Proceedings of the Edinburgh Mathematical Society (2025) **68**, 128–156
doi:10.1017/S0013091524000476

EXISTENCE AND NON-EXISTENCE RESULTS FOR A CLASS OF SYSTEMS UNDER CONCAVE-CONVEX NONLINEARITIES

JOÃO PABLO PINHEIRO DA SILVA

Universidade Federal do Pará, Departamento de Matemática, Belém, Pará, Brazil
Email: jpabloufpa@gmail.com

(Received 12 March 2023)

Abstract In this work, we are interested in studying the following class of problems:

$$\begin{cases} -\Delta u = f_\lambda(x,u,v) & \text{in } \Omega \\ -\Delta v = g_\mu(x,u,v) & \text{in } \Omega \\ 0 \not\equiv u \geq 0,\ 0 \not\equiv v \geq 0 & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \qquad (\mathcal{P}_{\lambda\mu})$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\lambda > 0$, $\mu > 0$, $t \mapsto f_\lambda(x,t,t)$ and $t \mapsto g_\mu(x,t,t)$ have concave-convex type nonlinearities. We present results related to the existence and non-existence of solutions for problem $(\mathcal{P}_{\lambda\mu})$.

Keywords: sub-super solution; non-existence for systems; concave-convex nonlinearities

2020 Mathematics subject classification: Primary 35A01; 35B09; 35J47
Secondary 35A16

1. Introduction

To contextualize our work, we will begin by discussing the scalar case (a single equation), which serves as motivation for the system addressed in our research. Let us consider the following problem:

$$\begin{cases} -\Delta u = \lambda a(x)u^q + b(x)u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (\mathcal{P}_\lambda)$$

Here, $\Omega \subset \mathbb{R}^N$ is a bounded domain, $a, b \in L^\infty(\Omega)$, $\lambda > 0$ and $0 < q < 1 < p$. When $a \equiv b \equiv 1$, Ambrosetti, Brézis and Cerami in [4] studied this problem. The authors showed that there exists $\Lambda > 0$ such that $(\mathcal{P}_\lambda)$ has a positive solution when $0 < \lambda \leq \Lambda$, and there is no positive solution when $\lambda > \Lambda$. Moreover, if $p+1 \leq 2^* := 2N/(N-2)$, then the solutions of $(\mathcal{P}_\lambda)$ correspond to critical points of the functional $I_\lambda \in C^1(H_0^1(\Omega), \mathbb{R})$ defined by

$$I_\lambda(u) = \int_\Omega |\nabla u|^2 - \frac{\lambda}{q+1}\int_\Omega |u|^{q+1} - \frac{1}{p+1}\int_\Omega |u|^{p+1}.$$

Using variational methods and techniques introduced by Brézis and Nirenberg in [12], they showed the existence of a second positive solution for $0 < \lambda < \Lambda$ (it is necessary to demand some regularity for the domain in order to guarantee the existence of the unit exterior normal vector to $\partial\Omega$. See, for example, [4, Lemma 4.1]). From a purely mathematical perspective, problems with concave-convex type nonlinearities have received great interest since the seminal work [4]. The list of references is extensive, among which we highlight [6–8, 10, 28–30, 34, 39, 44, 45, 47]. Recently, there has been considerable interest in problems with indefinite weights, that is, problems where the weight functions $a(\cdot)$ or $b(\cdot)$ may change sign. De Figueiredo, Gossez and Ubilla in the work [25, Corollary 2.2 and Corollary 2.7] showed existence and non-existence results for $(\mathcal{P}_\lambda)$ when $a(\cdot)$ and $b(\cdot)$ satisfy certain conditions, including the possibility of $a(\cdot)$ and $b(\cdot)$ changing sign. More specifically, they showed that there exists $\lambda_0 > 0$ and $\overline{c} > 0$ such that $(\mathcal{P}_\lambda)$ admits two solutions when $p \leq 2^* - 1$ and $0 < \lambda < \lambda_0$ and admits no solution if $p \leq 2^* - 1$ and $\lambda > \overline{c}$. In a later work [26, Theorem 4.1 and Theorem 4.2], assuming certain conditions, including: $a, b \in L^\infty(\Omega)$, $0 \not\equiv a(x) \geq 0$ in Ω, and $\inf_{B_1} a(x) > 0$ for some ball $B_1 \subset \Omega$, the same authors recovered the results obtained by Ambrosetti et al. [4], that is, they showed that there exists $0 < \Lambda \leq \infty$ such that: If $0 \leq q < 1 < p$, the problem $(\mathcal{P}_\lambda)$ has at least one solution when $0 < \lambda \leq \Lambda$ and admits no solution when $\lambda > \Lambda \neq +\infty$. If $0 \leq q < 1 < p \leq 2^* - 1$, the problem $(\mathcal{P}_\lambda)$ admits at least two solutions when $0 < \lambda < \Lambda$.

The results provided by the works [4, 26] include nonlinearities with supercritical growth. In the context of the Laplacian operator, when Ω is bounded, the problem $-\Delta u = f(x,u)$ in Ω and $u = 0$ on $\partial\Omega$, is said to have supercritical growth when there is no $C > 0$ such that $|f(x,t)| \leq C(1 + |t|^{2^*-1})$, a.e. $x \in \Omega$ and $t \in \mathbb{R}$, where $2^* = 2N/(N-2)$ is the critical exponent of the Sobolev Embedding and $N \geq 3$. In this sense, the problem $(\mathcal{P}_\lambda)$ has supercritical growth when $p > 2^* - 1$ and $N \geq 3$.

The literature concerning problems with concave-convex type nonlinearities, as can be seen in the works we mentioned earlier, is rich in problems whose nonlinearity can have supercritical growth. For problems with supercritical growth and nonlinearities that are not concave-convex type, see for example [3, 5, 15, 16, 21]. To the best of our knowledge, very few existence results have been determined for elliptic systems with supercritical growth (see [17, 22]).

Consider the following system:

$$-\Delta u = f_\lambda(x,u,v) \text{ and } -\Delta v = g_\mu(x,u,v) \text{ in } \Omega,\ u = v = 0 \text{ on } \partial\Omega. \tag{1.1}$$

Although there is a substantial literature related to scalar problems involving concave-convex type nonlinearities, to this day the results of Ambrosetti et al. [4] have not been

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

fully recovered for systems of type (1.1) in which:

$$\begin{cases} f_\lambda(x,t,t) = \lambda a(x)t^{q_1} + c(x)t^{p_1} \text{ and } g_\mu(x,t,t) = \mu b(x)t^{q_2} + d(x)t^{p_2} \\ a,b,c,d \in L^\infty(\Omega),\ 0 \le q_i < 1 < p_i. \end{cases} \tag{1.2}$$

To date, issues such as non-existence of solution, as well as the existence of solutions for systems with nonlinearities exhibiting supercritical growth (i.e. $p_i > 2^* - 1$ in Equation (1.2)), have not been addressed in the current literature. In general, research involving Equations (1.1)–(1.2) as well as its generalizations is almost entirely restricted to gradient-type systems, a concept we will explain next. Consider the following system:

$$\begin{cases} -\mathcal{L}_i u = \phi_i(x,u_1,u_2) & \text{in } \ \Omega \subset \mathbb{R}^N,\ i = 1,2 \\ u_i = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{1.3}$$

it will be called a gradient-type system if there exists $G(x,\cdot,\cdot) \in C^1(\mathbb{R}^2)$ a.e. in Ω, such that $\phi_i(x,t_1,t_2) = G_{t_i}(x,t_1,t_2)$. The importance of this type of system lies in the fact that it is possible to associate to them an Euler–Lagrange functional.

As far as we know, Wu in [46] was the first author to consider a system with nonlinearities of type (1.2) and Dirichlet boundary condition. The author studied the following gradient-type system: Equation (1.3) with $\mathcal{L}_i = \Delta$ and $\phi_i(x,u_1,u_2) = G_{u_i}(x,u_1,v_2)$ where

$$G(x,u_1,u_2) = \frac{\lambda a(x)}{q+1}|u_1|^{q+1} + \frac{\mu b(x)}{q+1}|u_2|^{q+1} + \frac{c(x)}{\alpha+\beta}|u_1|^\alpha |u_2|^\beta.$$

When $0 < q < 1$, $2 < \alpha+\beta < 2^*$, $a^+(x) := \max\{a(x),0\} \not\equiv 0$, $b^+(x) := \max\{b(x),0\} \not\equiv 0$, and $c \in C^0(\overline{\Omega})$ with $0 \not\equiv c(x) \ge 0$ in Ω, using variational methods and assuming certain conditions on the weight functions, the author provided results on existence and multiplicity of non-negative solutions, provided that $\lambda > 0$ and $\mu > 0$ are sufficiently small.

There are many works dealing with the existence or multiplicity of non-negative solutions for systems with concave-convex nonlinearities of type (1.2), or even for systems involving the p-Laplacian operator and nonlinearities that generalize Equation (1.2). However, the approach we encounter in these works, in general, is the same as that used to address the problem $(\mathcal{P}_\lambda)$ from a variational perspective, so they are restricted to cases where the system is of gradient type and the nonlinearities exhibit subcritical or critical growth, i.e. $p_i \le 2^*$ in Equation (1.2). Furthermore, the results are limited to local cases, in the sense that the existence of a solution is guaranteed only if the parameters $\lambda > 0$ and $\mu > 0$ in Equation (1.1) are sufficiently small. The same applies to systems that generalize Equation (1.1) to more general operators. Regarding gradient-type systems (1.2)–(1.1), as well as their generalizations involving the operators: p-Laplacian (in this case $0 \le q_i < p-1 < p_i$), fractional Laplacian and $p\&q$-Laplacian, we refer to the following works and the references contained therein [2, 3.2, 9, 18, 19, 32, 38, 42, 48] (see also [14, 41] for nonlinearities on $\partial\Omega$).

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

As mentioned earlier, the current literature involving Equations (1.1)–(1.2) and their generalizations is almost entirely restricted to gradient-type systems. This implies that the exponent p_i appearing in Equation (1.2) is bounded by a power associated with the space in which the Euler–Lagrange functional is well-defined.

Some works involving systems require additional comments. In [22], the author of the present work studied the system (1.1)–(1.2) with:

$$\begin{cases} f_\lambda(x,u,v) &=& \lambda a(x)u^{q_1} - \tau c(x)u^{\alpha-1}v^{\beta} \\ g_\mu(x,u,v) &=& \mu b(x)v^{q_2} - \delta d(x)u^{\alpha}v^{\beta-1}, \end{cases} \tag{1.4}$$

where $\lambda,\mu,\delta,\tau > 0$, $0 < q_i < 1$, $a^+(\cdot) \not\equiv 0$, $b^+(\cdot) \not\equiv 0$ in Ω and $c(x) \equiv d(x) \geq 0$ in Ω. In said work, we showed the existence of a solution (u,v) such that $0 \not\equiv u \geq 0$ and $0 \not\equiv v \geq 0$ in Ω, provided that $\alpha,\beta \geq 1$ and $\alpha+\beta > 2$ (in this case $p_i = \alpha+\beta-1 < +\infty$). This system, although only of gradient type for specific values of $\tau > 0$ and $\delta > 0$, can be converted into a gradient-type system for all $\tau,\delta > 0$ (see [22, Proof of Theorem 1.1]). However, the approach of this work cannot be applied to systems that do not have the specific form (1.4) or for cases where $c(\cdot)$ or $d(\cdot)$ in Equation (1.2) may change sign, or for cases where $c(\cdot) \not\equiv d(\cdot)$ in Ω.

The only work we are aware of that deals with a class of systems that are not of gradient type is [20]; in this work, Chhetri et al. showed the existence of a solution (u,v) for Equation (1.1) such that u and v are positive in Ω, without necessarily requiring the system to be of gradient type. However, f_λ and g_μ, by assumption, must have subcritical growth (see [20, Theorem 2.8 and Condition (H1)]). The conditions on f_λ and g_μ are more restrictive than those addressed in the present paper; for instance, it is required that there exists $R_0 > 0$ such that $f_\lambda(x,t,s) > \lambda_1(\Omega)s+1$ for every $t \geq 0$ and $s > R_0$, and $g_\mu(x,t,s) > \lambda_1(\Omega)t+1$ for every $s \geq 0$ and $t > R_0$ (see [20, p. 43]), where $\lambda_1(\Omega)$ is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. In the present work, f_λ and g_μ do not necessarily need to satisfy the latter condition nor Condition (H1) from [20]; the class of nonlinearities addressed in this paper is more general and allows for nonlinearities with supercritical growth.

Candela and Sportelli in [17] considered the following system:

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) = G_u(x,u,v) & \text{in } \Omega, \\ -\mathrm{div}(b(x,v,\nabla v)) + B_t(x,v,\nabla v) = G_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $a(x,t,\xi) = (\partial_{\xi_1}A,\ldots,\partial_{\xi_N}A)$ and $b(x,t,\xi) = (\partial_{\xi_1}B,\ldots,\partial_{\xi_N}B)$, and the Euler–Lagrange functional associated with this system is defined on the Banach space $X := (W_0^{1,p_1}(\Omega) \cap L^\infty(\Omega)) \times (W_0^{1,p_2}(\Omega) \cap L^\infty(\Omega))$ (see [17, (1.6)]). Assuming certain conditions (see [17, $(h_0)-(h_7)$ and $(g_0)-(g_4)$]), among which $A(x,t,\xi)$ and $B(x,t,\xi)$

grow at least as fast as $(1+|t|^{s_1p_1})|\xi|^{p_1}$ and $(1+|t|^{s_2p_2})|\xi|^{p_2}$ respectively, where $p_i>1$, $s_i\geq 0$, the authors showed that if

$$\limsup_{(u,v)\to(0,0)}\frac{G(x,u,v)}{|u|^{p_1}+|v|^{p_2}}<\alpha_2\min\{\lambda_{1,1},\lambda_{2,1}\}\quad\text{uniformly a.e. in }\Omega,\tag{1.6}$$

where $\lambda_{i,1}$ is the first eigenvalue of $(-\Delta_{p_i},W_0^{1,p_i})$ and $\alpha_2>0$ is a constant related to the growth of A and B (see [17, Theorem 4.1]), then Equation (1.5) admits a non-trivial solution (see [17, Theorems 1.1 and 4.1]), assuming additional conditions they showed the existence of infinitely many solutions for Equation (1.5). The notable characteristic of the work of these authors is that they provided an existence result for the system (1.5) with the possibility of $G(x,u,v)$ having a supercritical growth, in the sense of, the exponents q_1 and q_2 that appear in the growth of G_u and G_v respectively (see [17, (g_1)]), satisfy:

$$1\leq q_i<p_i^*(s_i+1):=Np_i(s_i+1)/(N-p_i).\tag{1.7}$$

In this case, depending on the choice of s_i, it is possible to have $p_i^*<q_i<p_i^*(s_i+1)$ where p_i^* is the critical Sobolev exponent associated with $W_0^{1,p_i}(\Omega)$; the multiplicity result provided by these authors also allows for supercritical growth for G. However, no information about the sign of the solutions was given. When $A=(1/p_1)|\xi|^{p_1}$ and $B=(1/p_2)|\xi|^{p_2}$, the system (1.5) becomes

$$-\Delta_{p_1}u=G_u(x,u,v)\text{ and }-\Delta_{p_2}v=G_v(x,u,v)\text{ in }\Omega,\ u=v=0\text{ on }\partial\Omega.$$

Thus $s_1=s_2=0$ and the nonlinearities G_u and G_v only have subcritical growth (see [17, Remark 3.1]). Although the techniques we employ in this work are not only directed at gradient-type systems, our results include nonlinearities that do not satisfy Equation (1.6). For example, if $\gamma_1,\gamma_2>0$ and $\gamma_1+\gamma_2<p_i$ (or $\gamma_i<p_i$), condition (1.6) prevents G, near the origin, from having nonlinearities that grow like $|u|^{\gamma_1}|v|^{\gamma_2}$ (or $|u|^{\gamma_1}+|v|^{\gamma_2}$), which is a common type of nonlinearity in systems that generalize the results of [4].

The work of Adriouch and El Hamidi [1] arose before [46]. In their work, the authors consider Equation (1.3) as a gradient-type system with subcritical growth, where $\mathcal{L}_1=-\Delta_p$ and $\mathcal{L}_2=-\Delta_q$. However, the nonlinearity the authors considered is concave-convex only in one of the equations. For example, when $p=q=2$, system (1.3) takes the form (1.1), the exponents of the nonlinearities in Equation (1.2) satisfy $0<q_1<1<p_1=p_2<2^*$ and $0<q_2=1<p_1=p_2<2^*$. That is, the second equation has a linear term instead of a concave one.

Another fact worth mentioning is that, as in Equation (1.1), in general, the nonlinearity $\phi_i(x,u,v)$ of Equation (1.3) also depends on a parameter. Typically, we have $\phi_1(x,u,v)=\phi_{1,\lambda}(x,u,v)$ and $\phi_2(x,u,v)=\phi_{2,\mu}(x,u,v)$. In all the works we are aware of, the authors ensure the existence of a solution only if the parameters $\lambda>0$ and $\mu>0$ are sufficiently small. In other words, regarding the parameters, there are only local existence results. This raises an important question, for example: when $\phi_1=\phi_{1,\lambda}$ and $\phi_2=\phi_{2,\mu}$ are nonlinearities that generalize Equation (1.2), is the set $\mathcal{O}:=\{(\lambda,\mu)\in\mathbb{R}^2_+|\ (1.3)$ has a solution (u,v) with $u,v>0$ in $\Omega\}$ bounded or unbounded? We will show in this

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

work that, surprisingly, this set can be unbounded. In relation to the non-existence of a positive solutions for Equations (1.1)–(1.2) (as well as for Equation (1.3) equipped with a condition that generalizes Equation (1.2)), while there is ample literature for the scalar case, there are no studies involving the non-existence issue for systems. None of the previously mentioned works addresses the non-existence of solutions for systems.

Although our work is inspired by that of Ambrosetti–Brézis–Cerami, our present contribution addresses interesting cases which, to the best of our knowledge, have not been considered before, such as non-gradient type systems. We will also consider nonlinearities with supercritical growth and present novel existence results even for critical and subcritical cases, as well as providing both existence and non-existence results.

In this work, we are dealing with the following systems:

$$(\mathcal{P}_1)\ \begin{cases} -\Delta u = \lambda a(x)u^q + c(x)u^\alpha v^\beta & \text{in } \ \Omega \\ -\Delta v = \mu b(x)u^p + d(x)u^\theta v^\gamma & \text{in } \ \Omega \\ 0 \not\equiv u \geq 0, \ 0 \not\equiv v \geq 0 & \text{in } \ \Omega \\ u = v = 0 & \text{on } \ \partial\Omega \end{cases}$$

or

$$(\mathcal{P}_2)\ \begin{cases} -\Delta u = a(x)u^q + \lambda c(x)u^\alpha v^\beta & \text{in } \ \Omega \\ -\Delta v = b(x)v^p + \mu d(x)u^\theta v^\gamma & \text{in } \ \Omega \\ 0 \not\equiv u \geq 0, \ 0 \not\equiv v \geq 0 & \text{in } \ \Omega \\ u = v = 0 & \text{on } \ \partial\Omega \end{cases}$$

where $p, q \in (0,1)$, concerning the exponents, throughout this work, we will always assume that $\max\{\alpha, \beta\} > 1$ and $\max\{\theta, \gamma\} > 1$. More specifically, we have the following hypotheses:

(PW_1) $\alpha, \gamma \geq 1$ and $\beta, \theta > 0$;
(PW_2) $0 < \alpha < 1 < \beta$ and $0 < \gamma < 1 < \theta$;
(PW_3) $\alpha \geq 1, \ \beta > 0$ and $0 < \gamma < 1 < \theta$.

We will comment on the differences between $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ by comparing them with their scalar versions. Observe that u is a solution of $(\mathcal{P}_\lambda)$ if and only if $\overline{u} = \lambda^{\frac{-1}{1-q}} u$ is a solution to $(\mathcal{P}^{\overline{\lambda}})$ where $\overline{\lambda} = \lambda^{\frac{p-1}{1-q}}$, and

$$\begin{cases} -\Delta u = a(x)u^q + \lambda b(x)u^p & \text{in } \ \Omega \\ u = v = 0 & \text{on } \ \partial\Omega. \end{cases} \qquad (\mathcal{P}^\lambda)$$

When $a(x) \geq 0$, the global results provided by [26] imply global results for $(\mathcal{P}^\lambda)$. Here, global means that it is possible to establish for which parameters $\lambda > 0$ the problem has a solution and for which it does not, which contrasts with the local case where existence can only be established for small parameters. However, when $b(x) \geq 0$ and $a(x)$ change

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

sign, we cannot derive global results for $(\mathcal{P}^\lambda)$ from the results of [26]. Global results for $(\mathcal{P}^\lambda)$ were obtained by De Paiva in [27] when $b(x) \geq 0$ and $1 < p \leq 2^* - 1$.

Now, suppose that $(\alpha-1)(\gamma-1) \neq \theta\beta$. Then, for all $\overline{\lambda} > 0$ and $\overline{\mu} > 0$, the following system has a solution

$$\begin{cases} (\alpha-1)x + \beta y &= \ln\overline{\lambda} \\ \theta x + (\gamma-1)y &= \ln\overline{\mu}. \end{cases} \tag{1.8}$$

Then, $\lambda := e^{x(1-q)} > 0$ and $\mu := e^{y(1-p)} > 0$ satisfy $\overline{\lambda} = \lambda^{\frac{\alpha-1}{1-q}}\mu^{\frac{\beta}{1-p}}$ and $\overline{\mu} = \lambda^{\frac{\theta}{1-q}}\mu^{\frac{\gamma-1}{1-p}}$. Setting $t = \lambda^{\frac{-1}{1-q}}$ and $s = \mu^{\frac{-1}{1-p}}$, we see that u and v are solutions for $(\mathcal{P}_1)$ with $\lambda, \mu > 0$ if and only if $\overline{u} = tu$ and $\overline{v} = sv$ are solutions for $(\mathcal{P}_2)$ with $\overline{\lambda}, \overline{\mu} > 0$. Therefore, the global results for $(\mathcal{P}_1)$ with certain hypotheses on the weights yield global results for $(\mathcal{P}_2)$ with the same hypotheses.

But when the system (1.8) has no solution, this approach does not work. Therefore, even in the case that all weights are non-negative, we cannot obtain a solution (u_2, v_2) to $(\mathcal{P}_2)$ with $u_2 = tu_1$ and $v_2 = sv_1$ for some $t, s > 0$ and (u_1, v_1) solution to $(\mathcal{P}_1)$. This does not happen in the scalar case, since when the weights are non-negative, global results for $(\mathcal{P}_\lambda)$ guarantee global results for $(\mathcal{P}^\lambda)$.

We will see that the conditions (PW_1)–(PW_3) affect the set of values $\lambda, \mu > 0$ for which the system has a solution (u, v). Before formalizing the concept of the solution we will employ, it is pertinent to note that since the weight functions lie in $L^\infty(\Omega)$, our optimal expectation is that solutions belong to $C^1(\overline{\Omega})$. Thus, we shall consider the following definition:

Definition 1.1. *We will say that (u, v) is a solution to $(\mathcal{P}_i)$ if, for all $s>1$, u, $v \in W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega) \cap C^1(\overline{\Omega})$ and u, v satisfy $(\mathcal{P}_i)$.*

Concerning the existence of solutions, throughout this work, we will assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain such that $\partial\Omega \in C^{1,1}$. Except for the solution of item (v) of Theorem 1.4, this regularity is sufficient to obtain solutions in $W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega) \cap C^1(\overline{\Omega})$ for all $s>1$. To this end, we will invoke the following classical results from elliptic regularity theory: Theorems 7.26 and 9.15 of [31], which require that $\Omega \subset \mathbb{R}^N$ be a domain such that $\partial\Omega \in C^{0,1}$ and $\partial\Omega \in C^{1,1}$, respectively. Regarding the non-existence of solutions, we require the existence of $\Omega_0 \subset \Omega$ such that $\partial\Omega_0 \in C^{1,1}$, so that we can use Green's identities (which require $\partial\Omega_0 \in C^1$) and the results of [31, Theorems 9.15 and Lemma 9.17] in Proposition 2.7 (which require $\partial\Omega_0 \in C^{1,1}$).

Let $\Omega_0 \subset \Omega$ be open and connected. We say that a function $h : \overline{\Omega} \to \mathbb{R}$ satisfies the condition (P_{Ω_0}) if $h(x) \geq 0$ for all $x \in \Omega_0$, and the set $\{x \in \Omega_0 \mid h(x) > 0\}$ has positive Lebesgue measure. The conditions on the weights are stated below.

- (P_1) $a, b, c, d \in L^\infty(\Omega) \setminus \{0\}$;
- (P_2) $a(x) \geq 0$ and $b(x) \geq 0$ in Ω;

Regarding condition (P_1), we do not know if the results of the present work are valid for the case where the weight functions are in an appropriate L^p space. We have not delved into this issue.

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

Certain constants will appear in the form of powers associated with the questions of existence and non-existence. To better organize the presentation of our first theorem, we present these constants in the table below.

$\tau_{11} = \frac{\alpha-1}{1-q}$	$\sigma_{11} = \frac{\beta}{1-p}$	$\overline{\tau}_{11} = \frac{\theta}{1-q}$	$\overline{\sigma}_{11} = \frac{\gamma-1}{1-p}$
$\tau_{21} = 1$	$\sigma_{21} = 0$	$\overline{\tau}_{21} = 0$	$\overline{\sigma}_{21} = 1$
$\tau_{12} = 1$	$\sigma_{12} = 0$	$\overline{\tau}_{12} = 0$	$\overline{\sigma}_{12} = 1$
$\tau_{22} = 1-\gamma$	$\sigma_{22} = \beta$	$\overline{\tau}_{22} = \theta$	$\overline{\sigma}_{22} = 1-\alpha$
$\tau_{13} = 1$	$\sigma_{13} = 0$	$\overline{\tau}_{13} = \frac{\alpha-1}{1-q}$	$\overline{\sigma}_{13} = \frac{\beta}{1-p}$
$\tau_{23} = 1$	$\sigma_{23} = 0$	$\overline{\tau}_{23} = 1-\gamma$	$\overline{\sigma}_{23} = \beta$

In our first result, we show that there exist $\tau, \sigma, \overline{\tau}, \overline{\sigma} \geq 0$ satisfying $\tau+\sigma > 0$, $\overline{\tau}+\overline{\sigma} > 0$, and $\Lambda_1^*, \Lambda_2^* > 0$ such that problem $(\mathcal{P}_1)$ has no solution if $\lambda^\tau \mu^\sigma > \Lambda_1^*$ or $\lambda^{\overline{\tau}} \mu^{\overline{\sigma}} > \Lambda_2^*$. The same occurs with $(\mathcal{P}_2)$. Specifically, we prove the following:

Theorem 1.2 (Non-existence) *Consider the system $(\mathcal{P}_i)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain, and suppose that the weights $a(\cdot)$, $b(\cdot)$, $c(\cdot)$, $d(\cdot)$ satisfy the (P_{Ω_0}) condition for some $C^{1,1}$ domain $\Omega_0 \subset \Omega$, and that (PW_j) holds. Then, there are positive numbers $\Lambda_{1ij}^* > 0$ and $\Lambda_{2ij}^* > 0$, possibly depending on $a(\cdot)$, $b(\cdot)$, $c(\cdot)$, $d(\cdot)$, α, β, γ, θ, Ω_0, Ω, such that there are no solutions (u, v) for $(\mathcal{P}_i)$ in the sense of Definition 1.1, with $u > 0$ and $v > 0$ in Ω_0 when $\lambda^{\tau_{ij}} \mu^{\sigma_{ij}} > \Lambda_{1ij}^*$ or $\lambda^{\overline{\tau}_{ij}} \mu^{\overline{\sigma}_{ij}} > \Lambda_{2ij}^*$.*

In [4], the non-existence of positive solutions to $(\mathcal{P}_\lambda)$ is a consequence of the simple inequality: Given $c > 0$, there exists $\lambda = \lambda_c > 0$ such that $\lambda t^q + t^p > ct$, $\forall\, t > 0$ (see [4, (3.1) with $c = \lambda_1(\Omega)$]). However, even for the case $f_\lambda = \lambda t^q + t^\alpha s^\beta$ and $g_\mu = \mu s^p + t^\theta s^\gamma$, for all $\lambda, \mu > 0$, there is no $c > 0$ satisfying $f_\lambda(x, t, s) > ct$ and $g_\mu(x, t, s) > cs$ simultaneously for all $t, s > 0$. Hence, the approach for the non-existence of solutions is different in the context of systems. Lemma 2.1 plays a key role in our proof. It is a slight adaptation of [4, Lemma 3.3]. (Observe that in the work [4], this lemma is not relevant to show the result of non-existence.) Our proof is based on technical arguments, and in some cases, we use the Krein–Rutman theorem [37].

The problem $(\mathcal{P}_2)$ is related to $(\mathcal{P}^\lambda)$. In [27, Theorem 1], the author showed a global non-existence result when $0 < q < 1 < p \leq 2^*-1$, $b(x) \geq 0$ in Ω, and $\overline{\{x \in \Omega \mid a(x) > 0\}} \cap \overline{\{x \in \Omega \mid a(x) < 0\}} = \emptyset$ with other conditions. Therefore, concerning the non-existence question, Theorem 1.2 complements this result since the hypotheses considered are weaker than [27, Theorem 1] and we are considering supercritical powers, that is, $\alpha+\beta > 2^*-1$ or $\theta+\gamma > 2^*-1$ in $(\mathcal{P}_2)$.

Definition 1.3. *We say that (u, v) satisfying $(\mathcal{P}_i)$, in the sense of Definition 1.1, is a minimal positive solution when $u > 0$ and $v > 0$ in Ω. Moreover, if $u^* > 0$ and $v^* > 0$ in Ω, and (u^*, v^*) is another solution for $(\mathcal{P}_i)$, then $u \leq u^*$ and $v \leq v^*$ in Ω.*

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

In order to present our next theorem, we define the following hypothesis:

$(H_1) c(x) \geq 0$ in Ω;	$(H_4) d(x) \geq 0$ in Ω;
$(H_2) \alpha \geq 1$;	$(H_5) \gamma \geq 1$;
$(H_3) q < \alpha < 1$ and $\inf_{\Omega} a(x) > 0$;	$(H_6) p < \gamma < 1$ and $\inf_{\Omega} b(x) > 0$.

In our second result, in the main item, we show that there exist $\tau, \sigma, \overline{\tau}, \overline{\sigma} \geq 0$ satisfying $\tau + \sigma > 0$, $\overline{\tau} + \overline{\sigma} > 0$, and $\Lambda_1, \Lambda_2 > 0$ such that problem $(\mathcal{P}_1)$ has at least one solution if $\lambda^{\tau}\mu^{\sigma} \leq \Lambda_1$ and $\lambda^{\overline{\tau}}\mu^{\overline{\sigma}} \leq \Lambda_2$. The same occurs with $(\mathcal{P}_2)$. Specifically, we prove the following:

Theorem 1.4 (Positive Solution) *Consider the system $(\mathcal{P}_i)$ with $\Omega \subset \mathbb{R}^N$ being a $C^{1,1}$ bounded domain. Suppose that (PW_j) and (P_1)–(P_2) hold. In regard to the existence of solutions for $(\mathcal{P}_i)$ in the sense of Definition 1.1, there are positive numbers $\Lambda_{1ij} > 0$ and $\Lambda_{2ij} > 0$, possibly depending on $a(\cdot), b(\cdot), c(\cdot), d(\cdot), p, q, \alpha, \beta, \gamma, \theta, \Omega$, such that:*

(i) *Suppose that (H_l) and (H_k) hold with $1 \leq l \leq 3$ and $4 \leq k \leq 6$. Then $(\mathcal{P}_i)$ has a positive solution when $\lambda^{\tau_{ij}}\mu^{\sigma_{ij}} \leq \Lambda_{1ij}$ and $\lambda^{\overline{\tau}_{ij}}\mu^{\overline{\sigma}_{ij}} \leq \Lambda_{2ij}$;*

(ii) *If $c(x) \geq 0$ and $d(x) \geq 0$ in Ω, there exists $0 < L^*_{ij} \leq +\infty$ such that for all $\lambda \in (0, L^*_{ij})$, there is $0 < \Lambda_\lambda < +\infty$ such that for all $\mu \in (0, \Lambda_\lambda)$, problem $(\mathcal{P}_i)$ has a minimal positive solution. If $\mu \in (\Lambda_\lambda, +\infty)$, then there is no positive solution for $(\mathcal{P}_i)$. Moreover, $L^*_{11} = L^*_{22} = +\infty$, and $L^*_{ij} < +\infty$ for the other cases;*

(iii) *If $c(x) \geq 0$ and $d(x) \geq 0$ in Ω, there exists $0 < M^*_{ij} \leq +\infty$ such that for all $\mu \in (0, M^*_{ij})$, there is Λ_μ such that for all $\lambda \in (0, \Lambda_\mu)$, problem $(\mathcal{P}_i)$ has a minimal positive solution. If $\lambda \in (\Lambda_\mu, +\infty)$, then there is no positive solution for $(\mathcal{P}_i)$. Moreover, $M^*_{11} = M^*_{22} = M^*_{13} = M^*_{23} = +\infty$, and $M^*_{ij} < +\infty$ for the other cases;*

(iv) *If $c(x) \geq 0$ and $d(x) \geq 0$ in Ω, and $(u_{\lambda\mu}, v_{\lambda\mu})$ is a minimal positive solution for $(\mathcal{P}_i)$ with the parameters $\lambda, \mu > 0$, then for all $0 < \lambda_1 \leq \lambda$ and $0 < \mu_1 \leq \mu$, problem $(\mathcal{P}_i)$ has a minimal positive solution $(u_{\lambda_1\mu_1}, v_{\lambda_1\mu_1})$ with the parameters $\lambda_1, \mu_1 > 0$. Moreover, $u_{\lambda_1\mu_1} \leq u_{\lambda\mu}$ and $v_{\lambda_1\mu_1} \leq v_{\lambda\mu}$ in Ω.*

(v) *Suppose that $c(x) \geq 0$ and $d(x) \geq 0$ in Ω. Then, if $\alpha > 1$ and $\gamma > 1$, we have a positive weak solution for $(\mathcal{P}_i)$ when $\lambda \in (0, L^*_{ij})$ and $\mu = \Lambda_\lambda$. Similarly, we have a positive weak solution for $(\mathcal{P}_i)$ when $\mu \in (0, L^*_{ij})$ and $\lambda = \Lambda_\mu$.*

With the purpose of exemplifying the previous theorem, let us consider the following system:

$$\begin{cases} -\Delta u = \lambda u^q + u^\alpha v^\beta & \text{in } \Omega \\ -\Delta v = \mu u^p + u^\theta v^\gamma & \text{in } \Omega \\ u > 0, \ v > 0 & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

Supposing that $p, q \in (0, 1)$ and $\lambda, \mu > 0$. Regarding existence, Theorem 1.4-(i) ensures the existence of $\Lambda_i > 0$ such that:

(E_1) Assuming that $\alpha \geq 1$, $\gamma \geq 1$, $\beta > 0$, $\theta > 0$, then Equation (1.9) has a solution if:

$$\lambda^{\frac{\alpha-1}{1-q}} \mu^{\frac{\beta}{1-p}} \leq \Lambda_1 \text{ and } \lambda^{\frac{\theta}{1-q}} \mu^{\frac{\gamma-1}{1-p}} \leq \Lambda_2$$

(E_2) Assuming that $0 < \alpha < 1 < \beta$ and $0 < \gamma < 1 < \theta$, then Equation (1.9) has a solution if:

$$\lambda \leq \Lambda_1 \text{ and } \mu \leq \Lambda_2$$

(E_3) Assuming that $\alpha \geq 1$, $\beta > 0$, and $0 < \gamma < 1 < \theta$, then Equation (1.9) has a solution if:

$$\lambda \leq \Lambda_1 \text{ and } \lambda^{\frac{\alpha-1}{1-q}} \mu^{\frac{\beta}{1-p}} \leq \Lambda_2.$$

Note that if $\alpha > 1$ or $\gamma > 1$, then the result of (E_1) ensures that the set $\mathcal{E} := \{(\lambda, \mu) \in \mathbb{R}^2_+ | \text{ (1.9) has a solution} \}$ is unbounded. Furthermore, if $\alpha > 1$, then the result of (E_3) also implies that $\mathcal{E}$ is unbounded. These results contrast with the scalar case, since $\{\lambda \in \mathbb{R}_+ \mid (\mathcal{P}_\lambda) \text{ has a solution} \} = (0, \Lambda]$ and $\Lambda < +\infty$ for the case where the weight functions of $(\mathcal{P}_\lambda)$ are constants.

To prove Theorem 1.4, we use the sub-super solution method. The way we define the concept of sub-super solution for systems is somewhat different than in the scalar case. For example, a solution of $(\mathcal{P}_i)$ may not be a sub-solution or a super-solution for $(\mathcal{P}_i)$ (see Remarks 3.1 and 3.5).

The super-solutions are obtained by solving the systems Equation (3.1) and (3.2). The greatest difficulty lies in obtaining solutions when one of the parameters $\lambda > 0$ or $\mu > 0$ can be arbitrarily large. The sub-solutions come in two types of functions. To obtain the first, we use the theory of principal eigenfunction for operators with indefinite weight functions. For that, we choose an appropriate weight function (see, for instance, Equation (3.4)). To obtain the second sub-solution, we solve an auxiliary problem employing variational techniques (see Proposition 3.6).

The novelty of this theorem lies in the fact that we solve the problem of the existence of a solution for a more general class of systems (not just gradient systems and not just with critical or subcritical growth powers), and we prove that the set of solutions $\mathcal{O}_i := \left\{(\lambda, \mu) \in \mathbb{R}^2_+ \mid (\mathcal{P}_i) \text{ has solution} \right\}$ is unbounded in some cases. To prove (v), we use the same approach as in [4, p. 528]. However, the same idea does not work when $\alpha \leq 1$ or $\gamma \leq 1$, although we believe that this statement holds for other cases. Unfortunately, we cannot prove it. Despite this, in [23], we have a complete proof for when the system is of the gradient type.

Originally, this work contained a study involving the multiplicity of solutions for gradient-type systems, as well as analyzing cases where all weights change sign and the nonlinearities have supercritical growth terms. However, we decided to address this in another paper [23]. The issue of local minimization for functionals with two variables was also addressed in another paper [24]. However, the question of multiplicity for a system that is not of the gradient type remains open. A major difficulty is that if the system cannot be studied via variational methods, we do not know how to approach the issue of multiplicity. Unfortunately, we were unable to answer some other questions. For example, can we have the maps $\mu \mapsto \Lambda_\mu$ and $\lambda \mapsto \Lambda_\lambda$ continuous? Is it possible that $\Lambda_{1ij} = \Lambda^*_{1ij}$ and $\Lambda_{2ij} = \Lambda^*_{2ij}$? If $0 < \lambda = L^*_{ij} < +\infty$, can we have $\Lambda_\lambda > 0$? If $0 < \mu = M^*_{ij} < +\infty$, can we have $\Lambda_\mu > 0$?

After the introduction, this work is divided into three other sections. The second section is dedicated to the question of non-existence of solutions for $(\mathcal{P}_i)$. In the third section, we present the method of sub-super solution, and finally, in the fourth and last section, we present the proof of Theorem 1.4.

2. Non-existence results

This section is dedicated to non-existence issues for the problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$. The proof of the following lemma is an adaptation of [4, Lemma 3.3].

Lemma 2.1. *Let $\Omega_0 \subset \mathbb{R}^N$ be a C^1 bounded domain, $m \in L^\infty(\Omega_0)$ which satisfies $0 \leq m(x) \not\equiv 0$ in Ω_0, assume that f(t) is a function such that $t^{-1}f(t)$ is decreasing for $t > 0$. Suppose that ϕ, $\varphi \in W^{2,s}(\Omega_0) \cap C^1(\overline{\Omega}_0)$ and $Q > -\lambda_1(\Omega_0)$ satisfies*

$$\begin{cases} -\Delta\varphi + Q\varphi \geq m(x)f(\varphi), & x \in \Omega_0 \\ \varphi > 0, & x \in \Omega_0 \end{cases} \quad \textit{and} \quad \begin{cases} -\Delta\phi + Q\phi \leq m(x)f(\phi), & x \in \Omega_0 \\ \phi > 0, & x \in \Omega_0 \\ \phi = 0, & x \in \partial\Omega_0 \end{cases}$$

then $\varphi \geq \phi$ in Ω_0.

Proof. Let $\Theta \in C^1(\mathbb{R})$ be a nondecreasing function such that $\Theta(t) = 0$ for $t \leq 0$ and $\Theta(x) = 1$ for $t \geq 1$. Setting $\Theta_\varepsilon(t) = \Theta(\frac{t}{\varepsilon})$, as Ω_0 is a C^1 bounded domain, we can then invoke the Green's formula and proceed as in [4, Lemma 3.3] to get

$$\int_{\Omega_0} m(x)\phi\varphi \left[\frac{f(\varphi)}{\varphi} - \frac{f(\phi)}{\phi}\right] \Theta_\varepsilon(\phi - \varphi) \leq -\int_{\Omega_0} \widehat{\Theta}_\varepsilon(\phi - \varphi)\Delta\phi,$$

where $\widehat{\Theta}_\varepsilon(t) := \int_0^t s\Theta'_\varepsilon(s)ds$. Since $0 \leq \widehat{\Theta}_\varepsilon(t) \leq \varepsilon$, then

$$0 \leq \int_{\Omega_0} m(x)\phi\varphi \left[\frac{f(\varphi)}{\varphi} - \frac{f(\phi)}{\phi}\right] \Theta_\varepsilon(\phi - \varphi)dx \leq \varepsilon \left(-\int_{\Omega_0} \Delta\phi\right).$$

Since $t^{-1}f(t)$ is decreasing for $t > 0$, taking $\varepsilon \to 0^+$ in the above expression, we obtain that

$$\int_{\Omega_1} m(x)\phi\varphi \left[\frac{f(\varphi)}{\varphi} - \frac{f(\phi)}{\phi}\right] dx = 0$$

where $\Omega_1 := \{x \in \Omega_0 \mid \phi(x) > \varphi(x)\} \subset\subset \Omega_0$, which is an open set. Since $t^{-1}f(t)$ is decreasing for $t > 0$, if $\Omega_1 \neq \emptyset$, we have that $m(x) = 0 \ \forall x \in \Omega_1$, so $-\Delta(\varphi - \phi) + Q(\varphi - \phi) \geq$

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

0 in Ω_1 and $\varphi - \phi = 0$ on $\partial\Omega_1$. Thus, $\varphi \geq \phi$ in Ω_1, which is a contradiction. Therefore, $\Omega_1 = \emptyset$, and the lemma is proved. □

Remark 2.2. Let Ω be a $C^{1,1}$ bounded domain and $m \in L^\infty(\Omega)$ with $m^+(x) \not\equiv 0$ in Ω. From [33], we can define $\lambda_{1m}(\Omega)$ and $\varphi_{1m\Omega}$ respectively as the principal eigenvalue and first eigenfunction to the problem

$$\begin{cases} -\Delta\varphi = \lambda m(x)\varphi & , x \in \Omega \\ \varphi = 0 & , x \in \partial\Omega. \end{cases}$$

Moreover,

$$0 < \frac{1}{\lambda_{1m}(\Omega)} = \sup_{w \in H_0^1(\Omega)} \frac{\int_\Omega m(x) w^2 \, \mathrm{d}x}{\int_\Omega |\nabla w|^2 \, \mathrm{d}x}$$

and $\varphi_{1m\Omega} > 0$ in Ω. Observe that by the standard regularity theory, we have $\varphi_{1m\Omega} \in W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega) \cap C^1(\overline{\Omega})$ for $s > 1$.

In what follows, for the set $\Omega_0 \subset \Omega$ and $m \in L^\infty(\Omega_0)$ with $m^+(x) \not\equiv 0$ in Ω_0, we will write λ_{1m} and φ_{1m} instead of $\lambda_{1m\Omega_0}$ and $\varphi_{1m\Omega_0}$. We will also consider throughout this work that $\|\varphi_{1m}\|_\infty = 1$.

Corollary 2.3. *Suppose that for some C^1 bounded domain $\Omega_0 \subset \Omega$, we have $-\Delta w \geq tm(x)w^r$ in Ω_0, where $w \in W^{2,s}(\Omega) \cap C^1(\overline{\Omega})$, $t > 0$, $0 < r < 1$, $0 \leq m(x) \not\equiv 0$, $x \in \Omega_0$, and $w > 0$ in Ω_0. Then we have*

$$w(x) \geq \left(\frac{t}{\lambda_{1m}}\right)^{\frac{1}{1-r}} \varphi_{1m}(x), \quad \forall x \in \Omega_0.$$

Proof. It is easy to verify that $\overline{\varphi}_{1m}(x) = \left(\frac{t}{\lambda_{1m}}\right)^{\frac{1}{1-r}} \varphi_{1m}(x)$ satisfies $-\Delta\overline{\varphi}_{1m} \leq tg(x)\overline{\varphi}_{1m}^r$, so the corollary is a direct consequence of Lemma 2.1. □

The following two propositions will provide sufficient conditions for systems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ to have no solution, in the sense of Definition 1.1. In this section, $\Omega \subset \mathbb{R}^N$ is a bounded domain, and $\Omega_0 \subset \Omega$ is a domain whose boundary will have regularity C^1 or $C^{1,1}$.

The only requirement we will impose on the signs of the weight functions is that they must satisfy condition (P_{Ω_0}), where $\Omega_0 \subset \Omega$. That is, these functions must be non-negative and non-zero in Ω_0. Outside of Ω_0, we are not imposing any specific behavior regarding the sign; in other words, the functions may or may not have an indefinite sign in Ω.

Proposition 2.4. *Suppose that for some C^1 domain $\Omega_0 \subset \Omega$, the weights $a(\cdot)$, $b(\cdot)$, $c(\cdot)$, $d(\cdot)$ satisfy the condition (P_{Ω_0}), and $\alpha \geq 1$, β, γ, $\theta > 0$. Then there is a number $\Lambda_1^* := \Lambda_1(a(\cdot), b(\cdot), c(\cdot), d(\cdot), p, q, \alpha, \beta) > 0$ such that:*

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

(i) *There are no solutions for* $(\mathcal{P}_1)$ *with* $\lambda^{\frac{\alpha-1}{1-q}}\mu^{\frac{\beta}{1-p}} > \Lambda_1^*$;
(ii) *There are no solutions for* $(\mathcal{P}_2)$ *with* $\lambda > \Lambda_1^*$.

Proof. In what follows we will consider $x \in \Omega_0$.
Case (i): Since $-\Delta u \geq \lambda a(x)u^q$ and $-\Delta v \geq \mu b(x)v^p$ by Corollary 2.3, we have

$$u \geq \left(\frac{\lambda}{\lambda_{1a}}\right)^{\frac{1}{1-q}}\varphi_{1a} \ \text{ and } \ v \geq \left(\frac{\mu}{\lambda_{1b}}\right)^{\frac{1}{1-p}}\varphi_{1b}. \tag{2.1}$$

Since $-\Delta u \geq c(x)u^\alpha v^\beta$ by Equation (2.1), we get $-\Delta u \geq \lambda^{\frac{\alpha-1}{1-q}}\mu^{\frac{\beta}{1-p}}\overline{c}(x)u$, where $\overline{c}(x) := \lambda_{1a}^{\frac{1-\alpha}{1-q}}\lambda_{1b}^{\frac{-\beta}{1-p}}\varphi_{1a}^{\alpha-1}(x)\varphi_{1b}^{\beta}(x)c(x)$ so $-\varphi_{1\overline{c}}\Delta u \geq \lambda^{\frac{\alpha-1}{1-q}}\mu^{\frac{\beta}{1-p}}\overline{c}(x)\varphi_{1\overline{c}}u$, a simple computation provides $\lambda^{\frac{\alpha-1}{1-q}}\mu^{\frac{\beta}{1-p}} \leq \lambda_{1\overline{c}}$.

Case (ii): In this case, we have $u \geq \lambda_{1a}^{\frac{-1}{1-q}}\varphi_{1a}$ and $v \geq \lambda_{1b}^{\frac{-1}{1-p}}\varphi_{1b}$, since $-\Delta u \geq \lambda c(x)u^\alpha v^\beta$ then $-\Delta u \geq \lambda\overline{c}(x)u$, as in the last case we have $\lambda \leq \lambda_{1\overline{c}}$. □

Remark 2.5. Suppose that (PW_1) holds, and $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ and $d(\cdot)$ satisfy the (P_{Ω_0}) condition,for some C^1 domain $\Omega_0 \subset \Omega$. From Proposition 2.4, there are numbers $\Lambda_i^* > 0$, $i = 1, 2$, such that:

(i) There is no solution (u, v) for $(\mathcal{P}_1)$ with $\lambda^{\frac{\alpha-1}{1-q}}\mu^{\frac{\beta}{1-p}} > \Lambda_1^*$ or $\mu^{\frac{\gamma-1}{1-p}}\lambda^{\frac{\theta}{1-q}} > \Lambda_2^*$;
(ii) There is no solution (u, v) for $(\mathcal{P}_2)$ with $\lambda > \Lambda_1^*$ or $\mu > \Lambda_2^*$.

Before we present the next non-existence theorem, we will present a version of the Krein–Rutman theorem [37]. The proof of this result can be found in [40] or [11, Problem 41, p. 499]. This theorem plays a crucial role in the proof of our non-existence theorem.

Theorem 2.6 (Krein–Rutman) *Let E be a Banach space and let P be a convex cone with vertex at 0, i.e. $\lambda u + \mu v \in P$, $\forall \lambda, \mu \geq 0$, $\forall u, v \in P$. Assume that P is closed, $IntP \neq \emptyset$ and $P \neq E$. Let $T : E \to E$ be a compact operator such that $T(P \setminus \{0\}) \subset IntP$. Then there exists some $u_0 \in intP$ and some $\lambda_0 > 0$ such that $Tu_0 = \lambda_0 u_0$.*

Proposition 2.7. *Suppose that (PW_2) holds, and for some $C^{1,1}$ bounded domain $\Omega_0 \subset \Omega$, the weights $a(\cdot), b(\cdot), c(\cdot), d(\cdot)$ satisfy the (P_{Ω_0}) condition. Then there are positive numbers $\Lambda_i^* := \Lambda_i^*(a(\cdot), b(\cdot), c(\cdot), d(\cdot), p, q, \alpha, \beta, \theta, \gamma)$ such that:*

(i) *There is no solution (u, v) for $(\mathcal{P}_1)$, in the sense of Definition 1.1, with $u(x), v(x) > 0$, $\forall x \in \Omega_0$, and $\lambda > \Lambda_1^*$ or $\mu > \Lambda_2^*$;*
(ii) *There is no solution (u, v) for $(\mathcal{P}_2)$, in the sense of Definition 1.1, with $u(x), v(x) > 0$, $\forall x \in \Omega_0$, and $\lambda^{1-\gamma}\mu^\beta > \Lambda_3^*$ or $\lambda^\theta\mu^{1-\alpha} > \Lambda_4^*$.*

Proof. In what follows, we will consider $x \in \Omega_0$.

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

Case (i): Suppose that (u, v) is a solution for $(\mathcal{P}_1)$ with $u(x), v(x) > 0, \ \forall x \in \Omega_0$. From Corollary 2.3, we have

$$u \geq \left(\frac{\lambda}{\lambda_{1a}}\right)^{\frac{1}{1-q}} \varphi_{1a}. \tag{2.2}$$

Since $-\Delta v \geq d(x) u^\theta v^\gamma$ by Equation (2.2), we get

$$-\Delta v \geq \lambda^{\frac{\theta}{1-q}} \sigma(x) v^\gamma$$

where $\sigma(x) := \lambda_{1a}^{\frac{-\theta}{1-q}} d(x) \varphi_{1a}^\theta(x)$. Since $0 < \gamma < 1$, from Corollary 2.3, we obtain

$$v(x) \geq \lambda^{\frac{\theta}{(1-q)(1-\gamma)}} \lambda_{1\sigma}^{\frac{-1}{1-\gamma}} \varphi_{1\sigma}(x). \tag{2.3}$$

Since $-\Delta u \geq c(x) u^\alpha v^\beta$, Equations (2.2) and (2.3) provide

$$-\Delta u \geq \lambda^{\tau_1} \sigma_1(x) v \tag{2.4}$$

where $\sigma_1(x) = \lambda_{1a}^{\frac{-\alpha}{1-q}} \lambda_{1\sigma}^{\frac{-(\beta-1)}{1-\gamma}} c(x) \varphi_{1a}^\alpha(x) \varphi_{1\sigma}^{\beta-1}(x)$ and $\tau_1 = \frac{\alpha}{1-q} + \frac{\theta(\beta-1)}{(1-q)(1-\gamma)}$. Once again using $-\Delta v \geq d(x) u^\theta v^\gamma$ with Equations (2.2) and (2.3), we get

$$-\Delta v \geq \lambda^{\tau_2} \sigma_2(x) u \tag{2.5}$$

where $\sigma_2(x) = \lambda_{1a}^{\frac{1-\theta}{q-1}} \lambda_{1\sigma}^{\frac{-\gamma}{1-\gamma}} d(x) \varphi_{1\sigma}^\gamma(x) \varphi_{1a}^{\theta-1}(x)$ and $\tau_2 = \frac{\theta-1}{1-q} + \frac{\theta\gamma}{(1-q)(1-\gamma)}$.

Claim: There is a principal eigenvalue $\lambda_{12} > 0$ and principal eigenfunctions $\phi_1, \ \phi_2 > 0, \ \forall x \in \Omega_0$ belonging to $W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega_0) \cap C^1(\overline{\Omega}_0), \ \forall s > 1$ for the problem:

$$\begin{cases} -\Delta \phi_1 &= \lambda_{12} \sigma_1(x) \phi_2, \ \Omega_0 \\ -\Delta \phi_2 &= \lambda_{12} \sigma_2(x) \phi_1, \ \Omega_0. \end{cases} \tag{$\mathcal{LP}$}$$

From Equation (2.4), we get

$$\lambda_{12} \int_{\Omega_0} \sigma_2(x) u \phi_1 \, \mathrm{d}x = \int_{\Omega_0} -u \Delta \phi_2 \, \mathrm{d}x \geq -\int_{\Omega_0} \phi_2 \Delta u \, \mathrm{d}x \geq \lambda^{\tau_1} \int_{\Omega_0} \sigma_1(x) v \phi_2 \, \mathrm{d}x.$$

Therefore,

$$\lambda_{12} \int_{\Omega_0} \sigma_2(x) u \phi_1 \, \mathrm{d}x \geq \lambda^{\tau_1} \int_{\Omega_0} \sigma_1(x) v \phi_2 \, \mathrm{d}x. \tag{2.6}$$

Similarly, we get

$$\lambda_{12} \int_{\Omega_0} \sigma_1(x) v \phi_2 \, \mathrm{d}x \geq \lambda^{\tau_2} \int_{\Omega_0} \sigma_2(x) u \phi_1 \, \mathrm{d}x. \tag{2.7}$$

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

Follows from Equations (2.6) and (2.7) that $\lambda^{\tau_1+\tau_2} \le \lambda_{12}^2$, so it is enough to consider $\Lambda_1^* = \lambda_{12}^{\frac{2}{\tau_1+\tau_2}}$. Now we will prove the Claim. Let $E = C^1(\overline{\Omega}_0) \cap H_0^1(\Omega_0)$ and $P = \{u \in E \mid u(x) \ge 0,\, x \in \Omega_0\}$. It is known that

$$\text{int } P = \left\{ u \in E \mid u(x) > 0,\, x \in \Omega_0,\; \frac{\partial u}{\partial \nu} < 0,\, x \in \partial\Omega_0 \right\}.$$

For $i = 1, 2$, we set the operators $T_i : E \to E$ by $T_i(u) = v$, where $v \in W^{2,s}(\Omega_0) \cap W_0^{1,s}(\Omega_0) \cap C^1(\overline{\Omega_0})$ is the unique solution of the problem

$$-\Delta v = \sigma_i(x)u \text{ in } \Omega_0, \quad v(x) = 0 \text{ on } \partial\Omega_0,$$

from [31, Theorems 7.26, 9.15 and Lemma 9.17], we have that T_i is well-defined and $T := T_2 \circ T_1$ is a compact operator. Moreover, utilizing the maximum principles [31, Theorems 8.1 and 8.19] along with Hopf's lemma [35] (see also [31, Lemma 3.4] and Remark 2.8), we have $T(P \setminus \{0\}) \subset \text{int } P$; therefore, Theorem 2.6 provides $u_0 \in \text{int } P$ and $\lambda_0 > 0$ such that $Tu_0 = \lambda_0 u_0$, hence $\lambda_{12} = \lambda_0^{-1/2}$, $\phi_1 = \lambda_0^{-1/2} T_1 u_0$ and $\phi_2 = u_0$ are solutions for $(\mathcal{LP})$. In a similar way, we prove that for some Λ_2^*, there are no solutions for $\mu > \Lambda_2^*$.
Case (ii): In what follows, we consider $x \in \Omega_0$. Suppose that (u, v) is a solution for $(\mathcal{P}_2)$. Since $-\Delta u \ge a(x)u^q$ and $-\Delta v \ge b(x)v^p$, we have

$$u \ge \lambda_{1a}^{\frac{-1}{1-q}} \varphi_{1a} \quad \text{and} \quad v \ge \lambda_{1b}^{\frac{-1}{1-p}} \varphi_{1b}. \tag{2.8}$$

Since $-\Delta u \ge \lambda c(x)u^\alpha v^\beta$ and $-\Delta v \ge \mu d(x)u^\theta v^\gamma$, we have

$$-\Delta u \ge \lambda\sigma(x)u^\alpha \quad \text{and} \quad -\Delta v \ge \mu\overline{\sigma}(x)v^\gamma, \tag{2.9}$$

where $\sigma(x) = \lambda_{1b}^{\frac{-\beta}{1-p}} c(x)\varphi_{1b}^\beta(x)$ and $\overline{\sigma}(x) = \lambda_{1a}^{\frac{-\theta}{1-q}} d(x)\varphi_{1a}^\theta(x)$. By Equation (2.9), we get

$$u \ge \left(\frac{\lambda}{\lambda_{1\sigma}}\right)^{\frac{1}{1-\alpha}} \varphi_{1\sigma} \quad \text{and} \quad v \ge \left(\frac{\mu}{\lambda_{1\overline{\sigma}}}\right)^{\frac{1}{1-\gamma}} \varphi_{1\overline{\sigma}}. \tag{2.10}$$

From Equations (2.8) and (2.10), $-\Delta u \ge \lambda c(x)u^\alpha v^\beta$ and $-\Delta v \ge \mu d(x)u^\theta v^\gamma$, we obtain

$$\begin{cases} -\Delta u \ge \lambda^{\frac{1}{1-\alpha}} \overline{c}(x)v \\ -\Delta v \ge \lambda^{\frac{\theta-1}{1-\alpha}} \mu\overline{d}(x)u \end{cases} \tag{2.11}$$

where $\overline{c}(x) = \lambda_{1\sigma}^{\frac{-\alpha}{1-\alpha}} \lambda_{1b}^{\frac{1-\beta}{1-p}} \varphi_{1\sigma}^\alpha(x)\varphi_{1b}^{\beta-1}(x)c(x)$ and $\overline{d}(x) = \lambda_{1\sigma}^{\frac{1-\theta}{1-\alpha}} \lambda_{1b}^{\frac{-\gamma}{1-p}} \sigma_{1\sigma}^{\theta-1}(x)\varphi_{1b}^\gamma(x)d(x)$. From Equation (2.11), we can proceed as in the proof of Case (i) to obtain Λ_3^* such that

$$(\Lambda_3^*)^{\frac{1}{1-\alpha}} \ge \lambda^{\frac{1}{1-\alpha}} \lambda^{\frac{\theta-1}{1-\alpha}} \mu = \lambda^{\frac{\theta}{1-\alpha}} \mu \implies \Lambda_3^* \ge \lambda^\theta \mu^{1-\alpha}.$$

In a similar way, we get $\Lambda_4^* > 0$ such that $\Lambda_4^* \ge \lambda^{1-\gamma}\mu^\beta$. □

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

Remark 2.8. Traditionally, the Hopf Lemma [35] is presented for functions of class C^2. However, this result can be easily extended to functions in $W^{2,s}(\Omega) \cap C^1(\overline{\Omega})$ with $\partial\Omega \in C^{1,1}$. Indeed, although the proof given in [31, Lemma 3.4] is for functions in $C^2(\Omega)$, it essentially employs comparison principles for functions in $C^2(\Omega) \cap C^0(\overline{\Omega})$. Since $\partial\Omega \in C^{1,1}$, Ω satisfies the interior sphere condition, ensuring that the proof of this result remains unchanged if we use the comparison principles for functions in $W^{1,2}(\Omega)$ (see [31, Theorems 8.1 and 8.19], and also [31, Sections 9.7–9.9] for maximum principles for functions in $W^{2,N}(\Omega)$).

Following the same idea of Proposition 2.4, part (i), and Proposition 2.7, part (ii), we have

Corollary 2.9. *Suppose that $a(\cdot)$, $b(\cdot)$, $c(\cdot)$, $d(\cdot)$ satisfy the (P_{Ω_0}) condition for some $C^{1,1}$ domain $\Omega_0 \subset \Omega$, and (PW_3) holds. Then there are $\Lambda_i^* > 0$, $i = 1, 2, 3, 4$, satisfying:*

(i) *There is no solution (u, v) for $(\mathcal{P}_1)$ with $\lambda > \Lambda_1^*$ or $\lambda^{\frac{\alpha-1}{1-q}} \mu^{\frac{\beta}{1-p}} > \Lambda_2^*$;*
(ii) *There is no solution (u, v) for $(\mathcal{P}_2)$ with $\lambda > \Lambda_3^*$ or $\lambda^{1-\gamma}\mu^{\beta} > \Lambda_4^*$.*

Proof of Theorem 1.2: The proof of this theorem is a direct consequence of Proposition 2.4, Proposition 2.7 and Corollary 2.9. □

3. The sub-supersolution method

In this section, we write $\phi^{\pm}(x) := \max\{0, \pm\phi(x)\}$ and $\|\phi\|_\infty := |\phi|_{L^\infty(\Omega)}$. We always assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with $\partial\Omega \in C^{1,1}$. The following notation is of fundamental importance for the definition of sub and super-solutions: If $\phi \in L^\infty(\Omega)$, we set

$$[0, \phi] := \{w \in L^\infty(\Omega) \mid \ 0 \le w(x) \le \phi(x)\}.$$

We define $(\overline{u}, \overline{v})$ as a supersolution for $(\mathcal{P}_i)$ if, for all $s > 1$, $\overline{u}, \overline{v} \in W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega) \cap C^1(\overline{\Omega})$, and $\overline{u}, \overline{v}$ satisfy:

$$\begin{cases} -\Delta\overline{u} \ge f_\lambda(x, z, w) & \text{in } \ \Omega, \quad \forall z \in [0, \overline{u}], \ \forall w \in [0, \overline{v}] \\ -\Delta\overline{v} \ge g_\mu(x, z, w) & \text{in } \ \Omega, \quad \forall z \in [0, \overline{u}], \ \forall w \in [0, \overline{v}] \\ \overline{u}, \ \overline{v} > 0 & \text{in } \ \Omega \\ \overline{u} = \overline{v} = 0 & \text{on } \ \partial\Omega \end{cases} \tag{$\bar{\mathcal{S}}_{\lambda\mu}$}$$

where $f_\lambda(x, z, w) = \lambda a(x) z^q + c(x) z^\alpha w^\beta$ and $g_\mu(x, z, w) = \mu b(x) w^p + d(x) z^\theta w^\gamma$ for $(\mathcal{P}_1)$, and $f_\lambda(x, z, w) = a(x) z^q + \lambda c(x) z^\alpha w^\beta$ and $g_\mu(x, z, w) = b(x) w^p + \mu d(x) z^\theta w^\gamma$ for $(\mathcal{P}_2)$.

Remark 3.1. When all weight functions $a, b, c, d \in L^\infty(\Omega)$ are non-negative, then a solution (u, v) to $(\mathcal{P}_i)$ with $\lambda, \mu > 0$ is also a **super-solution** satisfying $(\overline{\mathcal{S}}_{\lambda\mu})$. However, if at least one of the weight functions changes sign in Ω or is negative on a set of positive measure, then a solution (u,v) to $(\mathcal{P}_i)$ may not be a super-solution satisfying $(\overline{\mathcal{S}}_{\lambda\mu})$. This is a very relevant difference for the concept of super-solutions for systems, since in the scalar case, a solution is always a super-solution.

In the scalar case $(\mathcal{P}_\lambda)$ with $a \equiv b \equiv 1$, the authors of [4] obtained the first solution via the method of sub-super solutions. The super-solution for $(\mathcal{P}_\lambda)$ provided by these authors takes the form Me, where $e \in C^2(\Omega)\cap C(\overline{\Omega})$ satisfies $-\Delta e = 1$ in Ω and $e=0$ on $\partial\Omega$. For this super-solution to exist, it is sufficient that the inequality $M \geq \lambda M^q\|e\|_\infty^q + M^p\|e\|_\infty^p$ admits a solution $M = M(\lambda) > 0$ (see [4, Lemma 3.1]). In our approach, the super-solutions take the form $\overline{u} = Xe$ and $\overline{v} = Ye$, where $X = X(\lambda,\mu) > 0$ and $Y = Y(\lambda,\mu) > 0$ are real numbers, which depending on $(\mathcal{P}_1)$ or $(\mathcal{P}_2)$, satisfy one of the following systems:

$$\begin{cases} X \geq \lambda A X^q + C X^\alpha Y^\beta \\ Y \geq \mu B Y^p + D X^\theta Y^\gamma \\ X>0,\ Y>0 \end{cases} \tag{3.1}$$

$$\begin{cases} X \geq A X^q + \lambda C X^\alpha Y^\beta \\ Y \geq B Y^p + \mu D X^\theta Y^\gamma \\ X>0,\ Y>0, \end{cases} \tag{3.2}$$

where $A,B,C,D \in \mathbb{R}$, $p,q \in (0,1)$, $\alpha,\beta,\gamma,\theta > 0$, $\max\{\alpha,\beta\} > 1$ and $\max\{\gamma,\theta\} > 1$. For which values of $\lambda>0$ and $\mu>0$ do the systems (3.1) and (3.2) have a solution (X,Y)? In Lemma 3.3, we will delve into this question; however, it should be noted that systems (3.1) and (3.2) always have a solution when $\lambda>0$ and $\mu>0$ are sufficiently small. We will register this fact in a brief remark.

Remark 3.2. If we look for solutions of type $X = Y > 0$, since $p,q \in (0,1)$, $\max\{\alpha,\beta\} > 1$, and $\max\{\theta,\gamma\} > 1$, it is easy to see that

$$X = Y = \max\{(2A\lambda)^{\frac{1}{1-q}}, (2B\mu)^{\frac{1}{1-p}}\}$$

with $\max\{(2A\lambda)^{\frac{1}{1-q}}, (2B\mu)^{\frac{1}{1-p}}\} \leq \min\{(2C)^{\frac{-1}{\alpha+\beta-1}}, (2D)^{\frac{-1}{\theta+\gamma-1}}\}$ satisfies Equation (3.1) and

$$X = Y = \min\{(2\lambda C)^{\frac{-1}{\alpha+\beta-1}}, (2\mu D)^{\frac{-1}{\theta+\gamma-1}}\}$$

with $\max\{(2A)^{\frac{1}{1-q}}, (2B)^{\frac{1}{1-p}}\} \leq \min\{(2\lambda C)^{\frac{-1}{\alpha+\beta-1}}, (2\mu D)^{\frac{-1}{\theta+\gamma-1}}\}$ satisfies Equation (3.2). Then, there is $\Lambda > 0$ such that Equations (3.1) and (3.2) always have solutions for all $\lambda,\mu \in (0,\Lambda]$, so we are interested in analyzing cases where $\lambda>0$ or $\mu>0$ can be arbitrarily large.

Lemma 3.3. *Suppose that A, B, C, $D > 0$, then there are $\Lambda_{kij} = \Lambda_{kij}(A,B,C,D,\alpha,\beta,\gamma,\theta) > 0$ such that*

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

(i) *If* (PW_1) *holds, then the system* (3.1) *has a solution* (X, Y) *if*

$$\lambda^{\frac{\alpha-1}{1-q}}\mu^{\frac{\beta}{1-p}} \leq \Lambda_{111} \ \textit{and}\ \lambda^{\frac{\theta}{1-q}}\mu^{\frac{\gamma-1}{1-p}} \leq \Lambda_{211};$$

(ii) *If* (PW_1) *holds, then the system* (3.2) *has a solution if* $\lambda \leq \Lambda_{121}$ *and* $\mu \leq \Lambda_{221}$*;*

(iii) *If* (PW_2) *holds, then the system* (3.1) *has a solution if* $\lambda \leq \Lambda_{112}$ *and* $\mu \leq \Lambda_{212}$*;*

(iv) *If* (PW_2) *holds, then the system* (3.2) *has a solution if* $\lambda^{1-\gamma}\mu^{\beta} \leq \Lambda_{122}$ *and* $\lambda^{\theta}\mu^{1-\alpha} \leq \Lambda_{222}$*;*

(v) *If* (PW_3) *holds, then the system* (3.1) *has a solution if* $\lambda \leq \Lambda_{113}$ *and* $\lambda^{\frac{\alpha-1}{1-q}}\mu^{\frac{\beta}{1-p}} \leq \Lambda_{213}$*;*

(vi) *If* (PW_3) *holds, then the system* (3.2) *has a solution if* $\lambda \leq \Lambda_{123}$ *and* $\lambda^{1-\gamma}\mu^{\beta} \leq \Lambda_{223}$.

Proof. The Case (ii) and Case (iii) follow from Remark 3.2.
Case (i): It is enough to take $X = (2A\lambda)^{\frac{1}{1-q}}$ and $Y = (2B\mu)^{\frac{1}{1-p}}$. Thus, we have $(X/2) = \lambda AX^q$ and $(Y/2) = \mu BY^p$. On the other hand, if $\lambda^{\frac{\alpha-1}{1-q}}\mu^{\frac{\beta}{1-p}} \leq \Lambda_{111} := (2A)^{\frac{-(\alpha-1)}{1-q}}(2B)^{\frac{-\beta}{1-p}}(2C)^{-1}$ and

$$\lambda^{\frac{\theta}{1-q}}\mu^{\frac{\gamma-1}{1-p}} \leq \Lambda_{211} := (2A)^{\frac{-\theta}{1-q}}(2B)^{\frac{-(\gamma-1)}{1-p}}(2D)^{-1},$$

then $(X/2) \geq CX^{\alpha}Y^{\beta}$ and $(Y/2) \geq DX^{\theta}Y^{\gamma}$, and therefore X and Y satisfy Equation (3.1).

Case (iv): If

$$X := ((2D)^{\beta}(2C)^{1-\gamma}\lambda^{1-\gamma}\mu^{\beta})^{\frac{-1}{\beta\theta-(1-\alpha)(1-\gamma)}}$$

and

$$Y := ((2D)^{1-\alpha}(2C)^{\theta}\lambda^{\theta}\mu^{1-\alpha})^{\frac{-1}{\beta\theta-(1-\alpha)(1-\gamma)}},$$

then we see that $X/2 = \lambda CX^{\alpha}Y^{\beta}$ and $Y/2 = \mu DX^{\theta}Y^{\gamma}$. Since $\beta\theta - (1-\alpha)(1-\gamma) > 0$, we can choose $\Lambda_{122} > 0$ and $\Lambda_{222} > 0$ such that $\lambda^{1-\gamma}\mu^{\beta} \leq \Lambda_{122}$ and $\lambda^{\theta}\mu^{1-\alpha} \leq \Lambda_{222}$ imply $X/2 \geq AX^q$ and $Y/2 \geq BY^p$. Therefore, X and Y satisfy Equation (3.2). *Case (v):* If we take $X = (2A\lambda)^{\frac{1}{1-q}}$ and $Y = \max\{(2B\mu)^{\frac{1}{1-p}}, (2D(2A\lambda)^{\frac{\theta}{1-q}})^{\frac{1}{1-\gamma}}\}$, then we have $X/2 = \lambda AX^q$ and $Y \geq \mu BY^p + DX^{\theta}Y^{\gamma}$. Now we choose $\Lambda_{113} > 0$ and $\Lambda_{213} > 0$, in such a way that for $\lambda \leq \Lambda_{15}$ and $\lambda^{\frac{\alpha-1}{1-q}}\mu^{\frac{\beta}{1-p}} \leq \Lambda_{25}$ we have $X/2 \geq CX^{\alpha}Y^{\beta}$, and therefore X and Y satisfy Equation (3.1).

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

Case (vi): First of all, take $\overline{B} = \overline{B}(B, \gamma, p)$ such that $Y \geq \overline{B}Y^\gamma \Rightarrow Y \geq 2BY^p$. Then, the solutions of the system

$$\begin{cases} X & \geq & AX^q + \lambda C X^\alpha Y^\beta \\ Y & \geq & \overline{B}Y^\gamma + 2\mu D X^\theta Y^\gamma \\ & & X > 0,\ Y > 0, \end{cases} \tag{3.3}$$

are solutions of Equation (3.2). Since $\alpha \geq 1$, there are $\Lambda_{123} > 0$ and $\Lambda_{223} > 0$ such that, for all $\lambda \leq \Lambda_{123}$ and $\lambda \mu^{\frac{\beta}{1-\gamma}} \leq \Lambda_{223}^{\frac{1}{1-\gamma}}$ ($\lambda^{1-\gamma}\mu^\beta \leq \Lambda_{223}$), there is $X > 0$ satisfying

$$X \geq AX^q + \lambda 2^{\frac{\beta}{1-\gamma}} \overline{B}^{\frac{\beta}{1-\gamma}} C X^\alpha + \lambda \mu^{\frac{\beta}{1-\gamma}} 2^{\frac{\beta}{1-\gamma}} (2D)^{\frac{\beta}{1-\gamma}} C X^{\alpha + \frac{\beta\theta}{1-\gamma}}.$$

So

$$\frac{X - AX^q}{\lambda C X^\alpha} \geq 2^{\frac{\beta}{1-\gamma}} \left(\overline{B}^{\frac{\beta}{1-\gamma}} + (2D\mu)^{\frac{\beta}{1-\gamma}} X^{\frac{\beta\theta}{1-\gamma}} \right) \geq \left(\overline{B} + 2\mu D X^\theta\right)^{\frac{\beta}{1-\gamma}}.$$

Hence, if we take $Y > 0$ in such a way

$$\frac{X - AX^q}{\lambda C X^\alpha} \geq Y^\beta \geq \left(\overline{B} + 2\mu D X^\theta\right)^{\frac{\beta}{1-\gamma}},$$

then $X > 0$ and $Y > 0$ satisfy Equation (3.3). □

Corollary 3.4. (Existence of super-solution) *Let $e \in W^{2,s}(\overline{\Omega}) \cap W_0^{1,s}(\Omega) \cap C^1(\overline{\Omega})$ be the unique positive solution of the problem $-\Delta e = 1$ in Ω, where $e = 0$ on $\partial\Omega$ (see [31, Theorems 7.26 and 9.15]). Define $A := \|a\|_\infty \|e\|_\infty^q$, $B := \|b\|_\infty \|e\|_\infty^p$, $C = \|c\|_\infty \|e\|_\infty^{\alpha+\beta}$ and $D := \|d\|_\infty \|e\|_\infty^{\theta+\gamma}$. For Λ_{kij} found in Lemma 3.3, in each of the cases $(i) - (vi)$, where λ, $\mu > 0$ satisfy these conditions, there are $\overline{u}, \overline{v} \in W^{2,s}(\overline{\Omega}) \cap W_0^{1,s}(\Omega) \cap C^1(\overline{\Omega})$ satisfying $(\overline{S}_{\lambda\mu})$.*

Proof. If $X > 0$ and $Y > 0$ are solutions for Equation (3.1) or (3.2), then we see that $\overline{u} = Xe$ and $\overline{v} = Ye$ satisfy $(\overline{S}_{\lambda\mu})$. □

In a manner akin to the definition of a super-solution, we define $(\underline{u}, \underline{v})$ as a sub-solution for $(\mathcal{P}_i)$ subordinate to $\overline{u}, \overline{v} \in L^\infty(\Omega) \setminus \{0\}$ when $\underline{u}, \underline{v} \in W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega) \cap C^1(\overline{\Omega})$ for all $s > 1$ and $\underline{u}, \underline{v}$ satisfy:

$$\begin{cases} -\Delta \underline{u} \leq f_\lambda(x, \underline{u}, w) & \text{in } \Omega,\ \forall w \in [0, \overline{v}] \\ -\Delta \underline{v} \leq g_\mu(x, z, \underline{v}) & \text{in } \Omega,\ \forall z \in [0, \overline{u}] \\ \underline{v},\ \underline{v} \geq 0 & \text{in } \Omega \\ \underline{u} = \underline{v} = 0 & \text{on } \partial\Omega \\ \underline{u} \not\equiv 0,\ \underline{v} \not\equiv 0 & \text{in } \Omega. \end{cases} \tag{$\underline{S}_{\lambda\mu}$}$$

Remark 3.5. A similar observation to that in Remark 3.1 holds for sub-solutions.

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

The sets $\mathcal{A} := \{x \in \Omega \mid a(x) > a_1 := \|a\|_\infty/2\}$ and $\mathcal{B} := \{x \in \Omega \mid b(x) > b_1 := \|b\|_\infty/2\}$ have positive Lebesgue measure. Furthermore, $\Omega \setminus \mathcal{A}$ and $\Omega \setminus \mathcal{B}$ can have zero Lebesgue measure, despite that we can set h, $r \in L^\infty(\Omega)$ by

$$h(x) = \begin{cases} a_1 & , \quad x \in \mathcal{A} \\ -1 & , \quad x \in \Omega \setminus \mathcal{A} \end{cases} \qquad r(x) = \begin{cases} b_1 & , \quad x \in \mathcal{B} \\ -1 & , \quad x \in \Omega \setminus \mathcal{B}. \end{cases} \tag{3.4}$$

We will see that in some cases, for a small $\varepsilon > 0$, we have sub-solutions of the type $\underline{u} = \varepsilon\varphi_{1h\Omega}$ and $\underline{v} = \varepsilon\varphi_{1r\Omega}$. However, this approach does not work when $c^-(x) \not\equiv 0$, $d^-(x) \not\equiv 0$, $q < \alpha < 1$, or $p < \gamma < 1$. In order to overcome this obstacle, we will obtain another type of sub-solution. For that, we will use the variational method.

Proposition 3.6. *Let $m \in L^\infty(\Omega) \setminus \{0\}$ with $m(x) \geq 0$ in Ω. For all $0 < r_1 < r_2 \leq 1$, $K \geq 0$, and $\varepsilon > 0$, the auxiliary problem $(\mathcal{P}_\varepsilon)$:*

$$\begin{cases} -\Delta u = \varepsilon m(x) u^{r_1} - K u^{r_2} & in \;\; \Omega \\ 0 \not\equiv u \geq 0 & in \;\; \Omega \\ u = 0 & on \;\; \partial\Omega, \end{cases} \tag{$\mathcal{P}_\epsilon$}$$

has a solution $u \in W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega) \cap C^1(\overline{\Omega})$, for all $s > 1$. Moreover, we have $u \leq \varepsilon^{\frac{1}{1-r_1}} Me$ in Ω, where $e \in W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega) \cap C^1(\overline{\Omega})$ is the unique solution of $-\Delta e = 1$ in Ω, and $M = M(r_1, \|m\|_\infty, \|e\|_\infty)$.

Proof. Let $I_\varepsilon : H_0^1(\Omega) \to \mathbb{R}$, $I_\varepsilon \in C^1$ given by

$$I_\varepsilon(u) := \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \frac{2K}{r_2+1}|u|^{r_2+1}\right) \mathrm{d}x - \frac{\lambda}{r_1+1}\int_\Omega m(x)|u|^{r_1+1}\,\mathrm{d}x.$$

Since $H_0^1(\Omega) \hookrightarrow L^{r_1+1}(\Omega)$, then for some $R > 0$, we have

$$I_\varepsilon(u) \geq 0, \quad \text{for all} \;\; \|u\| := \|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}} = R. \tag{3.5}$$

Choosing $\varphi \in H_0^1(\Omega)$ with $\int_\Omega m(x)\varphi\,\mathrm{d}x > 0$, we see that $\lim_{t\to 0^+} t^{-(r_1+1)} I_\varepsilon(t\varphi) < 0$, then we get

$$c_0 := \inf\left\{I_\varepsilon(u) \,|\, u \in H_0^1(\Omega) \,\text{and}\, \|u\| \leq R\right\} < 0.$$

A standard compactness argument provides $u_0 \in H_0^1(\Omega)$ with $\|u_0\| \leq R$ such that $I_\varepsilon(u_0) = c_0 < 0$, so $u_0 \not\equiv 0$. Since $I_\varepsilon(u_0) = I_\varepsilon(|u_0|)$, we will suppose $u_0 \geq 0$. From Equation (3.5), we get $\|u_0\| < R$. Then $I'_\varepsilon(u_0) = 0$, hence u_0 is a weak solution for $(\mathcal{P}_\varepsilon)$. From [43, Lemma B3, p. 270], we obtain $u_0 \in L^s(\Omega)$ for all $s > 1$ (since $r_2 \leq 1$, this can also be derived using a bootstrap argument). By [31, Theorems 7.26 and 9.15], it follows that $u_0 \in W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega) \cap C^1(\overline{\Omega})$ for all $s > 1$. Now observe

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

that for $t = \varepsilon^{\frac{1}{1-r_1}}(\|m\|_\infty \|e\|_\infty^{r_1})^{\frac{1}{1-r_1}}$, we have $-\Delta(te) \geq \varepsilon m(x)(te)^{r_1}$ in Ω. Since $-\Delta u_0 \leq \varepsilon m(x) u_0^{r_1}$ in Ω, we can invoke Lemma 2.1 to obtain $u_0 \leq te$. □

Remark 3.7. Although we do not know if the solution u_0 is positive, observe that the set $\Omega_0 := \{x \in \overline{\Omega} \,|\, u_0(x) = 0\}$ has empty interior. Otherwise, if there is $B_r \subset \overline{\Omega}_0$, then we can take $\varphi \in C_c^\infty(\Omega)$ with $\operatorname{supp}\varphi \subset B_r$. Therefore, for a small $t > 0$, we obtain $I_\varepsilon(u_0 + t\varphi) = I_\varepsilon(u_0) + I_\varepsilon(t\varphi) < I_\varepsilon(u_0)$, which is a contradiction.

Remark 3.8. If $m_0 := \inf_\Omega m(x) > 0$, since $r_2 > r_1$, then we get for $M > 0$ large $-\Delta u = \varepsilon m(x) u^{r_1} - K u^{r_2} \geq \varepsilon m_0 u^{r_1} - K u^{r_2} \geq -Mu$ in Ω, so the strong maximum principle provides $u > 0$ in Ω (see [31, Theorem 8.19]).

Lemma 3.9. (Existence of sub-solutions) *Let $(\overline{u}, \overline{v}) = (\overline{u}_{\lambda\mu}, \overline{v}_{\lambda\mu})$ be the super-solution obtained in Corollary 3.4 under the conditions (i)–(vi) of Lemma 3.3. Suppose that (P_1)–(P_2) hold. Consider the following hypotheses:*

$$
\begin{array}{ll}
(\underline{H}_1) c(x) \geq 0 \text{ in } \Omega; & (\underline{H}_3) d(x) \geq 0 \text{ in } \Omega; \\
(\underline{H}_2) \alpha > q; & (\underline{H}_4) \gamma > p.
\end{array}
$$

If $(\underline{H}_1)$ or $(\underline{H}_2)$ holds and if $(\underline{H}_3)$ or $(\underline{H}_4)$ holds, then there are $\underline{u}$ and $\underline{v}$ satisfying $(\underline{\mathcal{S}}_{\lambda\mu})$.

Proof. Without loss of generality, we will only obtain sub-solutions to $(\mathcal{P}_1)$. To illustrate the other cases, let us initially consider that $\alpha > 1$ and $\gamma > 1$. Let h and r be defined in Equation (3.4). We set $\varphi_{1h} := \varphi_{1h\Omega}$ and $\varphi_{1r} := \varphi_{1r\Omega}$. We can choose $\varepsilon > 0$ small in such a way:

$$
\begin{cases}
\varepsilon\lambda_{1h}\varphi_{1h} & \leq \lambda a_1(\varepsilon\varphi_{1h})^q - \|c\|_\infty(\varepsilon\varphi_{1h})^\alpha \|\overline{v}\|_\infty^\beta \quad x \in \mathcal{A} \\
-\varepsilon\lambda_{1h}\varphi_{1h} & \leq -\|c\|_\infty(\varepsilon\varphi_{1h})^\alpha \|\overline{v}\|_\infty^\beta \quad x \in \Omega \setminus \mathcal{A} \\
\varepsilon\lambda_{1r}\varphi_{1r} & \leq \mu b_1(\varepsilon\varphi_{1r})^p - \|d\|_\infty \|\overline{u}\|_\infty^\theta (\varepsilon\varphi_{1r})^\gamma \quad x \in \mathcal{B} \\
-\varepsilon\lambda_{1r}\varphi_{1r} & \leq -\|d\|_\infty \|\overline{u}\|_\infty^\theta (\varepsilon\varphi_{1r})^\gamma \quad x \in \Omega \setminus \mathcal{B}.
\end{cases}
\tag{3.6}
$$

Taking $\underline{u} = \varepsilon\varphi_{1h}$ and $\underline{v} = \varepsilon\varphi_{1r}$ in view of Equation (3.6), we have that $\underline{u}$ and $\underline{v}$ satisfy $(\underline{\mathcal{S}}_{\lambda\mu})$.

If $q < \alpha \leq 1$ for $\varepsilon > 0$, we set u_ε as the solution of the problem $(\mathcal{P}_\varepsilon)$ with $K = \|c\|_\infty \|\overline{v}\|_\infty^\beta$ (see Proposition 3.6), $r_1 = q$, and $r_2 = \alpha$. Then for $\varepsilon > 0$ sufficiently small, we have

$$
-\Delta u_\varepsilon = \varepsilon a(x) u_\varepsilon^q - K u_\varepsilon^\alpha \leq \lambda a(x) u_\varepsilon^q + c(x) u_\varepsilon^\alpha w^\beta, \quad \forall w \in [0, \overline{v}].
$$

Since $u_\varepsilon \leq \varepsilon^{\frac{1}{1-q}} Me$ in Ω, then for $\varepsilon > 0$ sufficiently small, we have $u_\varepsilon \leq \overline{u}$, so we set $\underline{u} = u_\varepsilon$. If $p < \gamma \leq 1$, in the same way we set $0 \leq v_\varepsilon \in W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega) \cap C^1(\overline{\Omega})$ as the non-trivial solution of the problem $-\Delta v_\varepsilon = \varepsilon b(x) v_\varepsilon^p - \widehat{K} v_\varepsilon^\gamma$ with $\widehat{K} = \|d\|_\infty \|\overline{u}\|_\infty^\theta$. So, we set $\underline{v} = v_\varepsilon$ for a sufficiently small $\varepsilon > 0$. If $c(x) \geq 0$ in Ω, then it is easy to check that $\varepsilon\varphi_{1a\Omega}$ satisfies the first inequality of $(\underline{\mathcal{S}}_{\lambda\mu})$ for $\varepsilon > 0$ sufficiently small. If $d(x) \geq 0$ in Ω,

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

then $\varepsilon\varphi_{1b\Omega}$ satisfies the second inequality of $(\underline{\mathcal{S}}_{\lambda\mu})$. In short, we have that $\underline{u}$ assumes one of the following forms: $\varepsilon\varphi_{1a\Omega}$, $\varepsilon\varphi_{1h}$, u_ε, and $\underline{v}$ assumes one of the following forms: $\varepsilon\varphi_{1b\Omega}$, $\varepsilon\varphi_{1r}$, v_ε. □

Remark 3.10. Let $\overline{u}$ and $\overline{v}$ be obtained in Corollary 3.4, and $\underline{u}$ and $\underline{v}$ be obtained in Lemma 3.9. By the maximum principle, we can take $\varepsilon > 0$ sufficiently small in such way $\underline{u} \le \overline{u}$ and $\underline{v} \le \overline{v}$.

4. Existence of positive solution: global and local results

This section is devoted to the proof of Theorem 1.4.

Proof of Theorem 1.4-(i): Suppose that (PW_j) holds and take $\Lambda_{1ij}, \Lambda_{2ij} > 0$ given by Lemma 3.3 with $A = \|a\|_\infty + 1$, $B = \|b\|_\infty + 1$, $C = \|c\|_\infty$ and $D = \|d\|_\infty$. Under conditions (H_l) and $(\underline{H_s})$, we will get positive solutions for the problem $(\mathcal{P}_i)$ when $\lambda^{\tau_{1j}}\mu^{\sigma_{1j}} \le \Lambda_{1ij}$ and $\lambda^{\overline{\tau}_{2j}}\mu^{\overline{\sigma}_{2j}} \le \Lambda_{2ij}$. We will obtain the solutions by an iteration argument. In order to do this, for $Q > 0$ we define

$$\begin{aligned} \mathcal{F}_\lambda(x,u,v) &:= Qu + f_\lambda(x,u,v) \\ \mathcal{G}_\mu(x,u,v) &:= Qv + g_\mu(x,u,v), \end{aligned}$$

where f_λ and g_μ were defined in §3. The main difficulty here is that if $x \in \Omega$ satisfies $a(x) = 0$, $q < \alpha < 1$, $c(x) < 0$ and $v(x) > 0$, then for all $Q > 0$ the function $t \mapsto \mathcal{F}_\lambda(x,t,v(x))$ is strictly decreasing for $t > 0$ close to zero, while if $a(x) > 0$ and $v(x) > 0$, for all $t_0 > 0$ there is $Q > 0$ large such that the function $t \mapsto \mathcal{F}_\lambda(x,t,v(x))$ is non-decreasing for $t > 0$ in $[0,t_0]$. The same phenomenon occurs with the function $t \mapsto \mathcal{G}_\mu(x,u(x),t)$. In this situation, we cannot guarantee that the functions are non-decreasing uniformly in $x \in \Omega$. Therefore, the iteration argument does not work. In order to get around this problem, we defined $a_k(x) := a(x) + 1/k$ and $b_k(x) := b(x) + 1/k$. The functions $f_\lambda^k(x,u,v)$ and $g_\mu^k(x,u,v)$ are the functions $f_\lambda(x,u,v)$ and $g_\mu(x,u,v)$ with $a_k(x)$ and $b_k(x)$ instead of $a(x)$ and $b(x)$. In view of our choice of A, B, C, D, Corollary 3.4 provides $\overline{u}, \overline{v} \in W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega) \cap C^1(\overline{\Omega})$ for all $s > 1$, satisfying $(\overline{\mathcal{S}}^k_{\lambda\mu})$ where

$$\text{for all} \quad k \in \mathbb{N} \begin{cases} -\Delta\overline{u} \ge f_\lambda^k(x,z,w) & \text{in} \quad \Omega, \quad \forall z \in [0,\overline{u}], \ w \in [0,\overline{v}] \\ -\Delta\overline{v} \ge g_\mu^k(x,z,w) & \text{in} \quad \Omega, \quad \forall z \in [0,\overline{u}], \ w \in [0,\overline{v}] \\ \overline{u}, \ \overline{v} > 0 & \text{in} \quad \Omega \\ \overline{u} = \overline{v} = 0 & \text{on} \quad \partial\Omega. \end{cases} \qquad (\bar{\mathcal{S}}^{\mathrm{k}}{}_{\lambda\mu})$$

□

Since (H_l) and (H_s) hold, then $(\underline{H}_{l_1})$ and $(\underline{H}_{l_2})$ hold for some $l_1 \in \{1,2\}$ and $l_2 \in \{3,4\}$. Then we are under the hypotheses of Lemma 3.9. Therefore, there are $\underline{u} \le \overline{u}$ and $\underline{v} \le \overline{v}$ satisfying $(\underline{\mathcal{S}}_{\lambda\mu})$ (now we are considering the problem with $f_\lambda(x,u,v)$ and $g_\mu(x,u,v)$). For all $Q > 0$, we define

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

$$\begin{aligned}\mathcal{F}^k_\lambda(x,u,v) &:= Qu + f^k_\lambda(x,u,v)\\ \mathcal{G}^k_\mu(x,u,v) &:= Qv + g^k_\mu(x,u,v).\end{aligned}$$

If $c(x)\geq 0$ (i.e. $(\underline{H}_1)$ holds), then $\partial_t\mathcal{F}^k_\lambda(x,t,s)\geq 0$ for all $Q>0$. If $(\underline{H}_2)$ holds, then $\alpha>q$. Therefore, for all $t_0>0$ and $s_0>0$, for $Q>0$ sufficiently large, we have $\partial_t\mathcal{F}^k_\lambda(x,t,s)\geq 0$ for $x\in\Omega$, $t\in[0,t_0]$ and $s\in[0,s_0]$. Analogously, we can suppose for the same $Q>0$ that $\partial_s\mathcal{G}^k_\mu(x,t,s)>0$ for $x\in\Omega$, $t\in[0,t_0]$ and $s_0\in[0,s_0]$. The last two expressions give us for some $Q>0$

$$\begin{cases} 0\leq s_1\leq s_2\leq \|\overline{u}\|_\infty\\ 0\leq t_1\leq t_2\leq \|\overline{v}\|_\infty\\ 0\leq s\leq \|\overline{u}\|_\infty\\ 0\leq t\leq \|\overline{v}\|_\infty\end{cases} \implies \begin{cases} 0\leq \mathcal{F}^k_\lambda(x,s_1,t) \leq \mathcal{F}^k_\lambda(x,s_2,t)\\ 0\leq \mathcal{G}^k_\mu(x,s,t_1) \leq \mathcal{G}^k_\mu(x,s,t_2).\end{cases} \tag{4.1}$$

We define the monotone iteration for $n\geq 0$, with $u_0=\underline{u}$ and $v_0=\underline{v}$ as follows:

$$\begin{cases} -\Delta u_{n+1}+Qu_{n+1}=\mathcal{F}^k_\lambda(x,u_n,v_n) & \text{in } \Omega\\ -\Delta v_{n+1}+Qv_{n+1}=\mathcal{G}^k_\mu(x,u_n,v_n) & \text{in } \Omega\\ u_{n+1},\ v_{n+1}>0 & \text{in } \Omega\\ u_{n+1}=v_{n+1}=0 & \text{on } \partial\Omega.\end{cases} \tag{$\mathcal{P}^{\mathrm{nk}}{}_{\lambda\mu}$}$$

From Equation (4.1), $(\overline{\mathcal{S}}^k_{\lambda\mu})$, $(\underline{\mathcal{S}}_{\lambda\mu})$ and the maximum principle, by induction, we get $0\leq\underline{u}\leq u_n\leq u_{n+1}\leq\overline{u}$ and $0\leq\underline{v}\leq v_n\leq v_{n+1}\leq\overline{v}$ in Ω. Thus, there are $u^k_{\lambda\mu}$, $v^k_{\lambda\mu}\in C^1_0(\overline{\Omega})\cap W^{2,s}(\Omega)$ such that $\underline{u}\leq u^k_{\lambda\mu}\leq\overline{u}$ and $\underline{v}\leq v^k_{\lambda\mu}\leq\overline{v}$ in Ω with $-\Delta u^k_{\lambda\mu}+Qu^k_{\lambda\mu}=\mathcal{F}^k_\lambda(x,u^k_{\lambda\mu},v^k_{\lambda\mu})$ and $-\Delta v^k_{\lambda\mu}+Qv^k_{\lambda\mu}=\mathcal{G}^k_\mu(x,u^k_{\lambda\mu},v^k_{\lambda\mu})$ in Ω. So, we have

$$-\Delta u^k_{\lambda\mu}=f^k_\lambda(x,u^k_{\lambda\mu},v^k_{\lambda\mu})\quad\text{and}\quad -\Delta v^k_{\lambda\mu}=g^k_\mu(x,u^k_{\lambda\mu},v^k_{\lambda\mu})\quad\text{in}\quad\Omega.$$

Therefore, we have $\|u^k_{\lambda\mu}\|$ and $\|v^k_{\lambda\mu}\|$ bounded. Then, for some $u_{\lambda\mu}$, $v_{\lambda\mu}\in H^1_0(\Omega)$, $u^k_{\lambda\mu}\rightharpoonup u_{\lambda\mu}$ and $v^k_{\lambda\mu}\rightharpoonup v_{\lambda\mu}$ weakly in $H^1_0(\Omega)$ when $k\to+\infty$. Up to a subsequence, we have $\underline{u}\leq u_{\lambda\mu}\leq\overline{u}$ and $\underline{v}\leq v_{\lambda\mu}\leq\overline{v}$ in Ω. For $\varphi\in C^\infty_c(\Omega)$, we have

$$\int_\Omega\nabla u_{\lambda\mu}\nabla\varphi=\lim_{k\to\infty}\int_\Omega-\Delta u^k_{\lambda\mu}\varphi=\lim_{k\to\infty}f^k_\lambda(x,u^k_{\lambda\mu},v^k_{\lambda\mu})\varphi=\int_\Omega f_\lambda(x,u_{\lambda\mu},v_{\lambda\mu})\varphi.$$

In the same way, we have

$$\int_\Omega\nabla v_{\lambda\mu}\nabla\varphi=\lim_{k\to\infty}\int_\Omega-\Delta v^k_{\lambda\mu}\varphi=\lim_{k\to\infty}g^k_\mu(x,u^k_{\lambda\mu},v^k_{\lambda\mu})\varphi=\int_\Omega g_\mu(x,u_{\lambda\mu},v_{\lambda\mu})\varphi.$$

So, $(u_{\lambda\mu},v_{\lambda\mu})$ are weak solutions for $(\mathcal{P}_i)$. Since $u_{\lambda\mu},v_{\lambda\mu}\in L^\infty(\Omega)$, from [31, Theorems 7.26 and 9.15]) one has $u_{\lambda\mu},v_{\lambda\mu}\in C^1_0(\overline{\Omega})\cap W^{2,s}(\Omega)$, where $C^1_0(\overline{\Omega}):=C^1(\overline{\Omega})\cap H^1_0(\Omega)$. To finish the proof, we need to show that solutions are positive. The sub-solution

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

$\underline{u}$ assumes one of the forms $\varepsilon\varphi_{1a\Omega}$, $\varepsilon\varphi_{1h}$, u_ε (see the final comments in Lemma 3.9). In the first two cases, we have $\underline{u} > 0$ in Ω. If $\underline{u} = u_\varepsilon$, by Remark 3.8, we get $\underline{u} > 0$ in Ω. Similarly, $\underline{v} > 0$ in Ω, therefore $u_{\lambda\mu}$, $v_{\lambda\mu} > 0$ in Ω.

Proof of Theorem 1.4-(ii): We define $(\mathcal{P}^i_{\lambda\mu})$ as the problem $(\mathcal{P}_i)$ with $\lambda > 0$ and $\mu > 0$. Let

$$\mathcal{O} := \left\{(\lambda,\mu) \in \mathbb{R}^2_+ \ : \ (\mathcal{P}^i_{\lambda\mu}) \text{ has a solution} \right\}.$$

By statement (i), the set $\mathcal{O}$ is nonempty. Let $(\overline{\lambda},\overline{\mu}) \in \mathcal{O}$ and $(\overline{u},\overline{v})$ be some solution of $(\mathcal{P}^i_{\overline{\lambda}\overline{\mu}})$. We take $\lambda,\mu > 0$ in such a way that $0 < \lambda \leq \overline{\lambda}$, $0 < \mu \leq \overline{\mu}$. From [13], we can set $u_{\lambda 0} > 0$ and $v_{0\mu} > 0$ as the unique positive solutions of

$$\begin{cases} -\Delta u = f_\lambda(x,u,0) & \text{in } \ \Omega \\ u = 0 & \text{on } \ \partial\Omega \end{cases} \qquad \begin{cases} -\Delta v = g_\mu(x,0,v) & \text{in } \ \Omega \\ v = 0 & \text{on } \ \partial\Omega. \end{cases}$$

From Lemma 2.1, we have $0 < u_{\lambda 0} \leq \overline{u}$ and $0 < v_{0\mu} \leq \overline{v}$ in Ω. We define the monotone iteration for $n \geq 0$, with $u_0 = u_{\lambda 0}$ and $v_0 = v_{0\mu}$ as follows:

$$\begin{cases} -\Delta u_{n+1} = f_\lambda(x,u_n,v_n) & \text{in } \ \Omega \\ -\Delta v_{n+1} = g_\mu(x,u_n,v_n) & \text{in } \ \Omega \\ u_{n+1}, \ v_{n+1} > 0 & \text{in } \ \Omega \\ u_{n+1} = v_{n+1} = 0 & \text{on } \ \partial\Omega. \end{cases} \qquad (\mathcal{P}^{\mathrm{n}}{}_{\lambda\mu})$$

□

Since all weights $a(\cdot), b(\cdot), c(\cdot), d(\cdot)$ are nonnegative, as in statement (i), by the maximum principle and induction on $n \geq 0$, we get $0 < u_{\lambda 0} \leq u_n \leq u_{n+1} \leq \overline{u}$ and $0 < v_{0\mu} \leq v_n \leq v_{n+1} \leq \overline{v}$ in Ω. Then, (u_n, v_n) converge strongly in $H^1_0 \times H^1_0$ to some solution $(u_{\lambda\mu}, v_{\lambda\mu})$ of $(\mathcal{P}^i_{\lambda\mu})$. Furthermore, $0 < u_{\lambda 0} \leq u_{\lambda\mu} \leq \overline{u}$ and $0 < v_{0\mu} \leq v_{\lambda\mu} \leq \overline{v}$, so this solution is a minimal positive solution as defined in Equation (1.3).

If $(\widetilde{u},\widetilde{v})$ is a solution of $(\mathcal{P}^i_{\lambda\mu})$, by Lemma 2.1, we have $0 < u_{\lambda 0} \leq \widetilde{u}$ and $0 < v_{0\mu} \leq \widetilde{v}$ in Ω. Therefore, $0 < u_{\lambda 0} \leq u_n \leq \widetilde{u}$ and $0 < v_{0\mu} \leq v_n \leq \widetilde{v}$ in Ω, and so we get $u_{\lambda\mu} \leq \widetilde{u}$ and $v_{\lambda\mu} \leq \widetilde{v}$. In this way, we prove the existence of minimal solutions for all $(\lambda,\mu) \in \mathcal{O}$ and $(0,\lambda] \times (0,\mu] \subset \mathcal{O}$. Now we define $\mathcal{O}_1 := \{\lambda > 0 \ : \ \text{for some } \mu > 0 \text{ we have } \ (\lambda,\mu) \in \mathcal{O}\}$. From the last observation, $\mathcal{O}_1$ is a nonempty interval. Let $L^*_{ij} := \sup \mathcal{O}_1$. From statement (i), we see that $L^*_{ij} = +\infty$ if and only if $\sigma_{ij} > 0$ and $\overline{\sigma}_{ij} > 0$, which only holds when $i = j = 1$ and $i = j = 2$. Taking $\lambda \in (0, L^*_{ij})$, we set the nonempty interval $\mathcal{O}_\lambda = \{\mu > 0 \ : \ (\lambda,\mu) \in \mathcal{O}\}$ and $\Lambda_\lambda := \sup \mathcal{O}_\lambda$. From Theorem 1.2, $\Lambda_\lambda < +\infty$, and this ends the proof of (ii).

If $(\widetilde{u},\widetilde{v})$ is a solution of $(\mathcal{P}^i_{\lambda\mu})$, by Lemma 2.1, we have $0 < u_{\lambda 0} \leq \widetilde{u}$ and $0 < v_{0\mu} \leq \widetilde{v}$ in Ω. Therefore, $0 < u_{\lambda 0} \leq u_n \leq \widetilde{u}$ and $0 < v_{0\mu} \leq v_n \leq \widetilde{v}$ in Ω, and so we get $u_{\lambda\mu} \leq \widetilde{u}$ and $v_{\lambda\mu} \leq \widetilde{v}$. In this way, we prove the existence of minimal solutions for all $(\lambda,\mu) \in \mathcal{O}$ and $(0,\lambda] \times (0,\mu] \subset \mathcal{O}$. Now we define $\mathcal{O}_1 := \{\lambda > 0 \ : \ \text{for some } \mu > 0 \text{ we have } \ (\lambda,\mu) \in \mathcal{O}\}$. From the last observation, $\mathcal{O}_1$ is a nonempty interval. Let $L^*_{ij} := \sup \mathcal{O}_1$. From statement (i), we see that $L^*_{ij} = +\infty$ if and only if $\sigma_{ij} > 0$ and $\overline{\sigma}_{ij} > 0$, which only holds when

$i = j = 1$ and $i = j = 2$. Taking $\lambda \in (0, L^*_{ij})$, we set the nonempty interval $\mathcal{O}_\lambda = \{\mu > 0 \,:\, (\lambda, \mu) \in \mathcal{O}\}$ and $\Lambda_\lambda := \sup \mathcal{O}_\lambda$. From Theorem 1.2, $\Lambda_\lambda < +\infty$, and this ends the proof of (ii).

Proof of Theorem 1.4-(iii) and (iv): Follows immediately from (ii). □

Proof of Theorem 1.4-(v): The proof of the existence of a solution to $0 < \lambda < L^*_{ij}$ and $\mu = \Lambda_\lambda$ is similar to the proof of [4, Lemma 3.5]. We will only deal with the case related to the system $(\mathcal{P}_i)$ with $i = 1$, as the case $i = 2$ is quite similar. We will do a very short proof. For all $\lambda > 0$ and $\mu < \Lambda_\lambda$, we define $A_{\lambda\mu}(x) = \lambda a(x) u^{q-1}_{\lambda\mu} + \alpha c(x) u^{\alpha-1}_{\lambda\mu} v^{\beta}_{\lambda\mu}$. Despite the fact that the possibility of $A_{\lambda\mu}(x) = +\infty$ for some values $x \in \partial\Omega$, the spectral theory for $-\Delta - A_{\lambda\mu}(x)$ can still be carried over in $H^1_0(\Omega)$ (see [4, Remark 2.1]). We claim that ν_1, the first eigenvalue of $(\mathcal{LP})$, is nonnegative.

$$\begin{cases} -\Delta\phi - A_{\lambda\mu}(x)\phi = \nu\phi & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\mathcal{LP}$}$$

□

In fact, suppose that $\nu_1 < 0$, and let $\phi_1 > 0$ be the first eigenfunction of $(\mathcal{LP})$. Following the proof of Lemma 3.5 in [4, p. 528], we obtain for a small $\varepsilon > 0$:

$$-\Delta(u_{\lambda\mu} - \varepsilon\phi_1) \geq f_\lambda(x, u_{\lambda\mu} - \varepsilon\phi_1, v_{\lambda\mu}) \ \text{ in } \Omega.$$

On the other hand, we have:

$$-\Delta v_{\lambda\mu} = g_\mu(x, u_{\lambda\mu}, v_{\lambda\mu}) \geq g_\mu(x, u_{\lambda\mu} - \varepsilon\phi_1, v_{\lambda\mu}) \ \text{ in } \Omega.$$

Then, $(u_{\lambda\mu} - \varepsilon\phi_1, v_{\lambda\mu})$ satisfies $(\overline{\mathcal{S}}_{\lambda\mu})$. The iteration $(\mathcal{P}^n_{\lambda\mu})$ gives us:

$$0 < u_{\lambda 0} \leq u_{\lambda\mu} \leq u_{\lambda\mu} - \varepsilon\phi_1 \ \text{ in } \Omega,$$

which is not possible because $\phi_1 > 0$. Therefore, we have $\nu_1 \geq 0$. From $(\mathcal{LP})$ we get:

$$\int_\Omega |\nabla\phi|^2 - A_{\lambda\mu}(x)\phi^2 \geq 0, \ \ \forall \phi \in H^1_0(\Omega).$$

Taking $\phi = u_{\lambda\mu}$, we get:

$$\|u_{\lambda\mu}\|^2 \geq \lambda q \int_\Omega a(x) u^{q+1}_{\lambda\mu} + \alpha \int_\Omega c(x) u^{\alpha+1}_{\lambda\mu} v^{\beta}_{\lambda\mu}. \tag{4.2}$$

Since $(u_{\lambda\mu}, v_{\lambda\mu})$ is a solution to $(\mathcal{P}^i_{\lambda\mu})$, we have

$$\|u_{\lambda\mu}\|^2 = \lambda \int_\Omega a(x) u^{q+1}_{\lambda\mu} + \int_\Omega c(x) u^{\alpha+1}_{\lambda\mu} v^{\beta}_{\lambda\mu}. \tag{4.3}$$

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

Everything we have done so far works for the case $0 < \alpha \leq 1$. The only moment that we have used $\alpha > 1$ is now. By Equations (4.2) and (4.3), we get

$$\|u_{\lambda\mu}\|^2 \leq \lambda \left(\frac{\alpha - p}{\alpha - 1}\right) \int_\Omega a(x) u_{\lambda\mu}^{q+1}.$$

Then for some $C_q > 0$, we have

$$\|u_{\lambda\mu}\|^{1-q} \leq \lambda C_q \left(\frac{\alpha - q}{\alpha - 1}\right). \tag{4.4}$$

In the same way, if $\gamma > 1$ we get for some $C_p > 0$

$$\|v_{\lambda\mu}\|^{1-p} \leq \mu C_p \left(\frac{\gamma - p}{\gamma - 1}\right). \tag{4.5}$$

Now we will prove the existence of a solution for the case $\lambda > 0$ and $\mu = \Lambda_\lambda$. In order to do this, we take a sequence $\mu_n < \mu_{n+1} < \Lambda_\lambda$ such that $\mu_n \to \Lambda_\lambda$. We define $u_n = u_{\lambda\mu_n}$ and $v_n = v_{\lambda\mu_n}$ as the minimal solutions for $(\mathcal{P}^i_{\lambda\mu_n})$. Moreover, we have

$$0 < u_{\lambda 0} \leq u_n \leq u_{n+1} \quad \text{and} \quad 0 < v_{0\mu} \leq v_n \leq v_{n+1} \quad \text{in } \Omega. \tag{4.6}$$

Since $\alpha > 1$ and $\gamma > 1$, by Equations (4.4) and (4.5), we have that u_n and v_n are bounded in $H^1_0(\Omega)$. Then, we have for some $u,\, v \in H^1_0(\Omega)$,

$$u_n \rightharpoonup u \quad \text{and} \quad v_n \rightharpoonup v \quad \text{weakly in} \quad H^1_0(\Omega). \tag{4.7}$$

From Equation (4.6), we get

$$u_n \leq u = \lim_{n\to\infty} u_n \quad \text{and} \quad v_n \leq v = \lim_{n\to\infty} v_n. \tag{4.8}$$

It is not hard to see that

$$\begin{cases} \lim\limits_{n\to+\infty} \displaystyle\int_\Omega c(x) u_n^\alpha v_n^\beta \varphi &= \displaystyle\int_\Omega c(x) u^\alpha v^\beta \varphi, \quad \forall \varphi \in C_c^\infty(\Omega) \\ \lim\limits_{n\to+\infty} \displaystyle\int_\Omega d(x) u_n^\theta v_n^\gamma \varphi &= \displaystyle\int_\Omega d(x) u^\theta v^\gamma \varphi, \quad \forall \varphi \in C_c^\infty(\Omega). \end{cases} \tag{4.9}$$

Since (u_n, v_n) is a solution for $(\mathcal{P}^i_{\lambda\mu_n})$ and $\mu_n \to \Lambda_\lambda$, by Equations (4.6)–(4.9), we have that (u, v) is a weak solution for $(\mathcal{P}^i_{\lambda\mu})$ with $0 < \lambda < L^*_{ij}$ and $\mu = \Lambda_\lambda$. The other case is similar.

Acknowledgements. The author is grateful to the anonymous referee for her/his comments and suggestions, which contributed significantly to improving the manuscript's quality.

Funding Statement. The author was partially supported by CNPq/Brazil 104432/2020-9 and FAP-DF/Brazil 00193-00001820/2022-86.

References

(1) K. Adriouch and A. El Hamidi, The Nehari manifold for systems of nonlinear elliptic equations, *Nonlinear Anal.* **64**(10) (2006), 2149–2167.

(2) G. A. Afrouzi and S. H. Rasouli, A remark on the existence of multiple solutions to a multiparameter nonlinear elliptic system, *Nonlinear Anal.* **71**(1–2) (2009), 445–455.

(3) W. Albalawi, C. Mercuri and V. Moroz, Groundstate asymptotics for a class of singularly perturbed p-Laplacian problems in $\mathbb{R}^N$, *Ann. Mat. Pura Appl. (4)* **199**(1) (2020), 23–63.

(4) A. Ambrosetti, H. Brézis and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.* **122**(2) (1994), 519–543.

(5) D. Arcoya, L. Boccardo and L. Orsina, Critical points for functionals with quasilinear singular Euler-Lagrange equations, *Calc. Var. Partial Differential Equations* **47**(1–2) (2013), 159–180.

(6) B. Barrios, E. Colorado, A. De Pablo and U. Sánchez, On some critical problems for the fractional Laplacian operator, *J. Differential Equations* **252**(11) (2012), 6133–6162.

(7) B. Barrios, E. Colorado, R. Servadei and F. Soria, A critical fractional equation with concave-convex power nonlinearities, *Ann. Inst. H. Poincaré Anal. Non LinéAire* **32**(4) (2015), 875–900.

(8) T. Bartsch and M. Willem, On an elliptic equation with concave and convex nonlinearities, *Proc. Amer. Math. Soc.* **123**(11) (1995), 3555–3561.

(9) C. J. Batkam and F. Colin, The effects of concave and convex nonlinearities in some noncooperative elliptic systems, *Ann. Mat. Pura Appl. (4)* **193**(6) (2014), 1565–1576.

(10) C. Brändle, E. Colorado, A. De Pablo and U. Sánchez, A concave-convex elliptic problem involving the fractional Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A* **143**(1) (2013), 39–71

(11) H. Brézis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, p. xiv+599 (Universitext. Springer, New York, 2011).

(12) H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **36** (4) (1983), 437–477.

(13) H. Brézis and L. Oswald, Remarks on sublinear elliptic equations, *Nonlinear Anal.* **10**(1) (1986), 55–64.

(14) K. J. Brown and T. F. Wu, A semilinear elliptic system involving nonlinear boundary condition and sign-changing weight function, *J. Math. Anal. Appl.* **337**(2) (2008), 1326–1336.

(15) A. M. Candela, G. Palmieri and A. Salvatore, Multiple solutions for some symmetric supercritical problems, *Commun. Contemp. Math.* **22**(8) (2020), 20, 1950075.

(16) A. M. Candela, K. Perera and C. Sportelli, On a class of supercritical N-Laplacian problems, *Nonlinear Anal. Real World Appl.* **71** (2023), 16, Paper No. 103817.

(17) A. M. Candela and C. Sportelli, Multiple solutions for coupled gradient-type quasilinear elliptic systems with supercritical growth, *Ann. Mat. Pura Appl. (4)* **201**(5) (2022), 2341–2369.

(18) W. Chen and S. Deng, Multiple solutions for a critical fractional elliptic system involving concave-convex nonlinearities, *Proc. Roy. Soc. Edinburgh Sect. A* **146**(6) (2016), 1167–1193.

(19) X. Cheng, Z. Feng and L. Wei, Nontrivial solutions for a quasilinear elliptic system with weight functions, *Differential Integral Equations* **33**(11–12) (2020), 625–656.

(20) M. Chhetri, S. Raynor and S. Robinson, On the existence of multiple positive solutions to some superlinear systems, *Proc. Roy. Soc. Edinburgh Sect. A* **142**(1) (2012), 39–59.

(21) M. Clapp and S. Tiwari, Multiple solutions to a pure supercritical problem for the p-Laplacian, *Calc. Var. Partial Differ. Equ.* **55**(1) (2016), 23, Art. 7.

https://doi.org/10.1017/S0013091524000476 Published online by Cambridge University Press

(22) J. P. P. Da Silva, Existence of non-negative solutions for a system with concave-convex nonlinearity, *J. Elliptic Parabol. Equ.* **9**(2) (2023), 1319–1328.

(23) J. P. P. Da Silva, Global and local results for a gradient system with concave-convex nonlinearities. Preprint.

(24) J. P. P. Da Silva, $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ versus $C_0^1(\Omega)\times C_0^1(\Omega)$ local minimizers, *Asymptotic Analysis* **140**(1–2), 59–76. doi: 10.3233/ASY-241911.

(25) D. G. De Figueiredo, J. P. Gossez and P. Ubilla, Local superlinearity and sublinearity for indefinite semilinear elliptic problems, *J. Funct. Anal.* **199**(2) (2003), 452–467.

(26) D. G. De Figueiredo, J. P. Gossez and P. Ubilla, Multiplicity results for a family of semilinear elliptic problems under local superlinearity and sublinearity, *J. Eur. Math. Soc. (JEMS)* **8**(2) (2006), 269–286.

(27) F. O. De Paiva, Nonnegative solutions of elliptic problems with sublinear indefinite nonlinearity, *J. Funct. Anal.* **261**(9) (2011), 2569–2586.

(28) J. García-Azorero and I. Peral, Some results about the existence of a second positive solution in a quasilinear critical problem, *Indiana Univ. Math. J.* **43**(3) (1994), 941–957.

(29) J. García-Azorero, I. Peral and J. J. Manfredi, Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations, *Commun. Contemp. Math.* **2**(3) (2000), 385–404.

(30) J. García-Azorero, I. Peral and J. D. Rossi, A convex-concave problem with a nonlinear boundary condition, *J. Differential Equations* **198**(1) (2004), 91–128.

(31) D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics Math. (Springer-Verlag, Berlin, 2001) reprint of the 1998 edition.

(32) X. He, M. Squassina and W. Zou, The Nehari manifold for fractional systems involving critical nonlinearities, *Commun. Pure Appl. Anal.* **15**(4) (2016), 1285–1308.

(33) P. Hess and T. Kato, On some linear and nonlinear eigenvalue problems with an indefinite weight function, *Comm. Partial Differ. Equ.* **5**(10) (1980), 999–1030.

(34) N. Hirano, C. Saccon and N. Shioji, Existence of multiple positive solutions for singular elliptic problems with concave and convex nonlinearities, *Adv. Differ. Equ.* **9**(1–2) (2004), 197–220.

(35) E. Hopf. Elementare Bemerkungen über die Lösung partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus, *Sitzungber. Preuss. Akad. Wiss. Phys. Math. Kl.* **19**: (1927), 147–152.

(36) T. S. Hsu, Multiple positive solutions for a critical quasilinear elliptic system with concave-convex nonlinearities, *Nonlinear Anal.* **71**(7–8) (2009), 2688–2698.

(37) M. G. Krein and M. A. Rutman, Linear operators leaving invariant a cone in a Banach space, *Uspekhi Mat. Nauk* **3**(1) (1948), 3–95.

(38) Q. Li and Z. Yang, Multiplicity of positive solutions for a p-q-Laplacian system with concave and critical nonlinearities, *J. Math. Anal. Appl.* **423**(1) (2015), 660–680.

(39) S. A. Marano and N. S. Papageorgiou, Positive solutions to a Dirichlet problem with p-Laplacian and concave-convex nonlinearity depending on a parameter, *Commun. Pure Appl. Anal.* **12**(2) (2013), 815–829.

(40) P. H.RabinowitzTheorie du degré topologique et applications à des problèmes aux limites non linéaires. Lect. Notes, Analyse Numérique Fonctionnelle, Univ. Paris VI. (1975)

(41) S. Rasouli and G. A. Afrouzi, The Nehari manifold for a class of concave-convex elliptic systems involving the p-Laplacian and nonlinear boundary condition, *Nonlinear Anal.* **73**(10) (2010), 3390–3401.

(42) H. Song, C. Chen and Q. Yan, Existence of multiple solutions for a p-Laplacian system in RN with sign-changing weight functions, *Canad. Math. Bull.* **59**(2) (2016), 417–434.

(43) M. Struwe, Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems. *Ergebnisse der Mathematik und Ihrer Grenzgebiete (3) [Results*

in Mathematics and Related Areas (3)], 2nd edn, Volume 34, pp. xvi+272 (Springer-Verlag, Berlin, 1996).

(44) M. Tang, Exact multiplicity for semilinear elliptic Dirichlet problems involving concave and convex nonlinearities, *Proc. Roy. Soc. Edinburgh Sect. A* **133**(3) (2003), 705–717.

(45) Z. Q. Wang, Nonlinear boundary value problems with concave nonlinearities near the origin, *NoDEA Nonlinear Differ. Equ. Appl.* **8**(1) (2001), 15–33.

(46) T. F. Wu, The Nehari manifold for a semilinear elliptic system involving sign-changing weight functions, *Nonlinear Anal.* **68**(6) (2008), 1733–1745.

(47) T. F. Wu, Multiple positive solutions for a class of concave-convex elliptic problems in $\mathbb{R}^N$ involving sign-changing weight, *J. Funct. Anal.* **258**(1) (2010), 99–131.

(48) H. Yin, Existence of multiple positive solutions for a p-q-Laplacian system with critical nonlinearities, *J. Math. Anal. Appl.* **403**(1) (2013), 200–214.