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A NOTE ON LOCALLY EXPANSIVE AND LOCALLY ACCRETIVE OPERATORS

BY
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ABSTRACT. Let X be a Banach space, D an open subset of X and Y a complete metric space. Assume that Y is metrically convex. For $T:\bar{D}\to Y$ closed, locally m-expansive and mapping open subsets of D onto open subsets of Y, is is shown that $y\in T(D)$ if and only if there exists $x_0\in D$ such that $d(Tx_0, y)\le d(Tx, y)$ for all $x\in\partial D$.

Let X be a Banach space, J the interval $[0, +\infty)$ and $M^+(J)$ the class of all continuous functions $m: J\to J$ such that

(1) $m(r)>0$ for each $r\in J$, and

(2) $\int^{+\infty} m(r)\,dr=+\infty$.

It is a well-known fact ([1, P. 62]) that if a local homeomorphism T of X into a Banach space Y is a local expansion, in the sense that for a fixed constant $c>0$ each point x of X has a neighborhood U_x such that

$$c\,\|u-v\|\le\|Tu-Tv\| \tag{$*$}$$

for each u and v in U_x, then $T(X)=Y$.

In [2], Kirk and Schöneberg have proved that a similar result can be obtained within the class of mappings whose graphs are closed subsets of $X\times Y$. Their approach allowed them to carry out an exhaustive study of some discontinuous mappings defined only on the closure of an open subset of X.

This note is a continuation of the Browder–Kirk–Schöneberg program; unlike the methods used in [1] or [2], ours relies heavily on the theory of differential inequalities.

If D is a subset of X, then $\bar{D}$ and ∂D denote, respectively, the closure and boundary of D in X. Recall that a mapping $T: D\to Y$ is said to be *closed* if for any sequence $\{x_n : n\in\mathbb{N}\}\subseteq D$ with $x_n\to x\in D$ and $Tx_n\to y$ as $n\to\infty$, it follows that $Tx=y$.

DEFINITION. A nonlinear operator T mapping a subset D of a Banach space X into a metric space Y is said to be *locally m-expansive*, $m\in M^+(J)$, if each point $x\in D$ has a neighborhood U_x such that

$$m(\mathrm{Max}\{\|u\|, \|v\|\})\,\|u-v\|\le d(Tu, Tv) \tag{+}$$

for each u and v in U_x.

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Following Menger [3], a metric space Y is said to be *metrically convex* if for all x and y in Y with $x \neq y$ there exists z in Y, distinct from x and y, such that $d(x, y) = d(x, z) + d(z, y)$.

Our main purpose in this note is to prove the following:

THEOREM 1. *Let X be a Banach space, D an open subset of X and Y a complete metric space. Assume that Y is metrically convex. Let $T : \bar{D} \to Y$ be a closed locally m-expansive mapping on D. If T maps open subsets of D into open subsets of Y, then for $y \in Y$ the following are equivalent:*

(1) $y \in T(D)$

(2) *There exists $x_0 \in D$ such that $d(Tx_0, y) \leq d(Tx, y)$ for all $x \in \partial D$.*

As a consequence of Theorem 1, we have the following:

COROLLARY 1. *Let X be a Banach space and Y a complete metric space. Assume that Y is metrically convex. Let $T : X \to Y$ be a closed locally m-expansive mapping. If T maps open subsets of X into open subsets of Y, then $T(X) = Y$.*

Proof of Theorem 1. We only need to prove that $(2) \to (1)$. Following Kirk and Schönberg [2] we let $\lambda : [0, d(Tx_0, y)] \to Y$ be an isometry such that $\lambda(0) = Tx_0$ and $\lambda(d(Tx_0, y)) = y$. The existence of λ is assured by Menger's result [3]. Since T is assumed to be an open locally m-expansive mapping, we can conclude the existence of a positive number τ, $0 < \tau \leq d(Tx_0, y)$, and a unique continuous map $\sigma : [0, \tau) \to D$ such that $\sigma(0) = x_0$ and $T\sigma(t) = \lambda(t)$ for each t, $0 \leq t < \tau$.

LEMMA 1. *If $L\ (\sigma; \tau) \equiv \inf\{m(\|\sigma(r)\|) : 0 \leq r < \tau\}$ then*

$$L(\sigma; \tau) > 0.$$

Proof of Lemma 1. For fixed $t \in [0, \tau)$ let $s > 0$ be such that condition (+) is satisfied for each $\sigma(t + r)$, $0 \leq r < s$. Then

$$m(\text{Max}\{\|\sigma(t)\|, \|\sigma(t+r)\|\})\ \|\sigma(t) - \sigma(t+r)\| \leq d(\lambda(t), \lambda(t+r)) = r.$$

Consequently,

$$m(\|\sigma(t)\|)D^+\ \|\sigma(t)\| \leq 1 \qquad 0 \leq t < \tau$$

where D^+v is the right-upper Dini derivative of the function v. Let

$$S(t) = \int_{\|\sigma(0)\|}^{t} m(x)\, dx.$$

We can easily see that S is an increasing mapping whose range $R(S)$ contains the interval J. If for each $t \in [0, \tau)$ we let

$$\Sigma(t) \equiv S(\|\sigma(t)\|) \quad \text{and} \quad \Phi(t) \equiv t,$$

then

$$\Sigma(0)=\Phi(0)$$

and

$$D^+\Phi(t)\le 1\le D^+\Phi(t)$$

for each t in $[0, \tau)$. Therefore

$$S(\|\sigma(t)\|)\le\Phi(t)\le\tau \qquad 0\le t<\tau$$

and then

$$\|\sigma(t)\|\le S^{-1}(\tau) \qquad 0\le t<\tau.$$

Thus

$$\{\|\sigma(t)\|:0\le t<\tau\}\subseteq[0, S^{-1}(\tau)].$$

The conclusion of the lemma is now an immediate consequence of the continuity and positivity of m on J. ■

LEMMA 2. *If* $0\le t,\ s<\tau$, *then*

$$\|\sigma(t)-\sigma(s)\|\le L(\sigma;\tau)^{-1}\,|t-s|.$$

Proof of Lemma 2. Assume $t<s$. By compactness of $\{\sigma(r):t\le r\le s\}$, we can choose $\{t_i\}_{i=0}^n$ such that

$$t=t_0<t_1<t_2<\cdots<t_n=s$$

and

$$m(\text{Max}\{\|\sigma(t_i)\|, \|\sigma(t_{i+1})\|\})\,\|\sigma(t_i)-\sigma(t_{i+1})\|\le t_{i+1}-t_i$$

for $i=0, 1,\ldots, n-1$. By Lemma 1.

$$L(\sigma,\tau)\,\|\sigma(t_i)-\sigma(t_{i+1})\|\le t_{i+1}-t_i$$

for $i=0, 1,\ldots, n-1$. Therefore

$$\begin{aligned}\|\sigma(t)-\sigma(s)\|&\le\sum_{i=0}^{n-1}\|\sigma(t_i)-\sigma(t_{i+1})\|\\&\le L(\sigma;\tau)^{-1}\sum_{i=0}^{n-1}(t_{i+1}-t_i)\\&=L(\sigma;\tau)^{-1}\,|t-s|.\end{aligned}$$ ■

Proof of Theorem 1 *completed:* By Lemma 2 and the assumption that T is closed on $\bar{D}$ we can conclude that

$$\lim_{t\uparrow\tau}\sigma(t)=x\in\bar{D}$$

exists, and

$$Tx = \lambda(\tau).$$

If $x \in \partial D$, by assumption (2),

$$\begin{aligned} d(Tx, y) \geq d(Tx_0, y) &= |d(Tx_0, y) - \tau| + \tau \\ &= d(\lambda(d(Tx_0, y)), \lambda(\tau)) + \tau \\ &= d(y, Tx) + \tau \\ &> d(y, Tx). \end{aligned}$$

This contradiction shows that x is in the interior of D. Thus, by letting $\sigma(\tau) = x$, we have that $\sigma:[0, \tau] \to D$ is continuous and

$$T\sigma(s) = \lambda(s) \qquad 0 \leq s \leq \tau$$

Let M denote the set of all t in $[0, d(Tx_0, y)]$ for which there exists a unique continuous map $\sigma:[0, t] \to D$ such that

$$T\sigma(s) = \lambda(s) \qquad 0 \leq s \leq t.$$

Then M is nonempty ($[0, \tau] \subseteq M$) and since T is an open locally m-expansive mapping we also have that M is open in $[0, d(Tx_0, y)]$. A conjunction of Lemmas 1 and 2 and the argument above will also prove that M is closed. Therefore there exists $\sigma:[0, d(Tx_0, y)] \to D$ such that $T\sigma(t) = \lambda(t)$, $0 \leq t \leq d(Tx_0, y)$; hence $y = \lambda(d(Tx_0, y)) = T\sigma(d(Tx_0, y))$ and $\sigma(d(Tx_0, y)) \in D$, completing the proof of Theorem 1. ■

We conclude this note with a domain invariance result for locally m-expansive mappings of accretive type. It should be pointed out that this result is a corollary of Schöneberg's results [4].

Let D be a subset of a Banach space X and F the normalized duality mapping of X to 2^{X^*}. An operator $T: D \to X$ is *locally m-strongly accretive* if

(1) $m \in C^+(J)$, the class of all positive continuous functions on J.

(2) Each point $x \in D$ has a neighborhood U_x such that

$$m(\text{Max}\{\|u\|, \|v\|\}) \|u - v\|^2 \leq (Tu - Tv, w) \tag{**}$$

for each u and v in U_x and each w in $F(u - v)$.

THEOREM 2. *Let $D \subseteq X$ be open and $T: D \to X$ be a continuous locally m-strongly accretive operator. Then $T(D)$ is open.*

Proof. Let $x_0 \in D$ and $y_0 = Tx_0$. Let $r > 0$ be such that(**) is satisfied on $\overline{B}(x_0, r)$. Since

$$\inf\{m(\text{Max}[\|x\|, \|x_0\|]) : \|x - x_0\| = r\} > 0$$

and

$$\|Tx - y_0\| \geq m(\text{Max}[\|x\|, \|x_0\|]) \|x - x_0\|$$

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if $\|x-x_0\|=r$, we conclude that the number

$$\sigma=\inf\{\|TX-y_0\|:\|x-x_0\|=r\}$$

is strictly positive. As in Schöneberg [4], we can prove that if $\Sigma>0$, $\Sigma(1+r)<\delta$ and $0<c<\Sigma$, then the equation

$$(***) \qquad Tx+cx=y+cx_0$$

has a solution $x_c \in B(x_0, r)$ for each $y \in B(y_0, \Sigma)$.

Fix now $y \in B(y_0, \Sigma)$ and for each $0<c<\Sigma$ let $x_c \in B(x_0, r)$ be the solution of (***) corresponding to y and c. Then

$$L\,\|x_c-x_{\bar{c}}\|<|c-\bar{c}|\,\|x_0\|+\|cx_c-\bar{c}x_{\bar{c}}\|$$

for $0<c$, $\bar{c}<\Sigma$ and $L=\inf\{m(s):0\le s\le\|x_0\|+r\}$. Since $\|x_c\|\le\|x_0\|+r$, we conclude

(i) $x_c\to\bar{x}$ as $c\to0^+$, and
(ii) $Tx_c\to y$ as $c\to0^+$.

By continuity of T, $T\bar{x}=y$. The theorem will be proved if we show that $\bar{x}\in B(\bar{x}_0, r)$. In fact,

$$\begin{aligned}\|T\bar{x}-y_0\|&=\|y-y_0\|\\&\le\Sigma(1+r)\\&<\delta\\&=\inf\{\|Tz-y_0\|:\|z-x_0\|=r\}.\end{aligned}$$

This inequality shows that $\|\bar{x}-x_0\|<r$, and the proof is completed. ■

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