u-2}(t^u-1)^n)\end{pmatrix}.$$

Proposition 5.2 implies that

$$\Delta_n(z)=(-1)^{(u-1)}\times\frac{1}{[(n+1)(u-1)]!}\partial_z^{(n+1)(u-1)}\cdot P_{u-1}(z)\times\Theta_n.$$

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According to the definition of $P_{u-1}(z)$,

$$\frac{1}{[(n+1)(u-1)]!}\partial_z^{(n+1)(u-1)} \cdot P_{u-1}(z) = \frac{((n+1)u-1-n)_n}{n!}.$$

By the definition of f_l,

$$\varphi_l(t^k) = \begin{cases} \frac{(\frac{1+l}{u})_N}{(\frac{u+l}{u})_N} & \text{if } k = uN + l \text{ for some } N \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

The above equality shows

$$\Theta_n = \det\begin{pmatrix} \varphi_0((t^u-1)^{uN+s}) & 0 & \dots & 0 \\ 0 & \varphi_1(t(t^u-1)^{uN+s}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varphi_{u-2}(t^{u-2}(t^u-1)^{uN+s}) \end{pmatrix}$$
$$= \prod_{l=0}^{u-2} \varphi_l(t^l(t^u-1)^{uN+s}). \tag{7-1}$$

We now compute $\varphi_l(t^l(t^u-1)^{uN+s})$. Since we have

$$t^l(t^u-1)^{uN+s} = \sum_{v=0}^{uN+s} \binom{uN+s}{v} (-1)^{uN+l-v} t^{uv+l},$$

we obtain

$$\varphi_l(t^l(t^u-1)^{uN+s}) = \sum_{v=0}^{uN+s} \binom{uN+s}{v} (-1)^{uN+l-v} \frac{(\frac{1+l}{u})_v}{(\frac{u+l}{u})_v}.$$

For positive real numbers α, β with $\alpha < \beta$ and a nonnegative integer v,

$$\frac{(\alpha)_v}{(\beta)_v} = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 \xi^{\alpha+v-1}(1-\xi)^{\beta-\alpha-1} d\xi.$$

Applying the above equality for $\alpha = (1+l)/u, \beta = (u+l)/u$, we obtain

$$\begin{aligned} &\varphi_l(t^l(t^u-1)^{uN+s}) \\ &= \frac{\Gamma(\frac{u+l}{u})}{\Gamma(\frac{1+u}{u})\Gamma(\frac{u-1}{u})} \sum_{v=0}^{uN+s} \binom{uN+s}{v} (-1)^{uN+l-v} \int_0^1 \xi^{(1+l)/u+v-1}(1-\xi)^{(u-1)/u-1} d\xi \\ &= \frac{(-1)^{uN+s}\Gamma(\frac{u+l}{u})}{\Gamma((1+l)/u)\Gamma(\frac{u-1}{u})} \int_0^1 \xi^{\frac{1+l}{u}-1}(1-\xi)^{uN+s+(u-1)/u-1} d\xi \end{aligned}$$

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$$= \frac{(-1)^{uN+s}\Gamma(\frac{u+l}{u})}{\Gamma(uN+s+\frac{u+l}{u})} \frac{\Gamma(uN+s+\frac{u-1}{u})}{\Gamma(\frac{u-1}{u})}$$

$$= \frac{(-1)^{uN+s}(\frac{u-1}{u})_{uN+s}}{(\frac{u+l}{u})_{uN+s}}. \tag{7-2}$$

Substituting the above equality into Equation (7-1), we obtain the assertion. □

7.2. Estimates. Unless stated otherwise, the Landau symbols small o and large O refer when N tends to infinity.

For a finite set S of rational numbers and a rational number a, we define

$$\mathrm{den}\,(S) = \min\{n \in \mathbb{Z} \mid n \geq 1, ns \in \mathbb{Z} \text{ for all } s \in S\} \quad \text{and} \quad \mu(a) = \mathrm{den}\,(a) \prod_{\substack{q:\text{prime} \\ q \mid \mathrm{den}(a)}} q^{1/(q-1)}.$$

We now quote an estimate of the denominator of $((a)_k/(b)_k)_{0\leq k\leq n}$ for $n \in \mathbb{N}$ and $a, b \in \mathbb{Q}$ being nonnegative integers.

Lemma 7.3 [20, Lemma 5.1]. *Let $n \in \mathbb{N}$ and $a, b \in \mathbb{Q}$ be nonnegative integers. Put*

$$D_n = \mathrm{den}\Big(\frac{(a)_0}{(b)_0}, \dots, \frac{(a)_n}{(b)_n}\Big).$$

Then,

$$\limsup_{n\to\infty} \frac{1}{n} \log D_n \leq \log \mu(a) + \frac{\mathrm{den}(b)}{\varphi(\mathrm{den}(b))},$$

where φ is Euler's totient function.

For a rational number a and a nonnegative integer b, we denote $\binom{a}{b} = (-1)^k(-a)_b/b!$.

Lemma 7.4. *Let N, l, h be nonnegative integers with $0 \leq l \leq u-2$ and $0 \leq h \leq u-1$.*

(i) We have

$$P_{uN,h}(z) = (-1)^{uN} \sum_{k=0}^{N(u-1)} \Big[\sum_{s=0}^{k} \binom{uN-1/u}{s+N}\binom{u(s+N)+h}{uN}\binom{1/u}{k-s}\Big](-1)^k z^{uk+h}.$$

(ii) Put $\tilde{\varepsilon}_{l,h} = 1$ if $h < l+1$ and 0 if $l+1 \leq h$. We have

$$Q_{uN,l,h}(z) = (-1)^{uN} \sum_{v=\tilde{\varepsilon}_{l,h}}^{N(u-1)} \Big(\sum_{k=0}^{(u-1)N-v} (-1)^{k+v}$$

$$\times \Big[\sum_{s=0}^{k+v} \binom{uN-1/u}{s+N}\binom{u(s+N)+h}{uN}\binom{1/u}{k+v-s}\Big]\frac{(\frac{1+l}{u})_k}{(\frac{u+l}{u})_k}\Big) z^{uv+h-l-1}.$$

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(*iii*) *Put* $\varepsilon_{l,h} = 1$ *if* $l < h$ *and* $\varepsilon_{l,h} = 0$ *if* $h \leq l$. *We have*

$$\mathfrak{R}_{uN,l,h}(z) = \frac{(\frac{u-1}{u})_{uN}}{(\frac{u+l}{u})_{uN} z^{uN+l-h+1}} \sum_{k=\varepsilon_{l,h}}^{\infty} \binom{u(N+k)+l-h}{uN} \frac{(\frac{1+l}{u})_k}{(\frac{u+l}{u}+uN)_k} \frac{1}{z^{uk}}.$$

PROOF. (i) Put

$$w(z) = (1-z^u)^{-1/u} = \sum_{k=0}^{\infty} \binom{-1/u}{k} (-z^u)^k \in K[[z]].$$

Then $w(z)$ is a solution of $-(z^u-1)\partial_z - z^{u-1} \in K[z, \partial_z]$. Lemma 4.3 yields

$$\frac{1}{(uN)!}\left(\partial_z - \frac{z^{u-1}}{z^u-1}\right)^{uN} (z^u-1)^{uN} = \frac{1}{(uN)!} w(z)^{-1} \partial_z^{uN} w(z)(z^u-1)^{uN}$$

and therefore

$$\begin{aligned} P_{uN,h}(z) &= \frac{1}{(uN)!} w(z)^{-1} \partial_z^{uN} w(z)(z^u-1)^{uN} \cdot z^h \\ &= \frac{(-1)^{uN}}{(uN)!} w(z)^{-1} \partial_z^{uN} \cdot \sum_{k=0}^{\infty} \binom{uN-1/u}{k} (-1)^k z^{uk+h} \\ &= (-1)^{uN} \sum_{k=0}^{\infty} \binom{1/u}{k} (-1)^k z^{uk} \cdot \sum_{k=0}^{\infty} \binom{uN-1/u}{k+N} \binom{u(k+N)+h}{uN} (-1)^k z^{uk+h} \\ &= (-1)^{uN} \sum_{k=0}^{\infty} \left[\sum_{s=0}^{k} \binom{uN-1/u}{s+N} \binom{u(s+N)+h}{uN} \binom{1/u}{k-s} \right] (-1)^k z^{uk+h}. \end{aligned}$$

Since $\deg P_{uN,h} = u(u-1)N + h$, using the above equality, we obtain the assertion.

(ii) Put $P_{uN,h}(z) = \sum_{k=0}^{u(u-1)N+h} p_k z^k$. Notice that, by item (i),

$$p_k = \begin{cases} (-1)^{uN+k'} \displaystyle\sum_{s=0}^{k'} \binom{uN-1/u}{s+N} \binom{u(s+N)+h}{uN} \binom{1/u}{k'-s} & \begin{array}{l}\text{if there exists } k' \geq 0 \\ \text{such that } k = uk'+h,\end{array} \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} \frac{P_{uN,h}(z) - P_{uN,h}(t)}{z-t} &= \sum_{k'=1}^{u(u-1)N+h} p_{k'} \sum_{v'=0}^{k'-1} z^{v'} t^{k'-v'-1} = \sum_{k'=0}^{u(u-1)N+h-1} p_{k'+1} \sum_{v'=0}^{k'} z^{v'} t^{k'-v'} \\ &= \sum_{v'=0}^{u(u-1)N+h-1} \left[\sum_{k'=v'}^{u(u-1)N+h-1} p_{k'+1} t^{k'-v'} \right] z^{v'} \\ &= \sum_{v'=0}^{u(u-1)N+h-1} \left[\sum_{k'=0}^{u(u-1)N+h-v'-1} p_{k'+v'+1} t^{k'} \right] z^{v'}. \end{aligned}$$

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Since $\varphi_l(t^{k'}) = 0$ if $k' \not\equiv l \bmod u$, putting $k' = uk + l$, we obtain

$$
\begin{aligned}
Q_{uN,l,h}(z) &= \varphi_l\Big(\frac{P_{uN,h}(z) - P_{uN,h}(t)}{z - t}\Big) \\
&= \sum_{v'=0}^{u(u-1)N+h-1}\Big[\sum_{k'=0}^{u(u-1)N+h-v'-1} p_{k'+v'+1}\varphi_l(t^{k'})\Big]z^{v'} \\
&= \sum_{v'=0}^{u(u-1)N+h-1}\Big[\sum_{k=0}^{(u-1)N+\lfloor (h-v'-l-1)/u\rfloor} p_{uk+l+v'+1}\frac{(\frac{1+l}{u})_k}{(\frac{u+l}{u})_k}\Big]z^{v'}.
\end{aligned}
$$

Since we have $p_{uk+l+v'+1} = 0$ for $0 \le v' \le u(u-1)N + h - 1$ with $v' \notin u\mathbb{Z} + h - l - 1$, putting $v' = uv + h - l - 1$, we conclude

$$
Q_{uN,l,h}(z) = \sum_{v=\tilde{\varepsilon}_{l,h}}^{(u-1)N}\Big[\sum_{k=0}^{(u-1)N-v} p_{u(k+v)+h}\frac{(\frac{1+l}{u})_k}{(\frac{u+l}{u})_k}\Big]z^{uv+h-l-1}.
$$

This completes the proof of item (ii).

(iii) Lemma 5.1 yields

$$
\mathfrak{R}_{uN,l,h}(z) = \sum_{k=uN}^{\infty}\frac{\varphi_l(t^k P_{uN,h}(t))}{z^{k+1}}. \tag{7-3}
$$

We now compute $\varphi_l(t^k P_{uN,h}(t))$ for $k \ge uN$. Put $\mathcal{E} = \partial_t - t^{u-1}/(t^u - 1)$. Using Proposition 3.2(i) for $k \ge uN$, there exists a set of integers $\{c_{uN,k,v} \mid v = 0, 1, \dots, uN\}$ with

$$
\begin{aligned}
&c_{uN,k,uN} = (-1)^{uN}k(k-1)\cdots(k-uN+1) \ \text{ and} \\
&t^k\mathcal{E}^{uN}(t^u-1)^{uN} = \sum_{v=0}^{uN} c_{uN,k,v}\mathcal{E}^{uN-v}t^{k-v}(t^u-1)^{uN} \ \text{ in } \mathbb{Q}(t)[\partial_t].
\end{aligned}
$$

Since $\mathcal{E}(t^u - 1) \subseteq \ker\varphi_l$, using the above relation,

$$
\begin{aligned}
\varphi_l(t^k P_{uN,h}(t)) &= \varphi_l\Big(\frac{t^k}{(uN)!}\mathcal{E}^{uN}(t^u-1)^{uN}\cdot t^h\Big) = \varphi_l\Big(\sum_{v=0}^{uN}\frac{c_{uN,k,v}}{(uN)!}\mathcal{E}^{uN-v}t^{k-v}(t^u-1)^{uN}\cdot t^h\Big) \\
&= \varphi_l\Big(\frac{c_{uN,k,uN}}{(uN)!}t^{k-uN}(t^u-1)^{uN}\cdot t^h\Big) = (-1)^{uN}\binom{k}{uN}\varphi_l(t^{k-uN+h}(t^u-1)^{uN}).
\end{aligned} \tag{7-4}
$$

Note we have $\varphi_l(t^{k-uN+h}(t^u - 1)^{uN}) = 0$ if $k - uN + h \not\equiv l \bmod u$. Let $\tilde{k} \ge 0$ and put $k = u(\tilde{k} + N + \varepsilon_{l,h}) + l - h$. A similar computation which we performed in Equation (7-1) implies

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$$\varphi_l(t^{k-uN+h}(t^u-1)^{uN}) = \varphi_l(t^{u(\tilde{k}+\varepsilon_{l,h})+l}(t^u-1)^{uN})$$
$$= \frac{(-1)^{uN}(\frac{u-1}{u})_{uN}(\frac{1+l}{u})_{\tilde{k}+\varepsilon_{l,h}}}{(\frac{u+l}{u})_{uN+\tilde{k}+\varepsilon_{l,h}}} = \frac{(-1)^{uN}(\frac{u-1}{u})_{uN}(\frac{1+l}{u})_{\tilde{k}+\varepsilon_{l,h}}}{(\frac{u+l}{u})_{uN}(\frac{u+l}{u}+uN)_{\tilde{k}+\varepsilon_{l,h}}}.$$

Substituting the above equality into Equations (7-4) and (7-3), we obtain the desired equality. □

In the following, for a rational number a and a nonnegative integer n, we put

$$\mu_n(a) = \mathrm{den}(a)^n \prod_{\substack{q:\text{prime} \\ q \mid \mathrm{den}(a)}} q^{\lfloor n/(q-1) \rfloor}.$$

Notice that $\mu_n(a) = \mu_n(a+k)$ for $k \in \mathbb{Z}$ and

$$\mu_{n_2}(a) \text{ is divisible by } \mu_{n_1}(a) \quad \text{and} \quad \mu_{n_1+n_2}(a) \text{ is divisible by } \mu_{n_1}(a)\mu_{n_2}(a) \tag{7-5}$$

for $n, n_1, n_2 \in \mathbb{N}$ with $n_1 \le n_2$.

LEMMA 7.5. *Let K be an algebraic number field, v a place of K and $\alpha \in K \setminus \{0\}$.*

(i) We have

$$\max_{0 \le h \le u-1} \log |P_{uN,h}(\alpha)|_v \le o(N) + u(u-1)\mathrm{h}_v(\alpha)N + \begin{cases} u(u+1)\mathrm{h}_v(2)N & \text{if } v \mid \infty \\ \log |\mu_{uN}(1/u)|_v^{-1} & \text{if } v \nmid \infty. \end{cases}$$

(The function $o(N)$ is equal to 0 for almost all places v. This also holds in statement (ii).)

(ii) For $0 \le l \le u-2$, put

$$D_N = \mathrm{den}\left(\frac{(\frac{1+l}{u})_k}{(\frac{u+l}{u})_k}\right)_{\substack{0 \le l \le u-2 \\ 0 \le k \le (u-1)N}}.$$

Then,

$$\max_{\substack{0 \le l \le u-2 \\ 0 \le h \le u-1}} \log |Q_{uN,l,h}(\alpha)|_v \le o(N) + u(u-1)\mathrm{h}_v(\alpha)N$$
$$+ \begin{cases} u(u+1)\mathrm{h}_v(2)N & \text{if } v \mid \infty \\ \log |\mu_{uN}(1/u)|_v^{-1} + \log |D_N|_v^{-1} & \text{if } v \nmid \infty. \end{cases}$$

PROOF. (i) Let v be an archimedean place. Since

$$\binom{uN-1/u}{s+N} \le 2^{uN}, \quad \binom{u(s+N)+h}{uN} \le 2^{u(s+N)+h} \quad \text{and} \quad \left|\binom{1/u}{k-s}\right| \le 1,$$

for $0 \le k \le N(u-1)$ and $0 \le s \le k$, we obtain

$$\left|\sum_{s=0}^{k} \binom{uN-1/u}{s+N}\binom{u(s+N)+h}{uN}\binom{1/u}{k-s}\right| \le 2^{2uN+h} \sum_{s=0}^{k} 2^{us} \le 2^{2uN+h+u(k+1)}. \tag{7-6}$$

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Thus, by Lemma 7.4(i),

$$|P_{uN,h}(\alpha)|_v \le |2^{2uN+h}|_v \cdot \left| \sum_{k=0}^{N(u-1)} 2^{u(k+1)} \alpha^{uk+h} \right|_v \le e^{o(N)} |2|_v^{u(u+1)N} \max\{1, |\alpha|_v\}^{u(u-1)N}.$$

This completes the proof of the archimedean case.

Second, we consider the case of v is a nonarchimedean place. Note that

$$\binom{uN - 1/u}{s+N} = \frac{(-1)^{s+N}(1/u - uN)_{s+N}}{(s+N)!} \quad \text{and} \quad \binom{1/u}{k-s} = \frac{(-1)^{k-s}(-1/u)_{k-s}}{(k-s)!}$$

for $0 \le k \le N(u-1),\ 0 \le s \le k$. Combining

$$\left| \frac{(a)_k}{k!} \right|_v \le |\mu_n(a)|_v^{-1} \quad \text{for } a \in \mathbb{Q} \quad \text{and} \quad k, n \in \mathbb{N} \text{ with } k \le n,$$

(see [9, Lemma 2.2]) and Equation (7-5) yields

$$\left| \binom{uN - 1/u}{s+N} \binom{u(s+N)+h}{uN} \binom{1/u}{k-s} \right|_v \le |\mu_{k+N}(1/u)|_v^{-1} \quad \text{for } 0 \le k \le (u-1)N.$$

Therefore, the strong triangle inequality yields

$$\max_{0 \le k \le N(u-1)} \left| \sum_{s=0}^{k} \binom{uN - 1/u}{s+N} \binom{u(s+N)+h}{uN} \binom{1/u}{k-s} \right|_v \le |\mu_{uN}(1/u)|_v^{-1}. \tag{7-7}$$

Using Lemma 7.4(i) again, we conclude the desired inequality.

(ii) Let v be an archimedean place. We use the same notation as in the proof of Lemma 7.4(ii). Using Equation (7-6) again, we obtain

$$\begin{aligned} \left| \sum_{k=0}^{(u-1)N-v} p_{u(k+v)+h} \frac{(\frac{1+l}{u})_k}{(\frac{u+l}{u})_k} \right|_v &\le |2|_v^{2uN+h+u(v+1)} \sum_{k=0}^{N(u-1)-v} |2|_v^{uk} \\ &\le |2|_v^{2uN+u(u-1)N+u+h}. \end{aligned}$$

Lemma 7.4(ii) implies that

$$\begin{aligned} |Q_{uN,l,h}(\alpha)|_v &\le \sum_{v=0}^{(u-1)N} \left\| \left[\sum_{k=0}^{(u-1)N-v} p_{u(k+v)+h} \frac{(\frac{1+l}{u})_k}{(\frac{u+l}{u})_k} \right] \right\|_v |\alpha|_v^{uv+h-l-1} \\ &\le e^{o(N)} |2|_v^{u(u+1)N} \max\{1, |\alpha|_v\}^{u(u-1)N}. \end{aligned}$$

Let v be a nonarchimedean place. Then by the definition of D_N,

$$\max_{\substack{0 \le l \le u-2 \\ 0 \le k \le (u-1)N}} \left(\left| \frac{(\frac{1+l}{u})_k}{(\frac{u+l}{u})_k} \right|_v \right) \le |D_N|_v^{-1}$$

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for all $N \in \mathbb{N}$. Using the above inequality and Equation (7-7) for Lemma 7.4(ii), we obtain the desire inequality by the strong triangle inequality. This completes the proof of Lemma 7.5. □

LEMMA 7.6. *Let K be an algebraic number field, v_0 a place of K, $\alpha \in K$. Let N, l, h be nonnegative integers with $0 \le l \le u-2$ and $0 \le h \le u-1$.*

(i) Assume v_0 is an archimedean place and $|\alpha|_{v_0} > 2$. We have

$$\max_{\substack{0\le l\le u-2\\ 0\le h\le u-1}} \log|\mathfrak{R}_{uN,l,h}(\alpha)|_{v_0} \le -u(\mathrm{h}_{v_0}(\alpha)-\mathrm{h}_{v_0}(2))N + o(N).$$

(ii) Assume v_0 is a nonarchimedean place and $|\alpha|_{v_0} > 1$. Let p_{v_0} be the rational prime under v_0. Put $\varepsilon_{v_0}(u) = 1$ if u is coprime with p_{v_0} and $\varepsilon_{v_0}(u) = 0$ if u is divisible by p_{v_0}. We have

$$\max_{\substack{0\le l\le u-2\\ 0\le h\le u-1}} \log|\mathfrak{R}_{uN,l,h}(\alpha)|_{v_0} \le -u\Big(\mathrm{h}_{v_0}(\alpha) - \frac{\varepsilon_{v_0}(u)\log|p_{v_0}|_{v_0}}{p_{v_0}-1}\Big)N + o(N).$$

PROOF. (i) For a nonnegative integer k, we have $\binom{u(N+k)+l-h}{uN} \le 2^{u(N+k)+l-h}$. Thus,

$$\begin{aligned}\Big|\sum_{k=\varepsilon_{l,h}}^{\infty}\binom{u(N+k)+l-h}{uN}\frac{(\frac{1+l}{u})_k}{(\frac{u+l}{u}+uN)_k}\frac{1}{\alpha^{uk}}\Big|_{v_0} &\le |2^{uN+l-h}|_{v_0}\sum_{k=\varepsilon_{l,h}}^{\infty}\Big|\frac{(\frac{1+l}{u})_k}{(\frac{u+l}{u}+uN)_k}\Big|_{v_0}\Big|\frac{2}{\alpha}\Big|_{v_0}^{uk}\\ &\le |2^{uN+l-h}|_{v_0}\sum_{k=0}^{\infty}\Big|\frac{2}{\alpha}\Big|_{v_0}^{uk} = |2^{uN}|_{v_0}e^{o(N)}.\end{aligned}$$

Using the above inequality in Lemma 7.4(ii), we obtain the assertion.

(ii) By [14, Proposition 4, Lemma 4] (*loc. cit.* Section (6.1), (6.2)),

$$\max_{0\le l\le u-2}\left(\Big|\frac{(\frac{u-1}{u})_{uN}}{(\frac{u+l}{u})_{uN}}\Big|_{v_0}\right) \le |p_{v_0}|_{v_0}^{\varepsilon_{v_0}(u)v_{p_{v_0}}((uN)!)+o(N)},$$

$$\Big|\sum_{k=\varepsilon_{l,h}}^{\infty}\binom{u(N+k)+l-h}{uN}\frac{(\frac{1+l}{u})_k}{(\frac{u+l}{u}+uN)_k}\frac{1}{\alpha^{uk}}\Big|_{v} = e^{o(1)}.$$

Combining $v_p((uN)!) = uN/(p-1)+o(N)$ and the above inequality in Lemma 7.4(ii), we obtain the assertion. This completes the proof of Lemma 7.6. □

7.3. Proof of Theorem 7.1. We use the same notation as in Theorem 7.1. Let $\alpha \in K$ with $|\alpha|_{v_0} > 1$. For a nonnegative integer N, we define a matrix

$$\mathrm{M}_N = \begin{pmatrix} P_{uN,0}(\alpha) & \cdots & P_{uN,u-1}(\alpha)\\ Q_{uN,0,0}(\alpha) & \cdots & Q_{uN,0,u-1}(\alpha)\\ \vdots & \ddots & \vdots\\ Q_{uN,u-2,0}(\alpha) & \cdots & Q_{uN,u-2,u-1}(\alpha)\end{pmatrix} \in \mathrm{M}_u(K).$$

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By Lemma 7.2, the matrices M_N are invertible for every N. We define functions

$$F_v : \mathbb{N} \longrightarrow \mathbb{R}_{\geq 0};\ N \mapsto u(u-1)\mathrm{h}_v(\alpha)N$$
$$+\, o(N) + \begin{cases} u(u+1)\mathrm{h}_v(2)N & \text{if } v \mid \infty \\ \log |\mu_{uN}(1/u)|_v^{-1} + \log |D_N|_v^{-1} & \text{if } v \nmid \infty \end{cases}$$

for $v \in \mathfrak{M}_K$. By Lemma 7.3,

$$\lim_{N\to\infty} \frac{1}{N} \log D_N \leq (u-1)\Big(\log \nu(u) + \frac{u}{\varphi(u)}\Big),$$

where D_N is the integer defined in Lemma 7.5,

$$\lim_{N\to\infty} \frac{1}{N}\Big(\sum_{v\neq v_0} F_v(N)\Big) \leq u\mathbb{B}_{v_0}(\alpha),$$

and, by Lemma 7.5,

$$\max_{0\leq h\leq u-1} \log \max\{|P_{uN,h}(\alpha)|_{v_0}\} \leq uU_{v_0}(\alpha)N + o(N),$$
$$\max_{\substack{0\leq l\leq u-2\\ 0\leq h\leq u-1}} \log \max\{|P_{uN,h}(\alpha)|_v, |Q_{uN,l,h}(\alpha)|_v\} \leq F_v(N) \quad \text{for } v \in \mathfrak{M}_K.$$

By Lemma 7.6,

$$\max_{\substack{0\leq l\leq u-2\\ 0\leq h\leq u-1}} \log |\mathfrak{R}_{uN,l,h}(\alpha)|_{v_0} \leq -u\mathbb{A}_{v_0}(\alpha)N + o(N).$$

Using a quantitative linear independence criterion in [11, Proposition 5.6] for

$$\theta_l = \frac{1}{\alpha^{l+1}}\, {}_2F_1\Big(\frac{1+l}{u}, 1, \frac{u+l}{u} \Big| \frac{1}{\alpha^u}\Big) \quad \text{for } 0 \leq l \leq u-2,$$

and the invertible matrices $(\mathrm{M}_N)_N$, and applying the above estimates, we obtain Theorem 7.1. □

A. Appendix

Denote the algebraic closure of $\mathbb{Q}$ by $\overline{\mathbb{Q}}$. Let $a(z), b(z) \in \overline{\mathbb{Q}}[z]$ with $w := \max\{\deg a - 2, \deg b - 1\} \geq 0$ and $a(z) \neq 0$. Put $D = -a(z)\partial_z + b(z)$. The Laurent series $f_0(z), \dots, f_w(z)$ obtained in Lemma 4.1 for D become G-functions in the sense of Siegel when D is a G-operator (see [2, Section IV]). Here we refer below to a result due to Fischler and Rivoal [15] in which they gave a condition so that D becomes a G-operator.

LEMMA A.1 (cf. [15, Proposition 3(ii)]). *Let $m \geq 2$ be a positive integer, $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_{m-1}, \gamma \in \overline{\mathbb{Q}}$ with $\alpha_1, \dots, \alpha_m$ being pairwise distinct. In the case of $0 \in \{\alpha_1, \dots, \alpha_m\}$, we put $\alpha_m = 0$. Define $a(z) = \prod_{i=1}^m (z-\alpha_i)$, $b(z) = \gamma \prod_{j=1}^{m-1}(z-\beta_j)$ and $D = -a(z)\partial_z + b(z) \in \overline{\mathbb{Q}}[z, \partial_z]$. Then the following are equivalent.*

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(i) *D is a G-operator.*

(ii) *We have*

$$\gamma\frac{\prod_{j=1}^{m-1}(\alpha_i-\beta_j)}{\prod_{i'\neq i}(\alpha_i-\alpha_{i'})}\in\mathbb{Q}\quad \textit{for all } 1\le i\le m \textit{ if } 0\notin\{\alpha_1,\dots,\alpha_m\},$$
$$\gamma\frac{\prod_{j=1}^{m-1}(\alpha_i-\beta_j)}{\prod_{i'\neq i}(\alpha_i-\alpha_{i'})}\in\mathbb{Q}\quad \textit{for all } 1\le i\le m \textit{ and } \gamma\prod_{j=1}^{m-1}\frac{\beta_j}{\alpha_j}\in\mathbb{Q} \textit{ otherwise.}$$

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