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A new definition of discrete analytic functions

C.J. Harman

The concept of a *tetradiffric* function is introduced. This new scheme for defining discrete analytic functions is shown to retain the algebraic simplicity of *monodiffric* functions, while introducing to the theory a symmetry similar to the Schwarz Reflection Principle.

1. Introduction and definitions

Discrete analytic functions of the first kind (or *monodiffric* functions) are defined on the set of gaussian integers and satisfy the forward-difference equation

$$f(z+1) - f(z) = \frac{f(z+i)-f(z)}{i} ,$$

(see for example Isaacs [7, 8] and Berzsenyi [1, 2]). In [6], the monodiffric function $z^{(\alpha)}$ (the discrete analogue of z^{α}) was found. This function highlighted certain shortcomings in the monodiffric scheme. Monodiffric functions lack symmetry: for example $(-z)^{(\alpha)} \neq (-1)^{\alpha} z^{(\alpha)}$, and in the theory there is no analogue of the Schwarz Reflection Principle.

In this paper an alternative definition of discrete analytic functions is examined. The resulting functions demonstrate a symmetry similar to discrete functions of the second kind which were defined by Ferrand [4] and further developed by Duffin [3] and others. Unlike second kind functions however, it is seen that the simple algebraic form of monodiffric functions

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is retained. The function $z^{(\alpha)}$ is expressed in terms of divergent series when α is not an integer and as a polynomial when α is a non-negative integer. Finally an analogue of the Schwarz Reflection Principle is obtained.

The domain of definition to be considered is the set G of gaussian integers. Hence,

$$G = \{z;\ z = (x, y) = x + iy \text{ , where } x \text{ and } y \text{ are integers}\} .$$

Subsets of G in the four quadrants of the complex plane are defined by,

$$G_1 = \{z;\ z \in G,\ x > 0,\ y > 0\} ,\quad G_2 = \{z;\ z \in G,\ x < 0,\ y > 0\} ,$$

$$G_3 = \{z;\ z \in G,\ x < 0,\ y < 0\} ,\quad G_4 = \{z;\ z \in G,\ x > 0,\ y < 0\} ,$$

and on the axes,

$$X^+ = \{z;\ z \in G,\ x \geq 0,\ y = 0\} ,\quad X^- = \{z;\ z \in G,\ x \leq 0,\ y = 0\} ,$$

$$Y^+ = \{z;\ z \in G,\ x = 0,\ y \geq 0\} ,\quad Y^- = \{z;\ z \in G,\ x = 0,\ y \leq 0\} .$$

Forward and backward difference operators are defined by,

(1.1)
$$\Delta_1 f(z) = f(z) - f(z-1) ,$$

$$\Delta_2 f(z) = \frac{f(z)-f(z-i)}{i} ,$$

$$\Delta_3 f(z) = f(z+1) - f(z) ,$$

$$\Delta_4 f(z) = \frac{f(z+i)-f(z)}{i} .$$

2. Tetradiffric functions

A new type of discrete analytic function, based on the concept of a monodiffric function, is now defined. The definition involves a consideration of a separate monodiffric scheme in each of the four quadrants G_1, G_2, G_3, and G_4.

A function f is said to be *tetradiffric* at the point $z \in G_k$ ($k = 1, 2, 3$, or 4), if

$$\Delta_k f(z) = \Delta_{k+1} f(z) \ . \tag{2.1}$$

(For convenience of notation it has been assumed that in the case when $k = 4$, the operator Δ_5 means Δ_1.)

The importance of this method of definition is illustrated by the following theorem:- a tetradiffric function can be represented in any of the four quadrants of the complex plane by a linear combination of values from both the X and Y axes.

THEOREM 2.1. *The unique tetradiffric function* f, *with values prescribed on the axes* (*on* $X^+ \cup X^- \cup Y^+ \cup Y^-$) *is given by the following:-*

(*i*) *if* $z = (x, y) \in G_1$,

$$f(z) = (1-i)^{-(x+y)}\left\{\sum_{j=0}^{x} \binom{x+y}{j}(-i)^j(1-i\Delta_1)^{x-j} f(x-j, 0) + \sum_{j=x+1}^{x+y} \binom{x+y}{j}(-i)^j(1-\Delta_2)^{j-x} f(0, j-x)\right\} ;$$

(*ii*) *if* $z = (x, y) \in G_2$,

$$f(z) = (1+i)^{x-y}\left\{\sum_{j=0}^{-x} \binom{y-x}{j} i^j (1-i\Delta_3)^{-x-j} f(x+j, 0) + \sum_{j=1-x}^{y-x} \binom{y-x}{j} i^j (1+\Delta_2)^{x+j} f(0, x+j)\right\} ;$$

(*iii*) *if* $z = (x, y) \in G_3$,

$$f(z) = (1-i)^{x+y}\left\{\sum_{j=0}^{-x} \binom{-x-y}{j}(-i)^j(1+i\Delta_3)^{-x-j} f(x+j, 0) + \sum_{j=1-x}^{-x-y} \binom{-x-y}{j}(-i)^j(1+\Delta_4)^{x+j} f(0, -x-j)\right\} ;$$

(*iv*) *if* $z = (x, y) \in G_4$,

$$f(z) = (1+i)^{y-x}\left\{\sum_{j=0}^{x} \binom{x-y}{j} i^j (1+i\Delta_1)^{x-j} f(x-j,\ 0) + \sum_{j=x+1}^{x-y} \binom{x-y}{j} i^j (1-\Delta_4)^{j-x} f(0,\ x-j)\right\} .$$

The binomial operators in the above are defined in the usual way; for example

$$(1-i\Delta_1)^{x-j} = \sum_{j=0}^{x-j} \binom{x-j}{k} (-i)^k \Delta_1^k \ ; \quad (1-i\Delta_1)^0 = I \ ,$$

where I is the identity operator.

The proof of *(i)* above follows from [*6* , Theorem 2.3] and *(ii)*, *(iii)*, *(iv)* are proved in a similar way.

Hence a tetradiffric function $f(z)$ can be expressed in terms of a combination of specified values on the two half-axes which bound the quadrant containing the point z .

For example consider two simple cases:- from the above theorem it follows that for $z = (1,\ 1) \in G_1$,

(2.2) $$f(1,\ 1) = (1-i)^{-1}[f(1,\ 0)-if(0,\ 1)] \ ,$$

and for $z = (2,\ -1) \in G_4$,

$$f(2,\ -1) = (1+i)^{-3}[2if(2,\ 0)+(i-1)f(1,\ 0)-(1+i)f(0,\ -1)] \ .$$

3. The tetradiffric function $z^{(\alpha)}$

The monodiffric function $z^{(\alpha)}$ (α not a negative integer), given in [*6*], is now extended to tetradiffric functions. The resulting function highlights some important advantages of the tetradiffric scheme.

For points on the X-axis, the function $x^{(\alpha)}$ is to be defined by

(3.1) $$x^{(\alpha)} = \begin{cases} \dfrac{\Gamma(x+\alpha)}{\Gamma(x)} \ ; & x \in X^+ \ , \\[2ex] \dfrac{(-1)^{\alpha}\Gamma(\alpha-x)}{\Gamma(-x)} \ ; & x \in X^- \ , \end{cases}$$

and on the Y-axis

(3.2) $$(iy)^{(\alpha)} = i^{\alpha}y^{(\alpha)} \; ; \quad iy \in Y^{+} \cup Y^{-} ,$$

where $y^{(\alpha)}$ is given by (3.1).

Note that $x^{(\alpha)}$ satisfies $\Delta_1 x^{(\alpha)} = \alpha x^{(\alpha-1)}$ for $x \in X^{+}$, and $\Delta_3 x^{(\alpha)} = \alpha x^{(\alpha-1)}$ for $x \in X^{-}$. Also it can be shown that $x^{(\alpha)}$ is a very good asymptotic approximation to x^{α} on both X^{+} and X^{-}.

The tetradiffric analogue $z^{(\alpha)}$ of the classical function z^{α} is required to satisfy

(3.3) (i) $\Delta z^{(\alpha)} = z^{(\alpha-1)}$,

(ii) $0^{(\alpha)} = 0$, $\alpha > 0$, and

(iii) $z^{(0)} = 1$,

where $\Delta = \Delta_k$ or Δ_{k+1} for $z \in G_k$; $k = 1, 2, 3, 4$.

The case when $\alpha = n$, a non-negative integer, is quite simple. It can be shown that the function $z^{(n)}$ given by,

(3.4) $$z^{(n)} = \sum_{j=0}^{n} \binom{n}{j} x^{(n-j)} i^{j} y^{(j)} \; ; \quad z^{(0)} = 1 ,$$

is the tetradiffric function satisfying (3.3) and having the values $x^{(n)}$ and $(iy)^{(n)}$ on the axes.

When α is a negative integer, the function $x^{(\alpha)}$ has singularities at certain points on $X^{+} \cup X^{-}$. It will now be assumed that α is not an integer, but is otherwise an arbitrary constant.

By specifying $x^{(\alpha)}$ and $(iy)^{(\alpha)}$ on the axes, Theorem 2.1 provides the tetradiffric function $z^{(\alpha)}$ at any point in G, and as in [6, Theorem 3.1] it can easily be shown that $z^{(\alpha)}$ satisfies conditions (i) and (ii) of (3.3). However the resulting function $z^{(\alpha)}$ has a rather complicated form, and an alternative expression is now derived which has a remarkable analogy with the binomial expansion of the function $z^\alpha = (x+iy)^\alpha$.

THEOREM 3.1. *If* $z = (x, y) \in G$ *and* $x^{(\alpha)}$, $y^{(\alpha)}$ *are defined by* (3.1) *then the tetradiffric function* $z^{(\alpha)}$ *is given by*

$$z^{(\alpha)} = \sum_{j=0}^{\infty} \binom{\alpha}{j} x^{(\alpha-j)} i^j y^{(j)} + \sum_{j=0}^{\infty} \binom{\alpha}{j} x^{(j)} i^{\alpha-j} y^{(\alpha-j)} ,$$

where the two divergent series are summable (E, q) *in the Euler sense for* $q > 0$.

Proof. Define a function $z^{(\alpha)}$ by

$$z^{(\alpha)} = \sum_{j=0}^{\infty} \binom{\alpha}{j} x^{(\alpha-j)} i^j y^{(j)} + \sum_{j=0}^{\infty} \binom{\alpha}{j} x^{(j)} i^{\alpha-j} y^{(\alpha-j)} , \tag{3.5}$$

and let $z = (x, y) \in G_1$. For convenience consider the first of the above two sums and denote it by

$$S_\alpha(z) = \sum_{j=0}^{\infty} a_j , \text{ where } a_j = \binom{\alpha}{j} x^{(\alpha-j)} i^j y^{(j)} .$$

Now it can easily be verified that for $z \in G_1$, the series $S_\alpha(z)$ diverges. For S_α to be summable (E, q) it must be shown that

(a) $\sum_{j=0}^{\infty} a_j \rho^{j+1}$ converges for small ρ, and

(b) the series $s_\alpha(z)$ defined by

$$s_\alpha(z) = \sum_{n=0}^{\infty} (1+q)^{-n-1} \sum_{j=0}^{n} \binom{n}{j} a_j q^{n-j}$$

converges (see Hardy [5]).

If these conditions hold, the series S_α is said to be summable (E, q) to the sum s_α. That condition (a) holds in this case is readily checked. The following lemma shows that (b) is true.

LEMMA 3.1. *For* $z = (x, y) \in G_1$, *the series defined by*

$$s_\alpha(z) = \sum_{n=0}^{\infty} (1+q)^{-n-1} \sum_{j=0}^{n} \binom{n}{j}\binom{\alpha}{j} x^{(\alpha-j)} i^j y^{(j)} q^{n-j}$$

converges absolutely for $q > 0$.

Proof. For $z \in G_1$ it follows from the definitions of $x^{(\alpha)}$ and $y^{(\alpha)}$ that

$$\binom{\alpha}{j} x^{(\alpha-j)} y^{(j)} = \frac{(\alpha-j+1)(\alpha-j+2)\ldots(\alpha-j+x-1)(j+1)(j+2)\ldots(j+y-1)}{\Gamma(x)\Gamma(y)} .$$

This is a polynomial in j of degree $(x+y-2)$, and can be written as

$$\binom{\alpha}{j} x^{(\alpha-j)} y^{(j)} = \sum_{k=0}^{x+y-2} b_k j^k ,$$

where the coefficients b_k are determined by x, y and α . Hence $s_\alpha(z)$ becomes;

$$s_\alpha(z) = \sum_{n=0}^{\infty} (1+q)^{-n-1} \sum_{k=0}^{x+y-2} b_k \sum_{j=0}^{n} \binom{n}{j} q^{n-j} i^j j^k .$$

Now it can readily be shown by induction on k that for fixed n ,

$$\sum_{j=0}^{n} \binom{n}{j} q^{n-j} i^j j^k = \begin{cases} (q+i)^n & ; \quad k = 0 , \\ \displaystyle\sum_{r=1}^{k} \frac{S_r^{(k)} n!\, i^r (q+i)^{n-r}}{(n-r)!} & ; \quad k \ge 1 , \end{cases}$$

where $S_r^{(k)}$ are Stirling numbers of the second kind.

Hence, assuming for the moment that summation can be interchanged,

$$s_\alpha(z) = b_0 \sum_{n=0}^{\infty} \frac{(q+i)^n}{(q+1)^{n+1}} + \sum_{k=1}^{x+y-2} b_k \sum_{r=1}^{k} \frac{S_r^{(k)} i^r}{(q+1)^{r+1}} \sum_{n=0}^{\infty} \frac{n!}{(n-r)!} \left(\frac{q+i}{q+1}\right)^{n-r} ,$$

and since $\left|\frac{q+i}{q+1}\right| < 1$ when $q > 0$, it follows that the above series are absolutely convergent, which justifies the interchange of summation and proves the lemma.

Returning to the proof of the theorem; it has been shown that $s_\alpha(z)$

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converges and hence by (a) and (b) above, $S_\alpha(z)$ is summable (E, q) for $q > 0$ to the sum $s_\alpha(z)$.

Similarly the second series in (3.5), $\sum_{j=0}^{\infty} \binom{\alpha}{j} x^{(j)} i^{\alpha-j} y^{(\alpha-j)}$ is summable (E, q) , $q > 0$.

Now by (1.1),

$$\Delta_1 z^{(\alpha)} = z^{(\alpha)} - (z-1)^{(\alpha)}$$

$$= \sum_{j=0}^{\infty} \binom{\alpha}{j} x^{(\alpha-j)} i^j y^{(j)} - \sum_{j=0}^{\infty} \binom{\alpha}{j} (x-1)^{(\alpha-j)} i^j y^{(j)}$$

$$+ \sum_{j=0}^{\infty} \binom{\alpha}{j} x^{(j)} i^{\alpha-j} y^{(\alpha-j)} - \sum_{j=0}^{\infty} \binom{\alpha}{j} (x-1)^{(j)} i^{\alpha-j} y^{(\alpha-j)} ,$$

and by Hardy [5, p. 180, Properties α, β] it follows that

$$\Delta_1 z^{(\alpha)} = \alpha \sum_{j=0}^{\infty} \binom{\alpha-1}{j} i^j y^{(j)} x^{(\alpha-1-j)} + \alpha \sum_{j=0}^{\infty} \binom{\alpha-1}{j} i^{\alpha-1-j} y^{(\alpha-1-j)} x^{(j)}$$

$$= \alpha z^{(\alpha-1)}$$

Similarly $\Delta_2 z^{(\alpha)} = \alpha z^{(\alpha-1)}$ and so the function $z^{(\alpha)}$ is tetradiffric for $z \in G_1$. It evidently satisfies $0^{(\alpha)} = 0$.

On the axes, $z^{(\alpha)} = x^{(\alpha)}$ when $y = 0$, and $z^{(\alpha)} = i^{\alpha} y^{(\alpha)}$ when $x = 0$. Hence by Theorem 2.1, $z^{(\alpha)}$ is the unique tetradiffric function in G_1 with prescribed values $x^{(\alpha)}$ on X^+ and $(iy)^{(\alpha)}$ on Y^+ .

In a similar manner it can be shown that (3.5) represents the tetradiffric analogue of z^{α} in the other three quadrants G_2, G_3 and G_4 . This completes the proof of Theorem 3.1.

As an example of the method of Euler summability in the above theorem, consider the simple case $z = 1 + i$. From (3.5),

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$$z^{(\alpha)} = (1,\ 1)^{(\alpha)} = \sum_{j=1}^{\infty} \binom{\alpha}{j} 1^{(\alpha-j)} i^j 1^{(j)} + \sum_{j=0}^{\infty} \binom{\alpha}{j} 1^{(j)} i^{\alpha-j} 1^{(\alpha-j)}$$

$$= 1^{(\alpha)} \sum_{j=0}^{\infty} i^j + 1^{(\alpha)} \sum_{j=0}^{\infty} i^{\alpha-j} .$$

Defining S by

$$S \equiv \sum_{j=0}^{\infty} i^j = 1 + i - 1 - i + 1 + i - 1 - i + \ldots ,$$

then by Hardy [5, p. 180, Properties γ, δ] it follows that

$$S = 1 + i(1+i-1-i+1+ \ldots)$$
$$= 1 + iS$$

and so $S = (1-i)^{-1}$. Similarly

$$\sum_{j=0}^{\infty} i^{\alpha-j} = -i^{\alpha+1}(1-i)^{-1}$$

and hence $(1,\ 1)^{\alpha} = 1^{(\alpha)}\left(1-i^{\alpha+1}\right)(1-i)^{-1}$, which checks with (2.2) on substituting $f(1,\ 0) = 1^{(\alpha)}$, $f(0,\ 1) = i^{\alpha}1^{(\alpha)}$.

4. Properties

When α is not an integer, the tetradiffric function $z^{(\alpha)}$ given by (3.5) is evidently multi-valued. This demonstrates a good analogy with the classical function z^{α} .

Also by making use of backward differences on the positive half axes and forward differences on the negative half axes, the function $z^{(\alpha)}$ can be shown to be a very good approximation to z^{α} on $X^{+} \cup X^{-} \cup Y^{+} \cup Y^{-}$, even for small integer values of x and y .

The Schwarz Reflection Principle has an analogy for the tetradiffric function $z^{(\alpha)}$ as is indicated in the following theorem.

THEOREM 4.1. *When* α *is real, the tetradiffric function* $z^{(\alpha)}$ *is real for* $z \in X^{+}$ *and satisfies the symmetry condition* $z^{(\alpha)} = \overline{(\bar{z})^{(\alpha)}}$ *for*

$z \in G_1 \cup G_4$.

Proof. Let $z = (x, y) \in G_1$. By (3.4),

$$(\bar{z})^{(\alpha)} = (x, -y)^{(\alpha)} = \sum_{j=0}^{\infty} \binom{\alpha}{j} x^{(\alpha-j)} i^{j} (-y)^{(j)} + \sum_{j=0}^{\infty} \binom{\alpha}{j} x^{(j)} i^{\alpha-j} (-y)^{(\alpha-j)} .$$

Since $iy \in Y^+$, $-iy \in Y^-$ it follows from (3.1), (3.2) that

$$(-y)^{(\alpha)} = (-1)^{\alpha} \frac{\Gamma(\alpha+y)}{\Gamma(y)} = (-1)^{\alpha} y^{(\alpha)} .$$

Hence

$$(\bar{z})^{(\alpha)} = \sum_{j=0}^{\infty} \binom{\alpha}{j} x^{(\alpha-j)} (-i)^{j} y^{(j)} + \sum_{j=0}^{\infty} \binom{\alpha}{j} x^{(j)} (-i)^{\alpha-j} y^{(\alpha-j)} ,$$

and since $x^{(\alpha)}, y^{(\alpha)}$ are real for real α and $x \geq 0$, $y \geq 0$, it follows that

$$\overline{(\bar{z})^{(\alpha)}} = z^{(\alpha)} .$$

If $z \in G_4$ the above argument can be reversed, proving the theorem.

Another important property of $z^{(\alpha)}$ which demonstrates once again the symmetry of tetradiffric functions is given by the following.

THEOREM 4.2. *For* $z \in G$,

$$(-z)^{(\alpha)} = (-1)^{\alpha} z^{(\alpha)} .$$

The proof follows immediately from (3.1), (3.2), and (3.5).

Theorem 4.1 can be generalized to a wider class of tetradiffric functions as follows.

THEOREM 4.3. *If f is a tetradiffric function which is real on the X-axis and such that $\overline{f(\bar{z})} = f(z)$ for $z \in Y^+ \cup Y^-$, then for all $z \in G$,*

$$\overline{f(\bar{z})} = f(z) .$$

The proof follows readily from Theorem 2.1 and so is omitted.

When $\alpha = n$ a non-negative integer, Theorems 4.1 and 4.2 also apply to the function $z^{(n)}$ given by (3.4).

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For convenience it has been assumed throughout this paper that the functions concerned are tetradiffric on all of G . This restriction can of course be weakened to a consideration of functions tetradiffric on smaller domains.

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Department of Supply,
Weapons Research Establishment,
Salisbury,
South Australia.

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