This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to I. G. Connell, Department of Mathematics, McGill University, Montreal, P. Q.

COMPUTATION OF THE NUMBER OF SCORE SEQUENCES IN ROUND-ROBIN TOURNAMENTS

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We consider round-robin tournaments of n players in which, at each encounter, the winner is awarded 1 point and the loser 0 (ties are excluded).

Let

$$s_1 \leq s_2 \leq \ldots \leq s_n \tag{1}$$

be the n scores, ordered in a non-decreasing sequence. Clearly

$$s_1 + s_2 + \ldots + s_k \begin{cases} \geq \binom{k}{2}, & k = 1, 2, \ldots, n-1, \\ = \binom{n}{2}, & k = n. \end{cases} \tag{2}$$

Landau [2] has shown that (2) is a sufficient condition for a sequence of non-negative integers (1) to be a score sequence of some tournament. For related results, see [1], [3], [4].

Let $f_n(T, E)$, $n \geq 2$, be the number of sequences of non-negative integers satisfying (1) and

$$\sum_{i=1}^{n} s_i = T, \quad s_n = E,$$

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and such that

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2}, \qquad \text{for } k = 1, 2, \ldots, n-1.$$

It is easy to see that if we define

$$f_1(T, E) = \begin{cases} 1 & \text{if } T = E \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

then, for $n \geq 2$,

$$f_n(T, E) = \begin{cases} 0 & \text{if } T-E < \binom{n-1}{2} \\ \sum_{k=0}^{E} f_{n-1}(T-E, k) & \text{if } T-E \geq \binom{n-1}{2} \end{cases} .$$

Let t_n, $n \geq 2$, be the number of sequences of non-negative integers satisfying (1) and (2). Then we have immediately

$$t_n = \sum_{E=r}^{n-1} f_n\left(\binom{n}{2}, E\right), \qquad n \geq 2,$$

where

$$r = \left[\frac{n}{2}\right].$$

Using a computer, we used this recursive relation to evaluate t_n for $n = 2, 3, \ldots, 27$; these are listed in the accompanying table.

t_{2n} may be obtained from tables of $f_n(T, E)$ by the relation

$$t_{2n} = \sum_{T=\binom{n}{2}}^{\binom{2n}{2}} \sum_{E=r}^{2n-1-s} \left\{ f_n(T, E) \sum_{k=r}^{2n-1-E} f_n(T, k) \right\}$$

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and t_{2n+1} from

$$t_{2n+1} = \sum_{T=\binom{n+1}{2}}^{\binom{2n+1}{2}} \sum_{E=s}^{2n-s} \left\{ f_{n+1}(T,E) \sum_{k=r}^{2n-E} f_n(T-n,k) \right\} ,$$

where $s = \left[\frac{n-1}{2}\right] + 1$.

From these relations we obtained t_n for $n = 28, \ldots, 36$, and checked t_n for $n = 5, \ldots, 27$.

The number of distinct sets of scores in a round-robin tournament of n players.

n	t_n	n	t_n
2	1	20	259,451,116
3	2	21	951,695,102
4	4	22	3,251,073,303
5	9	23	11,605,141,649
6	22	24	41,631,194,766
7	59	25	150,021,775,417
8	167	26	542,875,459,724
9	490	27	1,972,050,156,181
10	1,486	28	7,189,259,574,618
11	4,649	29	26,295,934,251,565
12	14,805	30	96,478,910,768,821
13	48,107	31	354,998,461,378,719
14	158,808	32	1,309,755,903,513,481
15	531,469	33	4,844,523,965,710,167
16	1,799,659	34	17,961,489,379,744,400
17	6,157,068	35	66,742,666,423,989,519
18	21,258,104	36	248,530,319,605,591,021
19	73,996,100		

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