Zero entropy actions of amenable groups are not dominant

ADAM LOTT📕

Department of Mathematics, University of California, Los Angeles,
Los Angeles, CA 90095, USA
(e-mail: adamlott99@math.ucla.edu)

Abstract. A probability measure-preserving action of a discrete amenable group G is said to be *dominant* if it is isomorphic to a generic extension of itself. Recently, it was shown that for $G = \mathbb{Z}$, an action is dominant if and only if it has positive entropy and that for any G, positive entropy implies dominance. In this paper, we show that the converse also holds for any G, that is, that zero entropy implies non-dominance.

1. *Introduction*

1.1. *Definitions and results.* Let $(X, \mathcal{B}, \mu)$ be a standard Lebesgue space and let T be a free, ergodic, μ-preserving action of a discrete amenable group G on X. It is natural to ask what properties of T are preserved by a generic extension $(\overline{X}, \overline{\mu}, \overline{T})$ (a precise definition of 'generic extension' is discussed in §3). For example, it was shown in [**GTW21**] that a generic $\overline{T}$ has the same entropy as T and that if T is a non-trivial Bernoulli shift, then a generic $\overline{T}$ is also Bernoulli. A system (X, μ, T) is said to be *dominant* if it is isomorphic to a generic extension $(\overline{X}, \overline{\mu}, \overline{T})$. Thus, for example, the aforementioned results from [**GTW21**] together with Ornstein's famous isomorphism theorem [**Orn70**] imply that all non-trivial Bernoulli shifts are dominant. More generally, it has been shown in [**AGTW21**] that:

(1) if $G = \mathbb{Z}$, then (X, μ, T) is dominant if and only if it has positive Kolmogorov–Sinai entropy; and

(2) for any G, if (X, μ, T) has positive entropy, then it is dominant.

In this paper, we complete the picture by proving the following result.

THEOREM 1.1. *Let G be any discrete amenable group and let (X, μ, T) be any free ergodic action with zero entropy. Then (X, μ, T) is not dominant.*

The proof of result (2) is based on the theory of 'slow entropy' developed by Katok and Thouvenot in [**KT97**] (see also [**Fer97**]), and our proof of Theorem 1.1 uses the same ideas.

1.2. *Outline.* In §2, we introduce the relevant ideas from slow entropy. In §3, we describe a precise definition of 'generic extension' and begin the proof of Theorem 1.1. Finally, in §4, we prove the proposition that is the technical heart of Theorem 1.1.

2. *Slow entropy*

Fix a Følner sequence (F_n) for G. For $g \in G$, write $T^g x$ for the action of g on the point $x \in X$, and for a subset $F \subseteq G$, write $T^F x = \{T^f x : f \in F\}$. If $Q = \{Q_1, \ldots, Q_k\}$ is a partition of X, then for $x \in X$ denote by $Q(x)$ the index of the cell of Q containing x. Sometimes we use the same notation to mean the cell itself; which meaning is intended will be clear from the context. Given a finite subset $F \subseteq G$, the (Q, F)-*name* of x for the action T is the tuple $Q_{T,F}(x) := (Q(T^f x))_{f \in F} \in \{1, 2, \ldots, k\}^F$. Similarly, we also define the partition $Q_{T,F} := \bigvee_{f \in F} T^{f^{-1}} Q$, and in some contexts we use the same notation $Q_{T,F}(x)$ to refer to the cell of $Q_{T,F}$ containing x.

For a finite subset $F \subseteq G$ and any finite alphabet Λ, the symbolic space Λ^F is equipped with the normalized Hamming distance $d_F(w, w') = (1/|F|) \sum_{f \in F} 1_{w(f) \neq w'(f)}$.

Definition 2.1. Given a partition $Q = \{Q_1, \ldots, Q_k\}$, a finite set $F \subseteq G$, and $\epsilon > 0$, define

$$B_{\mathrm{Ham}}(Q, T, F, x, \epsilon) := \{y \in X : d_F(Q_{T,F}(y), Q_{T,F}(x)) < \epsilon\}.$$

We refer to this set as the '(Q, T, F)-Hamming ball of radius ϵ centered at x'. Formally, it is the preimage under the map $Q_{T,F}$ of the ball of radius ϵ centered at $Q_{T,F}(x)$ in the metric space $([k]^F, d_F)$.

Definition 2.2. Given $\epsilon > 0$, the *Hamming ϵ-covering number* of μ is defined to be the minimum number of (Q, T, F)-Hamming balls of radius ϵ required to cover a subset of X of μ-measure at least $1 - \epsilon$, and is denoted by $\mathrm{cov}(Q, T, F, \mu, \epsilon)$.

LEMMA 2.3. *Let $\varphi : (X, T, \mu) \to (Y, S, \nu)$ be an isomorphism. Also let Q be a finite partition of X, let F be a finite subset of G, and let $\epsilon > 0$. Then $\mathrm{cov}(Q, T, F, \mu, \epsilon) = \mathrm{cov}(\varphi Q, S, F, \nu, \epsilon)$.*

Proof. It is immediate from the definition of isomorphism that for μ-almost every $x, x' \in X$,

$$d_F(Q_{T,F}(x), Q_{T,F}(x')) = d_F((\varphi Q)_{S,F}(\varphi x), (\varphi Q)_{S,F}(\varphi x')).$$

Therefore, it follows that $\varphi(B_{\mathrm{Ham}}(Q, T, F, x, \epsilon)) = B_{\mathrm{Ham}}(\varphi Q, S, F, \varphi x, \epsilon)$ for μ-almost every x. Thus, any collection of (Q, T, F)-Hamming balls in X covering a set of μ-measure $1 - \epsilon$ is directly mapped by φ to a collection of $(\varphi Q, S, F)$-Hamming balls in Y

covering a set of ν-measure $1 - \epsilon$. Therefore, $\text{cov}(Q, T, F, \mu, \epsilon) \geq \text{cov}(\varphi Q, S, F, \nu, \epsilon)$. The reverse inequality holds by doing the same argument with φ^{-1} in place of φ. □

The goal of the rest of this section is to show that for a given action (X, T, μ), the sequence of covering numbers $\text{cov}(Q, T, F_n, \mu, \epsilon)$ grows at a rate that is bounded uniformly for any choice of partition Q. A key ingredient is an analog of the classical Shannon–McMillan theorem for actions of amenable groups [**MO85**, Theorem 4.4.2].

THEOREM 2.4. *Let G be a countable amenable group and let (F_n) be any Følner sequence for G. Let (X, T, μ) be an ergodic action of G and let Q be any finite partition of X. Then*

$$\frac{-1}{|F_n|} \log \mu(Q_{T, F_n}(x)) \xrightarrow{L^1(\mu)} h(\mu, T, Q) \quad \text{as } n \to \infty,$$

where h denotes the entropy. In particular, for any fixed $\gamma > 0$,

$$\mu\{x : \exp((-h - \gamma)|F_n|) < \mu(Q_{T, F_n}(x)) < \exp((-h + \gamma)|F_n|)\} \to 1 \quad \text{as } n \to \infty.$$

LEMMA 2.5. *For any partition P, any Følner sequence (F_n) and any $\epsilon > 0$, let $\ell(T, P, \mu, n, \epsilon)$ be the minimum number of P_{T, F_n}-cells required to cover a subset of X of measure more than $1 - \epsilon$. Then*

$$\limsup_{n \to \infty} \frac{1}{|F_n|} \log \ell(T, P, \mu, n, \epsilon) \leq h(\mu, T, P).$$

Proof. Let $h = h(\mu, T, P)$. Let $\gamma > 0$. By Theorem 2.4, for n sufficiently large depending on γ, we have

$$\mu\{x \in X : \mu(P_{T, F_n}(x)) \geq \exp((-h - \gamma)|F_n|)\} > 1 - \epsilon.$$

Let X' denote the set in the above equation. Let $\mathcal{G}$ be the family of cells of the partition P_{T, F_n} that meet X'. Then clearly $\mu(\bigcup \mathcal{G}) > 1 - \epsilon$ and $|\mathcal{G}| < \exp((h + \gamma)|F_n|)$. Therefore,

$$\limsup_{n \to \infty} \frac{1}{|F_n|} \log \ell(T, P, \mu, n, \epsilon) \leq h + \gamma,$$

and this holds for arbitrary γ, so we are done. □

At this point, fix for all time $\epsilon = 1/100$. We can also now omit ϵ from all of the notation defined previously, because it will never change. In addition, assume from now on that the system (X, T, μ) has zero entropy.

LEMMA 2.6. *If (F_n) is a Følner sequence for G and A is any finite subset of G, then (AF_n) is also a Følner sequence for G.*

Proof. First, because A is finite and (F_n) is Følner we have

$$\lim_{n \to \infty} \frac{|AF_n|}{|F_n|} = 1.$$

Now fix any $g \in G$ and observe that

$$\frac{|gAF_n \triangle AF_n|}{|AF_n|} \leq \frac{|gAF_n \triangle F_n| + |F_n \triangle AF_n|}{|F_n|} \cdot \frac{|F_n|}{|AF_n|} \to 0 \quad \text{as } n \to \infty,$$

which shows that (AF_n) is a Følner sequence. □

LEMMA 2.7. *Let $b(m, n) \geq 0$ be real numbers satisfying:*
- $\lim_{n \to \infty} b(m, n) = 0$ *for each fixed m; and*
- $b(m + 1, n) \geq b(m, n)$ *for all m, n.*

Then there exists a sequence (a_n) such that $a_n \to 0$ and for each fixed m, $b(m, n) \leq a_n$ for n sufficiently large (depending on m).

Proof. For each m, let N_m be such that $b(m, n) < 1/m$ for all $n > N_m$. Without loss of generality, we may assume that $N_m < N_{m+1}$. Then we define the sequence (a_n) by $a_n = b(1, n)$ for $n \leq N_2$ and $a_n = b(m, n)$ for $N_m < n \leq N_{m+1}$. We have $a_n \to 0$ because $a_n < 1/m$ for all $n > N_m$. Finally, the fact that $b(m + 1, n) \geq b(m, n)$ implies that for every fixed m, $a_n \geq b(m, n)$ as soon as $n > N_m$. $\square$

PROPOSITION 2.8. *There is a sequence (a_n) such that:*
(1) $\limsup_{n \to \infty} (1/|F_n|) \log a_n = 0$; *and*
(2) *for any finite partition Q, there exists an N such that $\mathrm{cov}(Q, T, F_n, \mu) \leq a_n$ for all $n > N$.*

Proof. Because T has zero entropy, there exists a finite generating partition for T (see, for example, [**Sew19**, Corollary 1.2] or [**Ros88**, Theorem 2']). Fix such a partition P and let $Q = \{Q_1, \ldots, Q_r\}$ be any given partition. Because P is generating, there is an integer m and another partition $Q' = \{Q'_1, \ldots, Q'_r\}$ such that Q' is refined by P_{T, F_m} and

$$\mu\{x : Q(x) \neq Q'(x)\} < \frac{\epsilon}{4}.$$

By the mean ergodic theorem, we can write

$$d_{F_n}(Q_{T, F_n}(x), Q'_{T, F_n}(x))$$
$$= \frac{1}{|F_n|} \sum_{f \in F_n} 1_{\{y : Q(y) \neq Q'(y)\}}(T^f x) \xrightarrow{L^1(\mu)} \mu\{y : Q(y) \neq Q'(y)\} < \frac{\epsilon}{4},$$

so, in particular, for n sufficiently large, we have

$$\mu\{x : d_{F_n}(Q_{T, F_n}(x), Q'_{T, F_n}(x)) < \epsilon/2\} > 1 - \frac{\epsilon}{4}.$$

Let Y denote the set $\{x : d_{F_n}(Q_{T, F_n}(x), Q'_{T, F_n}(x)) < \epsilon/2\}$.

Recall that Q' is refined by P_{T, F_m}, so Q'_{T, F_n} is refined by $(P_{T, F_m})_{T, F_n} = P_{T, F_m F_n}$. Let $\ell = \ell(m, n)$ be the minimum number of $P_{T, F_m F_n}$ cells required to cover a set of μ-measure at least $1 - \epsilon/4$, and let $C_1, \ldots, C_\ell$ be such a collection of cells satisfying $\mu(\bigcup_i C_i) \geq 1 - \epsilon/4$. If any of the C_i do not meet the set Y, then drop them from the list. Because $\mu(Y) > 1 - \epsilon/4$ we can still assume after dropping that $\mu(\bigcup_i C_i) > 1 - \epsilon/2$. Choose a set of representatives $y_1, \ldots y_\ell$ with each $y_i \in C_i \cap Y$.

Now we claim that $Y \cap \bigcup_i C_i \subseteq \bigcup_{i=1}^{\ell} B_{\mathrm{Ham}}(Q, T, F_n, y_i, \epsilon)$. To see this, let $x \in Y \cap \bigcup_i C_i$. Then there is one index j such that x and y_j are in the same cell of $P_{T, F_m F_n}$. We can then estimate

$$d_{F_n}(Q_{T,F_n}(x), Q_{T,F_n}(y_j)) \le d_{F_n}(Q_{T,F_n}(x), Q'_{T,F_n}(x)) + d_{F_n}(Q'_{T,F_n}(x), Q'_{T,F_n}(y_j))$$
$$+ d_{F_n}(Q'_{T,F_n}(y_j), Q_{T,F_n}(y_j))$$
$$< \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} = \epsilon.$$

The bounds for the first and third terms come from the fact that $x, y_j \in Y$. The second term is 0 because Q'_{F_n} is refined by $P_{T,F_m F_n}$ and y_j was chosen so that x and y_j are in the same $P_{T,F_m F_n}$-cell. Therefore, $\mathrm{cov}(Q, T, F_n, \mu) \le \ell(m, n)$. Thus, the proof is complete once we find a fixed sequence (a_n) that is subexponential in $|F_n|$ and eventually dominates $\ell(m, n)$ for each fixed m.

Because T has zero entropy, Lemmas 2.5 and 2.6 imply that

$$\limsup_{n \to \infty} \frac{1}{|F_n|} \log \ell(m, n) = \limsup_{n \to \infty} \frac{|F_m F_n|}{|F_n|} \cdot \frac{1}{|F_m F_n|} \log \ell(m, n) = 0 \quad \text{for each fixed } m.$$

Note also that because $P_{T,F_{m+1} F_n}$ refines $P_{T,F_m F_n}$, we have $\ell(m + 1, n) \ge \ell(m, n)$ for all m, n. Therefore, we can apply Lemma 2.7 to the numbers $b(m, n) = |F_n|^{-1} \log \ell(m, n)$ to produce a sequence (a'_n) satisfying $a'_n \to 0$ and $a'_n \ge b(m, n)$ eventually for each fixed m. Then $a_n := \exp(|F_n| a'_n)$ is the desired sequence. $\qquad\square$

3. *Cocycles and extensions*

Let I be the unit interval $[0, 1]$ and let m be Lebesgue measure on I. Denote by $\mathrm{Aut}(I, m)$ the group of invertible m-preserving transformations of I. A *cocycle* on X is a family of measurable maps $\alpha_g : X \to \mathrm{Aut}(I, m)$ indexed by $g \in G$ that satisfies the *cocycle condition*: for every $g, h \in G$ and μ-almost every x, $\alpha_{hg}(x) = \alpha_h(T^g x) \circ \alpha_g(x)$. A cocycle can equivalently be thought of as a measurable map $\alpha : R \to \mathrm{Aut}(I, m)$, where $R \subseteq X \times X$ is the orbit equivalence relation induced by T (that is, $(x, y) \in R$ if and only if $y = T^g x$ for some $g \in G$). With this perspective, the cocycle condition takes the form $\alpha(x, z) = \alpha(y, z) \circ \alpha(x, y)$. A cocycle α induces the skew product action T_α of G on the larger space $X \times I$ defined by

$$T_\alpha^g(x, t) := (T^g x, \alpha_g(x)(t)).$$

This action preserves the measure $\mu \times m$ and is an extension of the original action (X, T, μ).

By a classical theorem of Rokhlin (see, for example, [Gla03, Theorem 3.18]), any infinite-to-one ergodic extension of (X, μ, T) is isomorphic to T_α for some cocycle α. Therefore, by topologizing the space of all cocycles on X we can capture the notion of a 'generic' extension: a property is said to hold for a generic extension if it holds for a dense G_δ set of cocycles. Denote the space of all cocycles on X by $\mathrm{Co}(X)$. Topologizing $\mathrm{Co}(X)$ is done in a few stages.

(1) Let $\mathcal{B}(I)$ be the Borel sets in I and let (E_n) be a sequence in $\mathcal{B}(I)$ that is dense in the $m(\cdot \triangle \cdot)$ metric. For example, (E_n) could be an enumeration of the family of all finite unions of intervals with rational endpoints.

(2)   The group $\mathrm{Aut}(I, m)$ is completely metrizable via the metric

$$d_A(\phi, \psi) = \frac{1}{2} \sum_{n \geq 1} 2^{-n} [m(\phi E_n \triangle \psi E_n) + m(\phi^{-1} E_n \triangle \psi^{-1} E_n)].$$

Note that with this metric, $\mathrm{Aut}(I, m)$ has diameter at most 1. See, for example, [**Kec10**, §1.1]

(3)   If α_0, β_0 are maps $X \to \mathrm{Aut}(I, m)$, then define $\mathrm{dist}(\alpha_0, \beta_0) = \int d_A(\alpha_0(x), \beta_0(x))\, d\mu(x)$.

(4)   The metric defined in the previous step induces a topology on $\mathrm{Aut}(I, m)^X$. Therefore, because $\mathrm{Co}(X)$ is just a certain (closed) subset of $(\mathrm{Aut}(I, m)^X)^G$, it just inherits the product topology.

To summarize, if α is a cocycle, then a basic open neighborhood α is specified by two parameters: a finite subset $F \subseteq G$ and $\eta > 0$. The (F, η)-neighborhood of α is $\{\beta \in \mathrm{Co}(X) : \mathrm{dist}(\alpha_g, \beta_g) < \eta \text{ for all } g \in F\}$. In practice, we always arrange things so that there is a set of x of measure at least $1 - \eta$ on which $\alpha_g(x) = \beta_g(x)$ for all $g \in F$, which is sufficient to guarantee that β is in the (F, η)-neighborhood of α.

Let $\overline{Q}$ be the partition $\{X \times [0, 1/2], X \times (1/2, 1]\}$ of $X \times I$. We derive Theorem 1.1 from the following result about covering numbers of extensions, which is the main technical result of the paper.

THEOREM 3.1. *For any sequence (a_n) satisfying $\limsup_{n \to \infty} (1/|F_n|) \log a_n = 0$, there is a dense G_δ set $\mathcal{U} \subseteq \mathrm{Co}(X)$ such that for any $\alpha \in \mathcal{U}$, $\mathrm{cov}(\overline{Q}, T_\alpha, F_n, \mu \times m) > a_n$ for infinitely many n.*

Proof that Theorem 3.1 implies Theorem 1.1. Choose a sequence (a_n) as in Proposition 2.8 such that for any partition Q, $\mathrm{cov}(Q, T, F_n, \mu) \leq a_n$ for sufficiently large n. Let $\mathcal{U}$ be the dense G_δ set of cocycles associated to (a_n) as guaranteed by Theorem 3.1 and let $\alpha \in \mathcal{U}$, so we know that $\mathrm{cov}(\overline{Q}, T_\alpha, F_n, \mu \times m) > a_n$ for infinitely many n. Now if $\varphi : (X \times I, T_\alpha, \mu \times m) \to (X, T, \mu)$ were an isomorphism, then by Lemma 2.3, $\varphi \overline{Q}$ would be a partition of X satisfying $\mathrm{cov}(\varphi \overline{Q}, T, F_n, \mu) = \mathrm{cov}(\overline{Q}, T_\alpha, F_n, \mu \times m) > a_n$ for infinitely many n, contradicting the conclusion of Proposition 2.8. Therefore, we have produced a dense G_δ set of cocycles α such that $T_\alpha \not\simeq T$, which implies Theorem 1.1. $\square$

To prove Theorem 3.1, we need to show roughly that $\{\alpha \in \mathrm{Co}(X) : \mathrm{cov}(\overline{Q}, T_\alpha, F_n, \mu \times m) \text{ is large}\}$ is both open and dense. We address the open part here and leave the density part until the next section. Let π be the partition $\{[0, 1/2), [1/2, 1]\}$ of I.

LEMMA 3.2. *If $\beta^{(n)}$ is a sequence of cocycles converging to α, then for any finite $F \subseteq G$, we have*

$$(\mu \times m)\{(x, t) : \overline{Q}_{T_{\beta^{(n)}}, F}(x, t) = \overline{Q}_{T_\alpha, F}(x, t)\} \to 1 \quad as\ n \to \infty.$$

Proof. For the names $\overline{Q}_{T_{\beta^{(n)}}, F}(x, t)$ and $\overline{Q}_{T_\alpha, F}(x, t)$ to be the same means that for every $g \in F$,

$$\overline{Q}(T^g x, \beta_g^{(n)}(x)t) = \overline{Q}(T^g x, \alpha_g(x)t),$$

which is equivalent to

$$\pi(\beta_g^{(n)}(x)t) = \pi(\alpha_g(x)t). \tag{1}$$

The idea is the following. For fixed g and x, if $\alpha_g(x)$ and $\beta_g^{(n)}(x)$ are close in d_A, then (1) fails for only a small measure set of t. In addition, if $\beta^{(n)}$ is very close to α in the cocycle topology, then $\beta_g^{(n)}(x)$ and $\alpha_g(x)$ are close for all $g \in F$ and most $x \in X$. Then, by Fubini's theorem, we will get that the measure of the set of (x, t) failing (1) is small.

Here are the details. Fix $\rho > 0$; we show that the measure of the desired set is at least $1 - \rho$ for n sufficiently large. First, let σ be so small that for any $\phi, \psi \in \text{Aut}(I, m)$,

$$d_A(\phi, \psi) < \sigma \quad \text{implies} \quad m\{t : \pi(\phi t) = \pi(\psi t)\} > 1 - \rho/2.$$

This is possible because

$$\{t : \pi(\phi t) \neq \pi(\psi t)\} \subseteq (\phi^{-1}[0, 1/2) \triangle \psi^{-1}[0, 1/2)) \cup (\phi^{-1}[1/2, 1] \triangle \psi^{-1}[1/2, 1]).$$

Then, from the definition of the cocycle topology, we have

$$\mu\{x \in X : d_A(\beta_g^{(n)}(x), \alpha_g(x)) < \sigma \text{ for all } g \in F\} \to 1 \quad \text{as } n \to \infty.$$

Let n be large enough so that the above is larger than $1 - \rho/2$. Then, by Fubini's theorem, we have

$$(\mu \times m)\{(x, t) : \overline{Q}_{T_{\beta^{(n)}}, F}(x, t) = \overline{Q}_{T_\alpha, F}(x, t)\}$$

$$= \int m\{t : \pi(\beta_g^{(n)}(x)t) = \pi(\alpha_g(x)t) \text{ for all } g \in F\} \, d\mu(x).$$

We have arranged things so that the integrand above is greater than $1 - \rho/2$ on a set of x of μ-measure greater than $1 - \rho/2$, so the integral is at least $(1 - \rho/2)(1 - \rho/2) > 1 - \rho$ as desired. □

LEMMA 3.3. *For any finite $F \subseteq G$ and any $L > 0$, the set $\{\alpha \in \text{Co}(X) : \text{cov}(\overline{Q}, T_\alpha, F, \mu \times m) > L\}$ is open in $\text{Co}(X)$.*

Proof. Suppose $\beta^{(n)}$ is a sequence of cocycles converging to α and satisfying $\text{cov}(\overline{Q}, T_{\beta^{(n)}}, F, \mu \times m) \leq L$ for all n. We show that $\text{cov}(\overline{Q}, T_\alpha, F, \mu \times m) \leq L$ as well. The covering number $\text{cov}(\overline{Q}, T_{\beta^{(n)}}, F, \mu \times m)$ is a quantity which really depends only on the measure $(\overline{Q}_{T_{\beta^{(n)}}, F})_*(\mu \times m) \in \text{Prob}(\{0, 1\}^F)$, which we now call ν_n for short. The assumption that $\text{cov}(\overline{Q}, T_{\beta^{(n)}}, F, \mu \times m) \leq L$ for all n says that for each n, there is a collection of L words $w_1^{(n)}, \ldots, w_L^{(n)} \in \{0, 1\}^F$ such that the Hamming balls of radius ϵ centered at these words cover a set of ν_n-measure at least $1 - \epsilon$. As $\{0, 1\}^F$ is a finite set, there are only finitely many possibilities for the collection $(w_1^{(n)}, \ldots, w_L^{(n)})$. Therefore, by passing to a subsequence and relabeling, we may assume that there is a fixed collection of words $w_1, \ldots, w_L$ with the property that if we let B_i be the Hamming ball of radius ϵ centered at w_i, then $\nu_n(\bigcup_{i=1}^{L} B_i) \geq 1 - \epsilon$ for every n.

Now, by Lemma 3.2, the map $\overline{Q}_{T_{\beta^{(n)}}, F}$ agrees with $\overline{Q}_{T_\alpha, F}$ on a set of measure converging to 1 as $n \to \infty$. This implies that the measures ν_n converge in the total variation norm

on $\mathrm{Prob}(\{0, 1\}^F)$ to $\nu := (\overline{Q}_{T_\alpha, F})_*(\mu \times m)$. As $\nu_n(\bigcup_{i=1}^L B_i) \geq 1 - \epsilon$ for every n, we conclude that $\nu(\bigcup_{i=1}^L B_i) \geq 1 - \epsilon$ also, which implies that $\mathrm{cov}(\overline{Q}, T_\alpha, F, \mu \times m) \leq L$ as desired. $\square$

Define $\quad \mathcal{U}_N := \{\alpha \in \mathrm{Co}(X) : \mathrm{cov}(\overline{Q}, T_\alpha, F_n, \mu \times m) > a_n \text{ for some } n > N\}$. By Lemma 3.3, each $\mathcal{U}_N$ is a union of open sets and therefore open. In addition, $\bigcap_N \mathcal{U}_N$ is exactly the set of $\alpha \in \mathrm{Co}(X)$ satisfying $\mathrm{cov}(\overline{Q}, T_\alpha, F_n, \mu \times m) > a_n$ infinitely often. Therefore, by the Baire category theorem, in order to prove Theorem 3.1 it suffices to prove

PROPOSITION 3.4. *For each N, $\mathcal{U}_N$ is dense in $\mathrm{Co}(X)$.*

The proof of this proposition is the content of the next section.

4. *Proof of Proposition 3.4*

4.1. *Setup.* Let N be fixed and let α_0 be an arbitrary cocycle. Consider a neighborhood of α_0 determined by a finite set $F \subseteq G$ and $\eta > 0$. We can assume without loss of generality that $\eta \ll \epsilon = 1/100$. We produce a new cocycle $\alpha \in \mathcal{U}_N$ such that there is a set X' of measure at least $1 - \eta$ on which $\alpha_f(x) = (\alpha_0)_f(x)$ for all $f \in F$, implying that α is in the (F, η)-neighborhood of α_0. The construction of such an α is based on the fact that the orbit equivalence relation R is hyperfinite.

THEOREM 4.1. [**OW80**, Theorem 6] *There is an increasing sequence of equivalence relations $R_n \subseteq X \times X$ such that:*
- *each R_n is measurable as a subset of $X \times X$;*
- *every cell of every R_n is finite; and*
- *$\bigcup_n R_n$ agrees μ-almost everywhere with R.*

Fix such a sequence (R_n) and for $x \in X$, write $R_n(x)$ to denote the cell of R_n that contains x.

LEMMA 4.2. *There exists an m_1 such that $\mu\{x \in X : T^F x \subseteq R_{m_1}(x)\} > 1 - \eta$.*

Proof. Almost every x satisfies $T^G x = \bigcup_m R_m(x)$, so, in particular, for μ-almost every x, there is an m_x such that $T^F x \subseteq R_m(x)$ for all $m \geq m_x$. Letting $X_\ell = \{x \in X : m_x \leq \ell\}$, we see that the sets X_ℓ are increasing and exhaust almost all of X. Therefore, we can pick m_1 so that $\mu(X_{m_1}) > 1 - \eta$. $\square$

Now we drop $R_1, \ldots, R_{m_1-1}$ from the sequence and assume that $m_1 = 1$.

LEMMA 4.3. *There exists a K such that $\mu\{x : |R_1(x)| \leq K\} > 1 - \eta$.*

Proof. Every R_1-cell is finite, so if we define $X_k = \{x \in X : |R_1(x)| \leq k\}$, then the X_k are increasing and exhaust all of X. Thus, we pick K so that $\mu(X_K) > 1 - \eta$. $\square$

Continue to use the notation $X_K = \{x \in X : |R_1(x)| \leq K\}$.

LEMMA 4.4. *For all n sufficiently large, $\mu\{x \in X : |(T^{F_n}x) \cap X_K|/|F_n| > 1 - 2\eta\} > 1 - \eta$.*

Proof. We have $|(T^{F_n}x) \cap X_K| = \sum_{f \in F_n} 1_{X_K}(T^f x)$. By the mean ergodic theorem [Gla03, Theorem 3.33], we get

$$\frac{|(T^{F_n}x) \cap X_K|}{|F_n|} \to \mu(X_K) > 1 - \eta \quad \text{in probability as } n \to \infty.$$

Therefore, in particular, $\mu\{x \in X : |(T^{F_n}x) \cap X_K|/|F_n| > 1 - 2\eta\} \to 1$ as $n \to \infty$, so this measure is greater than $1 - \eta$ for all n sufficiently large. $\qquad \square$

From now on, let n be a fixed number that is large enough so that the above lemma holds, $n > N$, and $\frac{1}{2}\exp(1/8K^2 \cdot |F_n|) > a_n$. This is possible because (a_n) is assumed to be subexponential in $|F_n|$. The relevance of the final condition will appear at the end.

LEMMA 4.5. *There is an m_2 such that $\mu\{x \in X : T^{F_n}x \subseteq R_{m_2}(x)\} > 1 - \eta$.*

Proof. The proof follows the same lines as Lemma 4.2. $\qquad \square$

Again, drop $R_2, \ldots, R_{m_2-1}$ from the sequence of equivalence relations and assume $m_2 = 2$.

4.2. *Construction of the perturbed cocycle.* Let (R_n) be the relabeled sequence of equivalence relations from the previous section. The following measure-theoretic fact is well known. Recall that two partitions P and P' of I are said to be *independent* with respect to m if $m(E \cap E') = m(E)m(E')$ for any $E \in P$, $E' \in P'$.

LEMMA 4.6. *Let P and P' be two finite partitions of I. Then there exists a $\varphi \in \text{Aut}(I, m)$ such that P and $\varphi^{-1}P'$ are independent with respect to m.*

PROPOSITION 4.7. *For any $\alpha_0 \in \text{Co}(X)$, there is an $\alpha \in \text{Co}(X)$ such that:*
(1) $\alpha_g(x) = (\alpha_0)_g(x)$ *whenever $(x, T^g x) \in R_1$; and*
(2) *for μ-almost every x, the following holds. If C is an R_1-cell contained in $R_2(x)$, consider the map $Y_C : t \mapsto \overline{Q}_{T_\alpha, \{g:T^g x \in C\}}(x, t)$ as a random variable on the underlying space (I, m). Then as C ranges over all such R_1-cells, the random variables Y_C are independent.*

Proof. We give here only a sketch of the proof and leave the full details to Appendix A. It is more convenient to adopt the perspective of a cocycle as a map $\alpha : R \to \text{Aut}(I, m)$ satisfying the condition $\alpha(x, z) = \alpha(y, z) \circ \alpha(x, y)$.

Step 1. For $(x, y) \in R_1$, let $\alpha(x, y) = \alpha_0(x, y)$.

Step 2. Fix an R_2-cell $\overline{C}$. Enumerate by $\{C_1, \ldots, C_k\}$ all of the R_1-cells contained in $\overline{C}$ and choose from each a representative $x_i \in C_i$.

Step 3. Let π denote the partition $\{[0, 1/2), [1/2, 1]\}$ of I. Define $\alpha(x_1, x_2)$ to be an element of $\text{Aut}(I, m)$ such that

$$\bigvee_{y \in C_1} \alpha(x_1, y)^{-1}\pi \quad \text{and} \quad \alpha(x_1, x_2)^{-1}\left(\bigvee_{y \in C_2} \alpha(x_2, y)^{-1}\pi\right)$$

are independent. These expressions are well defined because α has already been defined on R_1 and we use Lemma 4.6 to guarantee that such an element of $\mathrm{Aut}(I, m)$ exists.

Step 4. There is now a unique way to extend the definition of α to $(C_1 \cup C_2) \times (C_1 \cup C_2)$ that is consistent with the cocycle condition. For arbitrary $y_1 \in C_1$, $y_2 \in C_2$, define

$$\alpha(y_1, y_2) = \alpha(x_2, y_2) \circ \alpha(x_1, x_2) \circ \alpha(y_1, x_1) \quad \text{and}$$
$$\alpha(y_2, y_1) = \alpha(y_1, y_2)^{-1}.$$

The middle term in the first equation was defined in the previous step and the outer two terms were defined in step 1.

Step 5. Extend the definition of α to the rest of the C_i inductively, making each cell independent of all the previous ones. Suppose α has been defined on $(C_1 \cup \cdots \cup C_j) \times (C_1 \cup \cdots \cup C_j)$. Using Lemma 4.6 again, define $\alpha(x_1, x_{j+1})$ to be an element of $\mathrm{Aut}(I, m)$ such that

$$\bigvee_{y \in C_1 \cup \cdots \cup C_j} \alpha(x_1, y)^{-1}\pi \quad \text{and} \quad \alpha(x_1, x_{j+1})^{-1}\left(\bigvee_{y \in C_{j+1}} \alpha(x_{j+1}, y)^{-1}\pi\right)$$

are independent. Then, just as in step 4, there is a unique way to extend the definition of α to all of $(C_1 \cup \cdots \cup C_{j+1}) \times (C_1 \cup \cdots \cup C_{j+1})$. At the end of this process, α has been defined on all of $\overline{C} \times \overline{C}$. This was done for an arbitrary R_2-cell $\overline{C}$, so now α is defined on R_2.

Step 6. For each $N \geq 2$, extend the definition of α from R_N to R_{N+1} with the same procedure, but there is no need to set up any independence. Instead, every time there is a choice for how to define α between two of the cell representatives, just take it to be the identity. This defines α on $\bigcup_{N \geq 1} R_N$, which is equal mod μ to the full orbit equivalence relation, so α is a well-defined cocycle.

Now we verify the two claimed properties of α. Property (1) is immediate from step 1 of the construction. To check property (2), fix x and let C_j be any of the R_1-cells contained in $R_2(x)$. Note that the name $\overline{Q}_{T_\alpha, \{g : T^g x \in C_j\}}(x, t)$ records the data $\overline{Q}(T_\alpha^g(x, t)) = \overline{Q}(T^g x, \alpha_g(x)t) = \pi(\alpha_g(x)t)$ for all g such that $T^g x \in C_j$, which, by switching to the other notation, is the same data as $\pi(\alpha(x, y)t)$ for $y \in C_j$. Thus, the set of t for which $\overline{Q}_{T_\alpha, \{g : T^g x \in C_j\}}(x, t)$ is equal to a particular word is given by a corresponding particular cell of the partition $\bigvee_{y \in C_j} \alpha(x, y)^{-1}\pi = \alpha(x, x_1)^{-1}(\bigvee_{y \in C_j} \alpha(x_1, y)^{-1}\pi)$. The construction of α was defined exactly so that the partitions $\bigvee_{y \in C_j} \alpha(x_1, y)^{-1}\pi$ are all independent and the names $\overline{Q}_{T_\alpha, \{g : T^g x \in_j\}}(x, t)$ are determined by these independent partitions pulled back by the fixed m-preserving map $\alpha(x, x_1)$, so they are also independent.

The reason this is only a sketch is because it is not clear that the construction described here can be done in a way so that the resulting α is a measurable function. To do it properly requires a slightly different approach; see Appendix A for full details. $\square$

Letting $\widetilde{X} = \{x \in X : T^F x \subseteq R_1(x)\}$, this construction guarantees that $\alpha_f(x) = (\alpha_0)_f(x)$ for all $f \in F$, $x \in \widetilde{X}$. By Lemma 4.2, $\mu(\widetilde{X}) > 1 - \eta$, so this shows that α is in the (F, η)-neighborhood of α_0.

4.3. *Estimating the size of Hamming balls.* Let α be the cocycle constructed in the previous section. We estimate the $(\mu \times m)$-measure of $(\overline{Q}, T_\alpha, F_n)$-Hamming balls in order to get a lower bound for the covering number. The following formulation of Hoeffding's inequality will be quite useful [**Ver18**, Theorem 2.2.6].

THEOREM 4.8. *Let* $Y_1, \dots, Y_\ell$ *be independent random variables such that each* $Y_i \in [0, K]$ *almost surely. Let* $a = \mathbb{E}[\sum Y_i]$. *Then for any* $t > 0$,

$$\mathbb{P}\left(\sum_{i=1}^{\ell} Y_i < a - t \right) \leq \exp\left(- \frac{2t^2}{K^2 \ell} \right).$$

Let $X_0 = \{x \in X : |(T^{F_n}x) \cap X_K|/|F_n| > 1 - 2\eta \text{ and } T^{F_n}x \subseteq R_2(x)\}$. By Lemmas 4.4 and 4.5, $\mu(X_0) > 1 - 2\eta$. In addition, write $\mu \times m = \int m_x \, d\mu(x)$, where $m_x = \delta_x \times m$.

PROPOSITION 4.9. *For any* $(x, t) \in X_0 \times I$,

$$m_x(B_{\text{Ham}}(\overline{Q}, T_\alpha, F_n, (x, t), \epsilon)) \leq \exp\left(- \frac{1}{8K^2} \cdot |F_n| \right). \tag{2}$$

Proof. Let $\mathcal{C}$ be the collection of R_1-cells C that meet $T^{F_n}x$ and satisfy $|C| \leq K$. For each $C \in \mathcal{C}$, let $F_C = \{f \in F_n : T^f x \in C\}$. Define

$$Y(t') = |F_n| \cdot d_{F_n}(\overline{Q}_{T_\alpha, F_n}(x, t), \overline{Q}_{T_\alpha, F_n}(x, t')) = \sum_{f \in F_n} 1_{\overline{Q}(T_\alpha^f(x,t)) \neq \overline{Q}(T_\alpha^f(x,t'))},$$

and for each $C \in \mathcal{C}$, define

$$Y_C(t') = \sum_{f \in F_C} 1_{\overline{Q}(T_\alpha^f(x,t)) \neq \overline{Q}(T_\alpha^f(x,t'))}.$$

Then we have

$$Y(t') \geq \sum_{C \in \mathcal{C}} Y_C(t'),$$

so to get an upper bound for $m_x(B_{\text{Ham}}(\overline{Q}, T_\alpha, F_n, (x, t), \epsilon)) = m\{t' : Y(t') < \epsilon|F_n|\}$, it is sufficient to control $m\{t' : \sum_{C \in \mathcal{C}} Y_C(t') < \epsilon|F_n|\}$.

View each $Y_C(t')$ as a random variable on the underlying probability space (I, m). Our construction of the cocycle α guarantees that the collection of names $\overline{Q}_{T_\alpha, F_C}(x, t')$ as C ranges over all of the R_1-cells contained in $R_2(x)$ is an independent collection. Therefore, in particular, the Y_C for $C \in \mathcal{C}$ are independent (the assumption that $x \in X_0$ guarantees that all $C \in \mathcal{C}$ are contained in $R_2(x)$).

We also have that each $Y_C \in [0, K]$ and the expectation of the sum is

$$a := \sum_{C \in \mathcal{C}} \int Y_C(t') \, dm(t') = \sum_{C \in \mathcal{C}} \sum_{f \in F_C} \int 1_{\overline{Q}(T_\alpha^f(x,t)) \neq \overline{Q}(T_\alpha^f(x,t'))} \, dm(t') = \sum_{C \in \mathcal{C}} \frac{1}{2}|F_C|$$

$$= \frac{1}{2} \sum_{C \in \mathcal{C}} |C \cap (T^{F_n}x)| > \frac{1}{2}(1 - 2\eta)|F_n|,$$

where the final inequality is true because $x \in X_0$. Thus, we can apply Theorem 4.8 with $t = a - \epsilon|F_n|$ to conclude

$$m\left\{t' : \sum_{C \in \mathcal{C}} Y_C(t') < \epsilon|F_n|\right\} \leq \exp\left(\frac{-2t^2}{K^2|\mathcal{C}|}\right) \leq \exp\left(\frac{-2(1/2 - \eta - \epsilon)^2|F_n|^2}{K^2|F_n|}\right)$$

$$\leq \exp\left(-\frac{1}{8K^2} \cdot |F_n|\right).$$

The final inequality holds because $\epsilon = 1/100$ and $\eta \ll \epsilon$ is small enough so that $1/2 - \eta - \epsilon > 1/4$. $\qquad\square$

COROLLARY 4.10. *Let* $y \in X_0$. *If* B *is any* $(\overline{Q}, T_\alpha, F_n)$-*Hamming ball of radius* ϵ, *then* $m_y(B) \leq \exp((-1/8K^2) \cdot |F_n|)$.

Proof. If B does not meet the fiber above y, then obviously $m_y(B) = 0$. Thus, assume $(y, s) \in B$ for some $s \in I$. Then applying the triangle inequality in the space $(\{0, 1\}^{F_n}, d_{F_n})$ shows that $B \subseteq B_{\mathrm{Ham}}(\overline{Q}, T_\alpha, F_n, (y, s), 2\epsilon)$. Now apply Proposition 4.9 with 2ϵ in place of ϵ. The proof goes through exactly the same and we get the same constant $1/8K^2$ in the final estimate because ϵ and η are small enough so that $1/2 - \eta - 2\epsilon$ is still greater than $1/4$. $\qquad\square$

COROLLARY 4.11. *We have* $\mathrm{cov}(\overline{Q}, T_\alpha, F_n, \mu \times m) \geq \frac{1}{2}\exp((1/8K^2) \cdot |F_n|)$.

Proof. Let $\{B_i\}_{i=1}^{\ell}$ be a collection of $(\overline{Q}, T_\alpha, F_n)$-Hamming balls of radius ϵ such that

$$(\mu \times m)\left(\bigcup B_i\right) > 1 - \epsilon.$$

Then

$$1 - \epsilon < (\mu \times m)\left(\bigcup B_i\right)$$

$$= (\mu \times m)\left(\bigcup B_i \cap (X_0 \times I)\right) + (\mu \times m)\left(\bigcup B_i \cap (X_0^c \times I)\right)$$

$$< \sum_{i=1}^{\ell} (\mu \times m)(B_i \cap (X_0 \times I)) + (\mu \times m)(X_0^c \times I)$$

$$< \sum_{i=1}^{\ell} \int_{y \in X_0} m_y(B_i) \, d\mu(y) + 2\eta$$

$$< \ell \cdot \exp\left(-\frac{1}{8K^2} \cdot |F_n|\right) + 2\eta,$$

implying that $\ell > (1 - \epsilon - 2\eta)\exp((1/8K^2) \cdot |F_n|) > (1/2)\exp((1/8K^2) \cdot |F_n|)$. $\qquad\square$

Our choice of n at the beginning now guarantees that $\mathrm{cov}(\overline{Q}, T_\alpha, F_n, \mu \times m) \geq \frac{1}{2}\exp(1/8K^2 \cdot |F_n|) > a_n$, showing that $\alpha \in \mathcal{U}_N$ as desired. This completes the proof of Proposition 3.4.

A. *Appendix. Measurability of the perturbed cocycle*

In this section, we give a more careful proof of Proposition 4.7 that addresses the issue of measurability. We need to use at some point the following measurable selector theorem [**Fre06**, Proposition 433F].

THEOREM A.1. *Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be standard Borel spaces. Let $\mathbb{P}$ be a probability measure on $(\Omega_1, \mathcal{F}_1)$ and suppose that $f : \Omega_2 \to \Omega_1$ is measurable and surjective. Then there exists a measurable selector $g : \Omega_1 \to \Omega_2$ which is defined $\mathbb{P}$-almost everywhere (meaning $g(\omega) \in f^{-1}(\omega)$ for $\mathbb{P}$-almost every $\omega \in \Omega_1$).*

Given $x \in X$, there is a natural bijection between $T^G x$ and G because T is a free action. We can also identify subsets: if $E \subseteq T^G x$, then we write $\widetilde{E} := \{g \in G : T^g x \in E\}$. Note that this set depends on the 'base point' x. If x and y are two points in the same G-orbit, then the set $\widetilde{E}$ based at x is a translate of the same set based at y. It will always be clear from context what the intended base point is.

Definition A.2. A *pattern* in G is a pair $(H, \mathcal{C})$, where H is a finite subset of G and $\mathcal{C}$ is a partition of H.

Definition A.3. For $x \in X$, define $\mathrm{pat}_n(x)$ to be the pattern $(H, \mathcal{C})$, where $H = \widetilde{R_n(x)}$ and $\mathcal{C}$ is the partition of H into the sets $\widetilde{C}$ where C ranges over all of the R_{n-1}-cells contained in $R_n(x)$.

LEMMA A.4. *The pattern $\mathrm{pat}_n(x)$ is a measurable function of x.*

Proof. Because there are only countably many possible patterns, it is enough to fix a pattern $(H, \mathcal{C})$ and show that $\{x : \mathrm{pat}_n(x) = (H, \mathcal{C})\}$ is measurable. Enumerate $\mathcal{C} = \{C_1, \ldots, C_k\}$. Saying that $\mathrm{pat}_n(x) = (H, \mathcal{C})$ is the same as saying that $T^H x = R_n(x)$ and each $T^{C_i} x$ is a cell of R_{n-1}. We can express the set of x satisfying this as

$$\left(\bigcap_{i=1}^{k} \bigcap_{g,h \in C_i} \{x : (T^g x, T^h x) \in R_{n-1}\} \quad \cap \bigcap_{(g,h) \in G^2 \setminus \bigcup (C_i \times C_i)} \{x : (T^g x, T^h x) \notin R_{n-1}\} \right)$$

$$\cap \left(\bigcap_{g \in H} \{x : (x, T^g x) \in R_n\} \quad \cap \bigcap_{g \notin H} \{x : (x, T^g x) \notin R_n\} \right).$$

Because each R_n is a measurable set and each T^g is a measurable map, this whole thing is measurable. $\square$

For each pattern $(H, \mathcal{C})$, let $X_{H,\mathcal{C}}^{(n)} = \{x \in X : \mathrm{pat}_n(x) = (H, \mathcal{C})\}$. We define our cocycle α inductively on the equivalence relations R_n. For each n, the sets $X_{H,\mathcal{C}}^{(n)}$ partition X into countably many measurable sets, so it will be enough to define α measurably on each $X_{H,\mathcal{C}}^{(n)}$. At this point, fix a pattern $(H, \mathcal{C})$, fix $n = 2$, and write $X_{H,\mathcal{C}}$ instead of $X_{H,\mathcal{C}}^{(2)}$. Define

$$\Omega_2^{H,\mathscr{C}} = \{\psi : H \times H \to \mathrm{Aut}(I,m) : \psi(h_1,h_3)$$
$$= \psi(h_2,h_3) \circ \psi(h_1,h_2) \text{ for all } h_1, h_2, h_3 \in H\},$$
$$\Omega_1^{H,\mathscr{C}} = \left\{\sigma : \bigcup_{C \in \mathscr{C}} C \times C \to \mathrm{Aut}(I,m) : \sigma(g_1,g_3)\right.$$
$$\left. = \sigma(g_2,g_3) \circ \sigma(g_1,g_2) \text{ for all } g_1, g_2, g_3 \in G\right\},$$
$$\Omega_2^{H,\mathscr{C},\,\mathrm{ind}} = \{\psi \in \Omega_2^{H,\mathscr{C}} : \psi \text{ is } (H,\mathscr{C})\text{-independent}\},$$

where $\psi \in \Omega_2^{H,\mathscr{C}}$ is said to be $(H,\mathscr{C})$-*independent* if for any fixed $h_0 \in H$, the partitions

$$\bigvee_{h \in C} \psi(h_0,h)^{-1}\pi$$

as C ranges over $\mathscr{C}$ are independent with respect to m.

PROPOSITION A.5. *For every $\sigma \in \Omega_1^{H,\mathscr{C}}$, there is some $\psi \in \Omega_2^{H,\mathscr{C},ind}$ that extends σ.*

Proof. The idea is exactly the same as the construction described in steps 3–5 in the sketched proof of Proposition 4.7, but we write it out here also for completeness.

Enumerate $\mathscr{C} = \{C_1, \ldots, C_k\}$ and for each i fix an element $g_i \in C_i$. First, obviously we define $\psi = \sigma$ on each $C_i \times C_i$. Next, define $\psi(g_1,g_2)$ to be an element of $\mathrm{Aut}(I,m)$ such that

$$\bigvee_{g \in C_1} \sigma(g_1,g)^{-1}\pi \quad \text{and} \quad \psi(g_1,g_2)^{-1}\left(\bigvee_{g \in C_2} \sigma(g_2,g)^{-1}\pi\right)$$

are independent. Then, define ψ on all of $(C_1 \cup C_2) \times (C_1 \cup C_2)$ by setting

$$\psi(h_1,h_2) = \sigma(g_2,h_2) \circ \psi(g_1,g_2) \circ \sigma(h_1,g_1) \quad \text{and}$$
$$\psi(h_2,h_1) = \psi(h_1,h_2)^{-1}$$

for any $h_1 \in C_1, h_2 \in C_2$. Continue this definition inductively, making each new step independent of all the steps that came before it. If ψ has been defined on $(C_1 \cup \cdots \cup C_j) \times (C_1 \cup \cdots \cup C_j)$, then define $\psi(g_1,g_{j+1})$ to be an element of $\mathrm{Aut}(I,m)$ such that

$$\bigvee_{g \in C_1 \cup \cdots \cup C_j} \psi(g_1,g)^{-1}\pi \quad \text{and} \quad \psi(g_1,g_{j+1})^{-1}\left(\bigvee_{g' \in C_{j+1}} \sigma(g_{j+1},g')^{-1}\pi\right)$$

are independent. Then extend the definition of ψ to all of $(C_1 \cup \cdots \cup C_{j+1}) \times (C_1 \cup \cdots \cup C_{j+1})$ in the exact same way.

At the end of this process, ψ has been defined on $(C_1 \cup \cdots \cup C_k) \times (C_1 \cup \cdots \cup C_k) = H \times H$, and it satisfies the cocycle condition by construction. To verify that it also satisfies the independence condition, note that the construction has guaranteed that

$$\bigvee_{h \in C} \psi(g_1,h)^{-1}\pi$$

are independent partitions as C ranges over $\mathscr{C}$. To get the same conclusion for an arbitrary base point h_0, pull everything back by the fixed map $\psi(h_0,g_1)$. Because

this map is measure preserving, pulling back all of the partitions by it preserves their independence.   □

Now we would like to take this information about cocycles defined on patterns and use it to produce cocycles defined on the actual space X. Define the map $\sigma^{H,\mathscr{C}} : X_{H,\mathscr{C}} \to \Omega_1^{H,\mathscr{C}}$ by $\sigma_x^{H,\mathscr{C}}(g_1, g_2) := \alpha_0(T^{g_1}x, T^{g_2}x)$. Note that this is a measurable map because α_0 is a measurable cocycle.

By Theorem A.1 applied to the measure $\mathbb{P} = (\sigma^{H,\mathscr{C}})_*(\mu(\cdot \mid X_{H,\mathscr{C}})) \in \mathrm{Prob}(\Omega_1^{H,\mathscr{C}})$, we get a *measurable* map $E^{H,\mathscr{C}} : \Omega_1^{H,\mathscr{C}} \to \Omega_2^{H,\mathscr{C},\mathrm{ind}}$ defined $\mathbb{P}$-almost everywhere such that $E^{H,\mathscr{C}}(\sigma)$ extends σ. Denote the composition $E^{H,\mathscr{C}} \circ \sigma^{H,\mathscr{C}}$ by $\psi^{H,\mathscr{C}}$ and write the image of x under this map as $\psi_x^{H,\mathscr{C}}$. To summarize, for every pattern $(H, \mathscr{C})$, there is a measurable map $\psi^{H,\mathscr{C}} : X_{H,\mathscr{C}} \to \Omega_2^{H,\mathscr{C},\mathrm{ind}}$ defined μ-almost everywhere with the property that $\psi_x^{H,\mathscr{C}}$ extends $\sigma_x^{H,\mathscr{C}}$.

It is now natural to define our desired cocycle α on the equivalence relation R_2 by the formula $\alpha(x, T^g x) := \psi_x^{\mathrm{pat}_2(x)}(e, g)$. It is then immediate to verify the two properties of α claimed in the statement of Proposition 4.7. The fact that α agrees with α_0 on R_1 follows from the fact that $\psi_{H,\mathscr{C}}$ extends $\sigma^{H,\mathscr{C}}$ and the claimed independence property of α translates directly from the independence property that the $\psi_x^{H,\mathscr{C}}$ were constructed to have (see also the discussion after step 6 in the sketched proof of Proposition 4.7). In addition, α is measurable because for each fixed g, the map $x \mapsto \alpha(x, T^g x)$ is simply a composition of other maps already determined to be measurable. The only problem is that α, when defined in this way, need not satisfy the cocycle condition. To see why, observe that the cocycle condition $\alpha(x, T^h x) = \alpha(T^g x, T^h x) \circ \alpha(x, T^g x)$ is equivalent to the condition

$$\psi_x^{\mathrm{pat}_2(x)}(e, h) = \psi_{T^g x}^{\mathrm{pat}_2(T^g x)}(e, h g^{-1}) \circ \psi_x^{\mathrm{pat}_2(x)}(e, g). \tag{A.1}$$

However, in defining the maps $\psi^{H,\mathscr{C}}$, we have simply applied Theorem A.1 arbitrarily to each pattern separately, so $\psi^{\mathrm{pat}_2(x)}$ and $\psi^{\mathrm{pat}_2(T^g x)}$ have nothing to do with each other. However, we can fix this problem with a little extra work, and once we do, we will have defined $\alpha : R_2 \to \mathrm{Aut}(I, m)$ with all of the desired properties.

Start by declaring two patterns as equivalent if they are translates of each other, and fix a choice of one pattern from each equivalence class. As there are only countably many patterns in total, there is no need to worry about how to make this choice. For each representative pattern $(H_0, \mathscr{C}_0)$, apply Theorem A.1 arbitrarily to get a map $\psi^{H_0,\mathscr{C}_0}$. This does not cause any problems because two patterns that are not translates of each other cannot appear in the same orbit (this follows from the easy fact that $\mathrm{pat}_2(T^g x) = g^{-1} \cdot \mathrm{pat}_2(x)$), so it does not matter that their ψ maps are not coordinated with each other. For convenience, let us denote the representative of the equivalence class of $\mathrm{pat}_2(x)$ by $\mathrm{rp}(x)$. Now, for every $x \in X$, let $g^*(x)$ be the unique element of G with the property that $\mathrm{pat}_2(T^{g^*(x)}x) = \mathrm{rp}(x)$. Note that the maps g^* and rp are both constant on each subset $X_{H,\mathscr{C}}$ and are therefore measurable.

Now for an arbitrary pattern $(H, \mathscr{C})$ and $x \in X_{H,\mathscr{C}}$, we define the map $\psi^{H,\mathscr{C}}$ by

$$\psi_x^{H,\mathscr{C}}(g, h) := \psi_{T^{g^*(x)}x}^{\mathrm{rp}(x)}(g \cdot g^*(x)^{-1}, h \cdot g^*(x)^{-1}).$$

Note that this is still just a composition of measurable functions, so $\psi^{H,\mathscr{C}}$ is measurable. All that remains is to verify that this definition satisfies (A.1). The right-hand side of (A.1) is

$$\psi^{\mathrm{rp}(T^g x)}_{T^{g*}(T^g x) T^g x}(eg^*(T^g x)^{-1}, hg^{-1}g^*(T^g x)^{-1}) \circ \psi^{\mathrm{rp}(x)}_{T^{g*}(x) x}(eg^*(x)^{-1}, gg^*(x)^{-1})$$

$$= \psi^{\mathrm{rp}(x)}_{T^{g*}(x)g^{-1} T^g x}((g^*(x)g^{-1})^{-1}, hg^{-1}(g^*(x)g^{-1})^{-1}) \circ \psi^{\mathrm{rp}(x)}_{T^{g*}(x) x}(g^*(x)^{-1}, gg^*(x)^{-1})$$

$$= \psi^{\mathrm{rp}(x)}_{T^{g*}(x) x}(gg^*(x)^{-1}, hg^*(x)^{-1}) \circ \psi^{\mathrm{rp}(x)}_{T^{g*}(x) x}(g^*(x)^{-1}, gg^*(x)^{-1})$$

$$= \psi^{\mathrm{rp}(x)}_{T^{g*}(x) x}(g^*(x)^{-1}, hg^*(x)^{-1}),$$

which is, by definition, equal to the left-hand side of (A.1) as desired.

This, together with the discussion surrounding (A.1), shows that if we construct the maps $\psi^{H,\mathscr{C}}$ in this way, then making the definition $\alpha(x, T^g x) = \psi^{\mathrm{pat}_2(x)}_x(e, g)$ gives us a true measurable cocycle with all of the desired properties. Finally, to extend the definition of α to R_n with $n \geq 3$, repeat the exact same process, except it is even easier because there is no need to force any independence. The maps $\psi^{H,\mathscr{C}}$ only need to be measurable selections into the space $\Omega_2^{H,\mathscr{C}}$, and then everything else proceeds in exactly the same way.

REFERENCES

[AGTW21] T. Austin, E. Glasner, J.-P. Thouvenot and B. Weiss. An ergodic system is dominant exactly when it has positive entropy. *Ergod. Th. & Dynam. Sys.* doi: 10.1017/etds.2022.69. Published online 4 November 2022.

[Fer97] S. Ferenczi. Measure-theoretic complexity of ergodic systems. *Israel J. Math.* **100** (1997), 189–207.

[Fre06] D. H. Fremlin. *Measure Theory*. Vol. 4. Torres Fremlin, Colchester, 2006. Topological Measure Spaces. Part I, II, Corrected second printing of the 2003 original.

[Gla03] E. Glasner. *Ergodic Theory via Joinings (Mathematical Surveys and Monographs, 101)*. American Mathematical Society, Providence, RI, 2003.

[GTW21] E. Glasner, J.-P. Thouvenot and B. Weiss. On some generic classes of ergodic measure preserving transformations. *Trans. Moscow Math. Soc.* **82** (2021), 15–36.

[Kec10] A. S. Kechris. *Global Aspects of Ergodic Group Actions (Mathematical Surveys and Monographs, 160)*. American Mathematical Society, Providence, RI, 2010.

[KT97] A. Katok and J.-P. Thouvenot. Slow entropy type invariants and smooth realization of commuting measure-preserving transformations. *Ann. Inst. Henri Poincaré Probab. Stat.* **33**(3) (1997), 323–338.

[MO85] J. M. Ollagnier. *Ergodic Theory and Statistical Mechanics (Lecture Notes in Mathematics, 1115)*. Springer-Verlag, Berlin, 1985.

[Orn70] D. Ornstein. Bernoulli shifts with the same entropy are isomorphic. *Adv. Math.* **4** (1970), 337–352.

[OW80] D. S. Ornstein and B. Weiss. Ergodic theory of amenable group actions. I. The Rohlin lemma. *Bull. Amer. Math. Soc. (N.S.)* **2**(1) (1980), 161–164.

[Ros88] A. Rosenthal. Finite uniform generators for ergodic, finite entropy, free actions of amenable groups. *Probab. Theory Related Fields* **77**(2) (1988), 147–166.

[Sew19] B. Seward. Krieger's finite generator theorem for actions of countable groups I. *Invent. Math.* **215**(1) (2019), 265–310.

[Ver18] R. Vershynin. *High-Dimensional Probability: An Introduction with Applications in Data Science (Cambridge Series in Statistical and Probabilistic Mathematics, 47)*. Cambridge University Press, Cambridge, 2018, with a foreword by S. van de Geer.