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# SPECTRUM OF THE LAPLACIAN OF AN ASYMMETRIC FRACTAL GRAPH

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*Abstract* We consider a simple self-similar sequence of graphs that does not satisfy the symmetry conditions that imply the existence of a spectral decimation property for the eigenvalues of the graph Laplacians. We show that, for this particular sequence, a very similar property to spectral decimation exists, and we obtain a complete description of the spectra of the graphs in the sequence.

*Keywords:* fractal graph; Laplacian; spectral decimation

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## 1. Introduction and definitions

### 1.1. Introduction

Many self-similar graphs, and related fractals, display a property known as *spectral decimation*: that the spectrum of the Laplacian can be described in terms of the iteration of a rational function $f$. Eigenvalues $\lambda$ of the Laplacian at a given stage of the construction are related to eigenvalues $\mu$ of the Laplacian at the following stage of the construction by the relationship

$$\lambda = f(\mu), \tag{1.1}$$

where $f$ is a rational function on $\mathbb{R}$, unless $\mu$ is a member of a small *exceptional set*, $\mathcal{E}$. This was first observed for the specific case of the Sierpiński gasket graph in [**8**], and this was given a rigorous mathematical treatment in [**4**, **11**, **12**]. In the case of the Sierpiński gasket, using our definition of the Laplacian (see § 1.3), the function $f(\mu) = \mu(5 - 4\mu)$ and the exceptional set is $\{\frac{1}{2}, \frac{5}{4}, \frac{3}{2}\}$.

A generalization of spectral decimation to a much larger class of self-similar graphs, including the Vicsek set graph, appears in [**7**], in which a symmetry condition is developed which, if satisfied, ensures that spectral decimation applies to the graph. Each self-similar graph in this class has a function $f$ and exceptional set $\mathcal{E}$ associated with it.

In this paper we consider a simple asymmetric self-similar graph which does not satisfy the symmetry condition of [**7**]. In § 1.2 we define a sequence of graphs $(G_n)_{n\in\mathbb{N}}$, which can be used to define a self-similar graph $G_\infty$ as for the self-similar graphs in [**7**].

In § 2 we show that, for this example, a property similar to spectral decimation exists, in which (1.1) is replaced by

$$2(1-\lambda)^2 = f(\mu), \tag{1.2}$$

where $f(\mu)$ is a quartic polynomial. Again there is an exceptional set of values of $\mu$ for which the relationship does not necessarily hold. This is proved in Theorems 2.1 and 2.2.

Another common spectral property of self-similar graphs and related fractals is that there are many eigenvalues of the Laplacian with high multiplicity and Dirichlet–Neumann eigenfunctions, i.e. eigenfunctions which are zero on the boundary. In [**9**] it is shown that the eigenvalues with Dirichlet–Neumann eigenfunctions dominate the spectrum in a large class of cases, that of the nested fractals introduced in [**5**]. In [**6**] a similar result is shown for two-point self-similar graphs, a class which includes the example in this paper.

In § 3 we calculate the number of linearly independent eigenfunctions of the Laplacian which are Dirichlet–Neumann or non-Dirichlet–Neumann. In § 4, we use the results of §§ 2 and § 3 to describe the spectra of the graphs in the self-similar sequence of finite graphs used in the construction of our graph. This is stated in Theorem 4.1, which gives a complete description of the spectrum, including the multiplicity of the eigenvalues and which eigenvalues are associated with Dirichlet–Neumann and non-Dirichlet–Neumann eigenfunctions.

An example of a self-similar graph which does not satisfy the symmetry conditions of [**7**], and for which spectral decimation appears not to apply, is associated with the pentagasket, as described in [**1**], in which numerical approximations for eigenvalues and eigenvectors are obtained, and some theoretical results are obtained that show how to construct eigenspaces of high multiplicity.

A more complicated method, using a rational map on a projective variety rather than on $\mathbb{R}$, which works for a larger class of self-similar graphs than that in [**7**] including some for which spectral decimation does not apply, is described in [**10**]. However, our graph does not meet all the conditions described in § 1.1.1 of [**10**].

### 1.2. The graph

In this section we define a self-similar sequence of finite graphs $(G_n)_{n\in\mathbb{N}}$.

We label the vertex sets and edge sets of a graph $G$ by $V(G)$ and $E(G)$, respectively, and, for a vertex $i \in V(G_n)$, we define $E_i^{(n)}$ to be the set of edges of $G_n$ connected to $i$.

We start with $G_0$, a single edge between two vertices 1 and 2, and proceed inductively, constructing $G_{n+1}$ from $G_n$. Our graphs will be defined in such a way that $V(G_{n-1}) \subseteq V(G_n)$.

To construct $G_{n+1}$, we assume as an induction hypothesis that, if $n \geqslant 1$, the graph $G_n$ is bipartite with the two parts being $V(G_{n-1})$ and $V(G_n) \setminus V(G_{n-1})$, and hence that each edge $e \in E(G_n)$ can be thought of as connecting two vertices $i(e)$ and $j(e)$, defined so that $i(e) \in V(G_{n-1})$ and $j(e) \in V(G_n) \setminus V(G_{n-1})$. To deal with the special case $G_0$, we set $i(e_0) = 1$ and $j(e_0) = 2$ for its single edge $e_0$.

For each $e \in E(G_n)$ we introduce a new vertex, which we label $k(e)$, and we let the vertex set $V(G_{n+1})$ of $G_{n+1}$ be the union of $V(G_n)$ with the set of new vertices

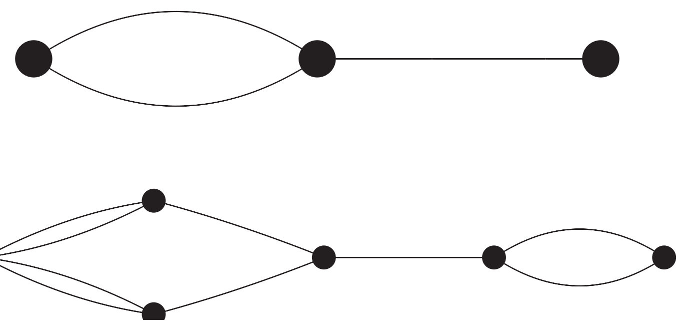

Figure 1. The graphs $G_1$ and $G_2$.

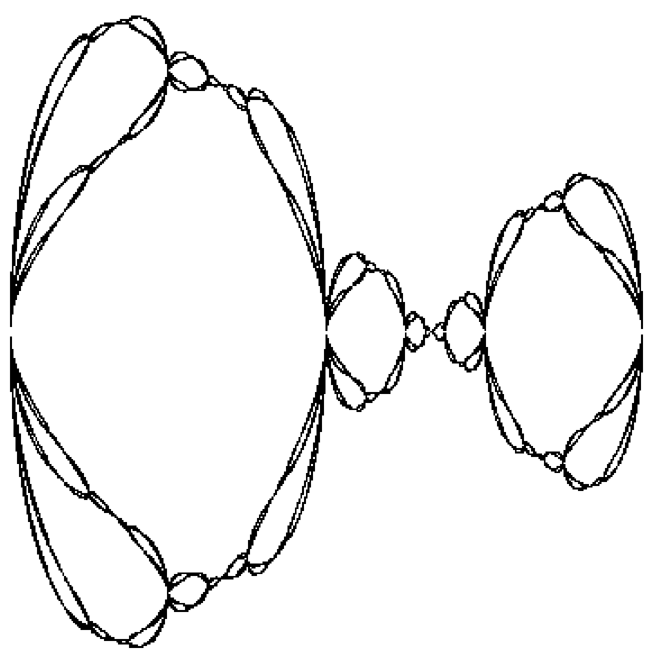

Figure 2. A later stage of the construction.

$\{k(e),\ e \in E(G_n)\}$. We then define $E(G_{n+1})$ to consist of, for each $e \in E(G_n)$, two edges connecting $k(e)$ with $i(e)$ and one edge connecting $k(e)$ with $j(e)$. This ensures that the new graph $G_{n+1}$ is bipartite with the two parts being $V(G_n)$ and $V(G_{n+1}) \setminus V(G_{n-1})$, so that we can continue the construction inductively.

Figure 1 shows $G_1$ and $G_2$, and Figure 2 shows a later stage of the construction, generated using MAPLE.

The same sequence of graphs can be obtained by using the framework of Definition 5.2 of [**7**], with the model graph being identical to $G_1$ above, but with conditions on the orientation to deal with the asymmetry.

It can also be obtained by a variation on the framework of § 1.1.1 of [**10**]. In that framework, the sequence of graphs is obtained from a basic cell $F = \{1, \ldots, N_0\}$, for some $N_0$, and an equivalence relation $\mathcal{R}$ defined on $\{1, \ldots, N\} \times F$, where $N$ is the number of cells and satisfies $N \geqslant N_0$. In our case, $N_0 = 2$ and $N = 3$, and our model graph $G_1$ can be defined by an equivalence relation $\mathcal{R}$ on $\{1, 2, 3\} \times \{1, 2\}$ with three equivalence classes $\{(1, 1), (3, 1)\}$, $\{(1, 2), (2, 2), (3, 2)\}$ and the singleton $\{(2, 1)\}$. However, in [**10**] the equivalence relation $\mathcal{R}$ is required to satisfy three conditions, one of which is that the equivalence class of $(i, i)$ for $1 \leqslant i \leqslant N_0$ should be a singleton, and our equivalence relation does not satisfy this condition, although it does satisfy the other two. As a result of this, the definitions of the equivalence relations $\mathcal{R}_{\langle\infty\rangle}$ and $\mathcal{R}_{\langle n\rangle}$, used in [**10**] to define the infinite graph and its subsets, need to be modified to deal with the more complicated behaviour of the boundary points.

When $1 \leqslant m < n$, the graph $G_n$ contains $3^{n-m}$ subgraphs isomorphic to $G_m$. We will call these subgraphs *m-cells*. Using this, we can define a sequence $(\tilde{G}_n)_{n\in\mathbb{N}}$ such that $\tilde{G}_n$ is isomorphic to $G_n$ and $\tilde{G}_m$ is a subgraph of $\tilde{G}_n$ for $m < n$. We then define the infinite graph $G_\infty = \bigcup_{n=0}^{\infty} \tilde{G}_n$. This is analogous to Definition 5.5 of [**7**].

We define maps $f_i : V(G_{n-1}) \to V(G_n)$, $i = 1, 2, 3$, mapping each vertex of $G_{n-1}$ to the corresponding vertex in each $(n-1)$-cell. We will label these so that $f_1$ and $f_2$ correspond to the two parallel cells.

We note that this graph is similar to that described in [**3**], although in the context of that paper the orientation of the cells is not important.

### 1.3. The Laplacian

There are a number of different definitions of the Laplacian of a graph. The definition of the graph Laplacian used in [**7**] is the generator matrix of a continuous-time random walk on the graph, while in [**2**] a related symmetric matrix is used. However, the eigenvalues of the different definitions differ by at most a simple transformation.

For convenience in describing the eigenfunctions, we use the following definition of the Laplacian: the Laplacian $\mathcal{L}_G$ of a graph $G$ (which may have multiple edges but with no loops) is a $|V(G)| \times |V(G)|$ matrix with, for a vertex $i \in V(G)$, $\mathcal{L}_G(i, i) = 1$, and, for $i, j \in V(G)$ with $i \neq j$, $\mathcal{L}_G(i, j) = -e_{i,j}/\delta_i$, where $e_{i,j}$ is the number of edges linking $i$ and $j$ in $G$ and $\delta_i$ is the degree of vertex $i$ in $G$. This gives the same eigenvalues as the symmetric Laplacian described in [**2**], and the eigenfunctions are the 'harmonic eigenfunctions' described in [**2**]. Our definition of the graph Laplacian differs from that in [**7**] only in that the sign of each entry (and hence of the eigenvalues) is reversed.

## 2. The relationship between the eigenvalues

We set $f(\mu) = 9(\mu - 1)^4 - 9(\mu - 1)^2 + 2$, so that (1.2) becomes

$$2(1 - \lambda)^2 = 9(\mu - 1)^4 - 9(\mu - 1)^2 + 2. \tag{2.1}$$

We first show how to construct eigenvalues $\mu$ of $\mathcal{L}_{G_{n+1}}$ from eigenvalues $\lambda$ of $\mathcal{L}_{G_n}$ when $\lambda \notin \{0, 1, 2\}$.

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Given $\lambda$ and $\mu$, we set

$$\gamma = \frac{3(\mu - 1)^2 - 2}{1 - \lambda}. \tag{2.2}$$

**Theorem 2.1.** *Given an eigenfunction $x$ of $\mathcal{L}_{G_n}$ with eigenvalue $\lambda \notin \{0, 1, 2\}$, we can do the following.*

(i) *Solve (2.1) for $\mu$ to obtain four roots.*

(ii) *For each possible $\mu$, set $\gamma$ using (2.2).*

(iii) *Then define $x'$ by*

$$x'_i = \begin{cases} x_i & i \in V(G_{n-1}), \\ \gamma x_i & i \in V(G_n) \setminus V(G_{n-1}), \end{cases} \tag{2.3}$$

*and, for a vertex $k = k(e) \in V(G_{n+1}) \setminus V(G_n)$, we set*

$$x'_k = \frac{2x_{i(e)}}{3(1 - \mu)} + \frac{\gamma x_{j(e)}}{3(1 - \mu)}. \tag{2.4}$$

*Then $x'$ is an eigenfunction of $\mathcal{L}_{G_{n+1}}$ with eigenvalue $\mu$.*

**Proof.** To check this, we just calculate $\mathcal{L}_{G_{n+1}} x'$. For $i \in V(G_{n-1})$,

$$\begin{aligned}
(\mathcal{L}_{G_{n+1}} x')_i &= x_i + \frac{1}{\delta_i^{(n)}} \sum_{e \in E_i^{(n)}} \left( \frac{2x_i}{3(\mu - 1)} + \frac{\gamma x_{j(e)}}{3(\mu - 1)} \right) \\
&= x_i + \frac{2x_i}{3(\mu - 1)} + \frac{\gamma x_i (1 - \lambda)}{3(\mu - 1)} \\
&= x_i \left( \frac{3(\mu - 1) + 2 + 3(\mu - 1)^2 - 2}{3(\mu - 1)} \right) \\
&= \mu x_i = \mu x'_i.
\end{aligned}$$

For $j \in V(G_n) \setminus V(G_{n-1})$,

$$\begin{aligned}
(\mathcal{L}_{G_{n+1}} x')_j &= \gamma x_j + \frac{1}{\delta_j^{(n)}} \sum_{e \in E_j^{(n)}} \left( \frac{2x_{i(e)}}{3(\mu - 1)} + \frac{\gamma x_j}{3(\mu - 1)} \right) \\
&= \gamma x_j + \frac{\gamma x_j}{3(\mu - 1)} + \frac{2x_j (1 - \lambda)}{3(\mu - 1)} \\
&= x_j \left( \frac{3(\mu - 1)\gamma + \gamma + 2(1 - \lambda)}{3(\mu - 1)} \right).
\end{aligned}$$

Using (2.1) and (2.2),

$$\begin{aligned}
2(1 - \lambda) &= \frac{9(\mu - 1)^4 - 9(\mu - 1)^2 + 2}{1 - \lambda} \\
&= \frac{(3(\mu - 1)^2 - 1)(3(\mu - 1)^2 - 2)}{1 - \lambda} \\
&= (3(\mu - 1)^2 - 1)\gamma,
\end{aligned}$$

and so

$$(\mathcal{L}_{G_{n+1}}x')_j = x_j\gamma\biggl(\frac{3(\mu-1)+1+3(\mu-1)^2-1}{3(\mu-1)}\biggr) \\ = \mu\gamma x_j \\ = \mu x'_j,$$

and, finally, for $k \in j \in V(G_n) \setminus V(G_{n-1})$, which satisfies $k = k(e)$ for some edge $e$ of $G_n$, we have

$$(\mathcal{L}_{G_{n+1}}x')_k = x'_k - \tfrac{2}{3}x'_i(e) - \tfrac{1}{3}x'_j(e) \\ = \biggl(\frac{1}{1-\mu} - 1\biggr)(\tfrac{2}{3}x_i + \tfrac{1}{3}\gamma x_j) \\ = \mu\biggl(\frac{2x_i}{3(1-\mu)} + \frac{\gamma x_j}{3(1-\mu)}\biggr) \\ = \mu x'_k,$$

so $x'$ is indeed an eigenfunction of $\mathcal{L}_{G_{n+1}}$ with eigenvalue $\mu$. □

**Theorem 2.2.** *If*

$$\mu \notin \biggl\{1,\ 1+\sqrt{\tfrac{2}{3}},\ 1-\sqrt{\tfrac{2}{3}},\ 1+\sqrt{\tfrac{1}{3}},\ 1-\sqrt{\tfrac{1}{3}}\biggr\}$$

*and $\lambda$ and $\mu$ satisfy (2.1), then $\mu$ is an eigenvalue of $G_{n+1}$ if and only if $\lambda$ is an eigenvalue of $G_n$, with the same multiplicity.*

**Proof.** If we have an eigenfunction $x'$ of $\mathcal{L}_{G_{n+1}}$ with eigenvalue $\mu \neq 1$, then, for each edge $e$ of $G_{n+1}$,

$$x'_{k(e)} = \frac{2x'_{i(e)}}{3(1-\mu)} + \frac{x'_{j(e)}}{3(1-\mu)},$$

so that, for each $i \in V(G_{n-1})$,

$$x'_i(1-\mu) = \frac{1}{\delta_i^{(n)}}\sum_{e\in E_i^{(n)}}\biggl(\frac{2x'_i}{3(1-\mu)} + \frac{x'_{j(e)}}{3(1-\mu)}\biggr) \\ = \frac{2x'_i}{3(1-\mu)} + \frac{1}{\delta_i^{(n)}}\sum_{e\in E_i^{(n)}}\frac{x'_{j(e)}}{3(1-\mu)},$$

giving

$$x'_i(3(1-\mu)^2-2) = \frac{1}{\delta_i^{(n)}}\sum_{e\in E_i^{(n)}} x'_{j(e)}. \tag{2.5}$$

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Similarly, for $j \in V(G_n) \setminus V(G_{n-1})$,

$$x'_j(1-\mu) = \frac{1}{\delta_j^{(n)}} \sum_{e \in E_j^{(n)}} \left( \frac{x'_j}{3(1-\mu)} + \frac{2x'_{i(e)}}{3(1-\mu)} \right)$$
$$= \frac{x'_j}{3(1-\mu)} + \frac{1}{\delta_j^{(n)}} \sum_{e \in E_j^{(n)}} \frac{2x'_{i(e)}}{3(1-\mu)},$$

giving

$$x'_j(3(1-\mu)^2 - 1) = \frac{1}{\delta_j^{(n)}} \sum_{e \in E_j^{(n)}} x'_{i(e)}. \tag{2.6}$$

By the conditions of the theorem, $(1-\mu)^2 \neq \frac{2}{3}$. Then (2.5) implies that, for any $\lambda$,

$$x'_i(1-\lambda) = \frac{1}{\delta_i^{(n)}} \sum_{e \in E_i^{(n)}} \frac{1-\lambda}{3(1-\mu)^2 - 2} x'_{j(e)}, \tag{2.7}$$

while (2.6) gives, if $\lambda \neq 1$,

$$\frac{1-\lambda}{3(1-\mu)^2 - 2x'_j} \frac{(3(1-\mu)^2 - 2)(3(1-\mu)^2 - 1)}{2(1-\lambda)} = \frac{1}{\delta_j^{(n)}} \sum_{e \in E_j^{(n)}} x'_{i(e)}. \tag{2.8}$$

Then, if we set $x_i = x'_i$ for $i \in V(G_{n-1})$ and

$$x_j = x'_j \frac{1-\lambda}{3(1-\mu)^2 - 2},$$

(2.7) and (2.8) become

$$x_i(1-\lambda) = \frac{1}{\delta_i^{(n)}} \sum_{e \in E_i^{(n)}} x_{j(e)}$$

and

$$x_j \frac{(3(1-\mu)^2 - 2)(3(1-\mu)^2 - 1)}{2(1-\lambda)} = \frac{1}{\delta_j^{(n)}} \sum_{e \in E_j^{(n)}} x_{i(e)},$$

which imply that $x$ is an eigenfunction of $\mathcal{L}_{G_n}$ with eigenvalue $\lambda$ if

$$(1-\lambda) = \frac{(3(1-\mu)^2 - 2)(3(1-\mu)^2 - 1)}{2(1-\lambda)},$$

which is equivalent to the quartic (2.1).

This eigenfunction can be degenerate only if $1 - \lambda = 0$, i.e. if either $(1-\mu)^2 = \frac{1}{3}$ or $(1-\mu)^2 = \frac{2}{3}$. □

The set

$$\left\{1,\ 1+\sqrt{\tfrac{2}{3}},\ 1-\sqrt{\tfrac{2}{3}},\ 1+\sqrt{\tfrac{1}{3}},\ 1-\sqrt{\tfrac{1}{3}}\right\}$$

of values of $\mu$ where Theorem 2.2 does not apply plays a similar role to that of the *exceptional set* in [**7**].

We note that the eigenvalues $\lambda$ and $2-\lambda$ produce the same values of $\mu$, with the same eigenfunctions. This is related to the bipartite nature of the graph; in fact, if $x$ is an eigenfunction with eigenvalue $\lambda$, then, following [**2**], we can obtain an eigenfunction with eigenvalue $2-\lambda$ by simply changing the sign of $x$ on $V(G_n)\setminus V(G_{n-1})$. These two eigenfunctions will then produce the same new eigenfunction using the above construction.

We now consider the cases when Theorems 2.1 and 2.2 do not apply, i.e. when $\lambda \in \{0,1,2\}$ or

$$\mu \in \left\{1,\ 1+\sqrt{\tfrac{2}{3}},\ 1-\sqrt{\tfrac{2}{3}},\ 1+\sqrt{\tfrac{1}{3}},\ 1-\sqrt{\tfrac{1}{3}}\right\}.$$

We note that if $\mu = 1$ and $\lambda$ and $\mu$ satisfy (2.1), then $\lambda = 0$ or 2.

When $\lambda = 1$ and $x_i \neq 0$ for some $i \in V(G_{n-1})$, we use the same method, but with $\gamma = 0$ and the quartic (2.1) replaced by

$$(\mu-1)^2 = \tfrac{2}{3}. \tag{2.9}$$

We cannot use this method if $x_i = 0$ for all $i \in V(G_{n-1})$, because the constructed eigenfunction would be zero everywhere.

However, in the case where $\lambda = 1$ and $x_i = 0$ for all $i \in V(G_{n-1})$, we can construct an eigenfunction $x'$ by setting $x'_i = 0$ for $i \in V(G_{n-1})$, and $x'_i = x_i$ for $i \in V(G_n)\setminus V(G_{n-1})$. This gives eigenvalues $\mu$ with

$$(\mu-1)^2 = \tfrac{1}{3}, \tag{2.10}$$

using similar methods to those above.

## 3. Dirichlet–Neumann and non-Dirichlet–Neumann eigenfunctions

The graph $G_n$ has $v_n$ vertices and $e_n$ edges where $v_0 = 2$, $e_0 = 1$ and $e_n = 3e_{n-1}$, $v_n = v_{n-1} + e_{n-1}$. Hence $e_n = 3^n$ and $v_n = \frac{1}{2}(3^n + 3)$.

The following lemma provides a means of constructing Dirichlet–Neumann eigenfunctions, which are zero on the two boundary vertices 1 and 2.

**Lemma 3.1.** *Let $\Gamma_0$ be a connected graph with $m$ vertices, including distinguished endpoints 1 and 2, and let $\Gamma$ be the graph formed by defining $\Gamma_1$ and $\Gamma_2$ to be two identical copies of $\Gamma_0$ and connecting them in parallel by identifying their endpoints. Then the Laplacian of $\Gamma$ has $m-2$ linearly independent eigenfunctions which are zero on the endpoints.*

*The associated eigenvalues are the eigenvalues of the Laplacian $\mathcal{L}_\Gamma$ restricted to the set $\{2j : 2 \leqslant j \leqslant m-1\}$ of vertices in $\Gamma_2$.*

**Proof.** We label the vertices of $\Gamma_0$, $1, 2, \ldots, m$. Then we label the vertices in $\Gamma$ so that, for $j \geqslant 3$, vertex $j$ in $\Gamma_0$ corresponds to vertex $2j-3$ in $\Gamma_1$ and vertex $2j-2$ in $\Gamma_2$.

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Now consider the Laplacian $\mathcal{L}_\Gamma$. For $2 \leqslant j, k \leqslant m-1$ we have

$$\begin{aligned}\mathcal{L}_\Gamma(1, 2j-1) &= \mathcal{L}_\Gamma(1, 2j),\\ \mathcal{L}_\Gamma(2, 2j-1) &= \mathcal{L}_\Gamma(2, 2j),\\ \mathcal{L}_\Gamma(2j-1, 2k-1) &= \mathcal{L}_\Gamma(2j, 2k),\\ \mathcal{L}_\Gamma(2j-1, 2k) &= \mathcal{L}_\Gamma(2j, 2k-1) = 0\end{aligned}$$

and consider functions $x$ satisfying

$$\begin{aligned}x(1) &= x(2) = 0,\\ x(2j-1) &= -x(2j) \quad \text{for } 2 \leqslant j \leqslant m-1.\end{aligned}$$

Now

$$(\mathcal{L}_\Gamma x)(1) = \sum_{j=2}^{m-1} (\mathcal{L}_\Gamma(1, 2j-1)x(2j-1) + \mathcal{L}_\Gamma(1, 2j)x(2j)) = 0$$

and, similarly, $(\mathcal{L}_\Gamma x)(2) = 0$, while

$$\begin{aligned}(\mathcal{L}_\Gamma x)(2j-1) &= \sum_{k=2}^{m-1} \mathcal{L}_\Gamma(2j-1, 2k-1)x(2k-1)\\ &= -\sum_{k=2}^{m-1} \mathcal{L}_\Gamma(2j, 2k)x(2k) = (\mathcal{L}_\Gamma x)(2j).\end{aligned}$$

So the Laplacian $\mathcal{L}_\Gamma$ preserves vectors of this form, which form a vector space of dimension $m-2$, and it acts on them in a similar way to the Laplacian restricted to the interior vertices of $\Gamma_0$. As the Laplacian is symmetric, there are $m-2$ linearly independent Dirichlet–Neumann eigenfunctions of the Laplacian. □

If $G_n$ contains a subgraph $\Gamma$ of this form, where the vertices of $\Gamma$ other than the endpoints have no edges linking them to $G_n \backslash \Gamma$, then we can take one of the eigenfunctions $x$ on $\Gamma$ constructed by the above lemma and extend it to an eigenfunction $\tilde{x}$ on $G_n$ by setting

$$\tilde{x}(v) = \begin{cases} x(v) & \text{for } v \in V(\Gamma),\\ 0 & \text{otherwise.}\end{cases}$$

If neither of the endpoints 1 or 2 is in the interior of the subgraph $\Gamma$, then this $\tilde{x}$ will be Dirichlet–Neumann.

**Proposition 3.2.** *The graph $G_n$ has at least $\frac{1}{2}(3^n + 3) - 2^n - 1$ linearly independent Dirichlet–Neumann eigenvalues.*

**Proof.** The model graph contains parallel edges, so $G_n$ contains a subgraph consisting of two copies of $G_{n-1}$ with their boundary points identified as in Lemma 3.1. This gives $v_{n-1} - 2$ eigenfunctions. For each eigenfunction $x$ obtained thus, we have $x(f_1(v)) = -x(f_2(v))$ and $x(f_3(v)) = 0$ for each $v \in V(G_{n-1})$.

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Furthermore, given a Dirichlet–Neumann eigenfunction $x$ of $G_{n-1}$, we can obtain three Dirichlet–Neumann eigenfunctions $x_1$, $x_2$, $x_3$ of $G_n$ by extending them from $(n-1)$-cells to the whole graph, i.e.

$$x_i(f_j(v)) = \delta_{ij}x(v) \quad \text{for all } v \in V(G_{n-1}), \quad i,j = 1,2,3.$$

However, we only obtain two linearly independent eigenfunctions, because the linear combination $x_1 - x_2$ is of the form obtained using Lemma 3.1.

Hence, if $l_n$ is the number of Dirichlet–Neumann eigenvalues constructed by these methods, it satisfies

$$l_n = 2l_n + \tfrac{1}{2}(3^{n-1} + 3) - 2$$

and $l_2 = 1$, which gives the result. □

Let the total number of Dirichlet–Neumann eigenfunctions of $G_n$ be $l_n + \hat{l}_n$, so that $\hat{l}_n$ is the number of eigenfunctions that are not constructed by the methods of Proposition 3.2.

For this graph, we can also describe a set of non-Dirichlet–Neumann eigenfunctions.

**Proposition 3.3.** *The graph $G_n$ has $2^n + 1$ linearly independent eigenfunctions which are not Dirichlet–Neumann.*

**Proof.** We consider the set of functions $x$ on $V(G_1)$ which are zero on the central vertex of $V(G_1)$, i.e. they satisfy $x(3) = 0$ and, for each $v \in V(G_{n-1})$, $x(f_1(v)) = x(f_2(v)) = \frac{1}{2}x(f_3(v))$; these form a $(v_{n-1} - 1)$-dimensional subspace. This is preserved by the Laplacian of $G_n$, so we can find $v_{n-1} - 1 = \frac{1}{2}(3^{n-1} + 1)$ linearly independent eigenfunctions satisfying these properties.

To exclude those which are Dirichlet–Neumann, this is equivalent to the condition that $x(1) = x(2) = 0$. So a Dirichlet–Neumann eigenfunction satisfying the above conditions reduces to a Dirichlet–Neumann eigenfunction on each $(n-1)$-cell. Hence there are $\frac{1}{2}(3^{n-1} + 1) - l_{n-1} - \hat{l}_{n-1} = 2^{n-1} - \hat{l}_{n-1}$ non-Dirichlet–Neumann eigenfunctions satisfying the above conditions.

Now, given any eigenfunction $x$ of $G_{n-1}$, we can extend it to an eigenfunction $x'$ of $G_n$ by setting $x'(1) = \sqrt{2}x(1)$, $x'(2) = x(1)$, $x'(3) = \sqrt{3}x(2)$, $x'(f_1(v)) = x'(f_2(v)) = x'(f_3(v)) = x(v)$ for $v \geqslant 3$, from the structure of the graph. This means that each of the eigenfunctions constructed in Proposition 3.3 for $G_{n-1}$ can be extended to a non-Dirichlet–Neumann eigenfunction of $G_n$, which will be linearly independent of those already found (because $x'(3) \neq 0$). Inductively, this also applies to those constructed for $G_{n-m}$, $m > 1$.

The graphs are bipartite, so eigenfunctions (which are zero nowhere) for eigenvalues 0 and 2 exist as described in [**2**].

Hence the total number of linearly independent non-Dirichlet–Neumann eigenfunctions is at least $2^n + 1 - \sum_{m=0}^{n-1} \hat{l}_n$. But we know that there are exactly $\frac{1}{2}(3^n + 3)$ linearly independent eigenfunctions. Hence

$$\tfrac{1}{2}(3^n + 3) \geqslant 2^n + 1 - \sum_{m=0}^{n-1} \hat{l}_m + \tfrac{1}{2}(3^n + 3) - 2^n - 1 + \hat{l}_n$$

and so $\hat{l}_n \leqslant \sum_{m=0}^{n-1} \hat{l}_m$. But $\hat{l}_m = 0$ for $m \leqslant 2$, and hence for all $m$.

Hence the eigenfunctions constructed are all that exist, and there are $2^n + 1$ linearly independent non-Dirichlet–Neumann ones, the remainder being Dirichlet–Neumann. □

## 4. The spectra of the graphs

We can now use the relationships between eigenvalues and the information on Dirichlet–Neumann and non-Dirichlet–Neumann eigenfunctions to obtain a complete description of the spectra of the graphs $G_n$.

**Theorem 4.1.** *Set*

$$\alpha_1^{(1)} = 1, \quad \alpha_1^{(2)} = 1 - \sqrt{\tfrac{2}{3}} \quad \text{and} \quad \alpha_2^{(2)} = 1 + \sqrt{\tfrac{2}{3}}.$$

*We extend this to define*

$$\{\alpha_i^{(n)};\ 1 \leqslant i \leqslant 2^{n-1}\}$$

*to be the* $2^{n-1}$ *values* $\mu$ *satisfying the quartic (2.1), with* $\lambda = \alpha_j^{(n-1)}$ *for some* $j$*.*

*Similarly, set*

$$\beta_1^{(1)} = 1, \quad \beta_1^{(2)} = 1 - \sqrt{\tfrac{1}{3}} \quad \text{and} \quad \beta_2^{(2)} = 1 + \sqrt{\tfrac{1}{3}}.$$

*We extend this to define*

$$\{\beta_i^{(n)};\ 1 \leqslant i \leqslant 2^{n-1}\}$$

*to be the* $2^{n-1}$ *values* $\mu$ *satisfying the quartic (2.1), with* $\lambda = \beta_j^{(n-1)}$ *for some* $j$*.*

*Then we have the following.*

(a) *If* $n \geqslant m$*, then* $\mathcal{L}_{G_n}$ *has a non-Dirichlet–Neumann eigenfunction with eigenvalue* $\alpha_i^{(m)}$*,* $1 \leqslant i \leqslant 2^{m-1}$*. Together with eigenfunctions with eigenvalues* 0 *and* 2*, this describes the non-Dirichlet–Neumann spectrum of* $\mathcal{L}_{G_n}$*.*

(b) *If* $n \geqslant m+1$*, then* $\mathcal{L}_{G_n}$ *has* $\frac{1}{2}(3^{n-m} - 1)$ *linearly independent Dirichlet–Neumann eigenvalues with eigenvalue* $\beta_i^{(m)}$*, for* $1 \leqslant i \leqslant 2^{m-1}$*.*

**Proof.** We note that $G_1$ has a non-Dirichlet–Neumann eigenfunction with eigenvalue 1. As described in the proof of Proposition 3.3, this can then be extended to give a non-Dirichlet–Neumann eigenfunction $x$ with eigenvalue 1 for each $G_n$, $n \geqslant 1$.

Because this eigenfunction is non-Dirichlet–Neumann, it has $x_i \neq 0$ for at least some $i \in V(G_{n-1})$. Hence the construction of eigenfunctions of $G_{n+1}$ with eigenvalues $\mu$ satisfying $(1-\mu)^2 = \frac{2}{3}$ produces non-degenerate eigenfunctions, which are also non-Dirichlet–Neumann. So $G_{n+1}$ has non-Dirichlet–Neumann eigenfunctions with eigenvalues

$$1 \pm \sqrt{\tfrac{2}{3}}.$$

We have already shown that $G_n$ has a non-Dirichlet–Neumann eigenfunction with eigenvalue $\alpha_i^{(2)}$, $1 \leqslant i \leqslant 2^{n-1}$, for $n \geqslant 2$. Now, if $G_{n-1}$ has a non-Dirichlet–Neumann

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eigenfunction with eigenvalue $\alpha_i^{(m-1)}$ for each $1 \leqslant i \leqslant 2^{m-2}$, then the construction of eigenfunctions gives us a non-Dirichlet–Neumann eigenfunction of $G_n$ with eigenvalue $\alpha_i^{(m)}$ for each $1 \leqslant i \leqslant 2^{m-1}$. Using this inductively, we find that $G_n$ has a non-Dirichlet–Neumann eigenfunction with eigenvalue $\alpha_i^{(m)}$, $1 \leqslant i \leqslant 2^{m-1}$, for $n \geqslant m$.

Along with the eigenfunctions with eigenvalues 0 and 2, this gives us all $2^n + 1$ non-Dirichlet–Neumann eigenfunctions from Proposition 3.3, and hence completes the proof of (a).

We now consider the Dirichlet–Neumann eigenfunctions. We note that $G_2$ has a Dirichlet–Neumann eigenfunction with eigenvalue 1. Such an eigenfunction is zero on $V(G_1)$, so our main construction produces a degenerate eigenfunction. However, the alternative construction with $(1-\mu)^2 = \frac{1}{3}$ does produce two eigenfunctions of $G_3$. Hence, as there are $\frac{1}{2}(3^n + 3) - 2^n - 1$ Dirichlet–Neumann eigenfunctions of $G_n$, we can use the constructions to obtain $3^n + 3 - 2^{n+1} - 2$ Dirichlet–Neumann eigenfunctions, with eigenvalues other than 1, of $G_{n+1}$.

We now show that, for $n \geqslant 2$, $G_n$ has $\frac{1}{2}(3^{n-1} - 1)$ linearly independent Dirichlet–Neumann eigenfunctions with eigenvalue 1. This is the case for $n = 2$. For each such eigenfunction of $G_n$, we can construct three eigenfunctions of $G_{n+1}$ using the methods in the proof of Proposition 3.2.

Assuming that $G_n$ has $\frac{1}{2}(3^n + 3) - 2^n - 1$ linearly independent Dirichlet–Neumann eigenfunctions of which $\frac{1}{2}(3^{n-1} - 1)$ have eigenvalue 1, we have $\frac{1}{2}(3^n - 3)$ linearly independent Dirichlet–Neumann eigenfunctions of $G_{n+1}$ with eigenvalue 1, and $3^n + 3 - 2^{n+1} - 2$ with other eigenvalues. However, we know from Proposition 3.2 that there are $\frac{1}{2}(3^{n+1} + 3) - 2^{n+1} - 1$ in total, and

$$\tfrac{1}{2}(3^n - 3) + 3^n + 3 - 2^{n+1} - 2 = \tfrac{1}{2}(3^{n+1} + 3) - 2^{n+1} - 2.$$

The one unexplained eigenfunction must also have eigenvalue 1 because, if it had eigenvalue $\lambda \neq 1$, we would also have an unexplained eigenfunction with eigenvalue $2 - \lambda$. Hence we have $\frac{1}{2}(3^n - 1)$ linearly independent Dirichlet–Neumann eigenfunctions of $G_{n+1}$ with eigenvalue 1, giving the result by induction.

We know that, for $n \geqslant 2$, $G_n$ has $\frac{1}{2}(3^{n-1} - 1)$ linearly independent Dirichlet–Neumann eigenfunctions with eigenvalue 1. We now use our constructions $m$ times to show that, for $n \geqslant m + 1$, $G_n$ has $\frac{1}{2}(3^{n-m} - 1)$ linearly independent Dirichlet–Neumann eigenfunctions with eigenvalue $\beta_i^{(m)}$, for $1 \leqslant i \leqslant 2^{m-1}$. This completes the proof of (b). □

## 5. Reversing the orientation

We remark that very similar results can be obtained if we reverse the orientation of the model graphs in the definitions of § 1.2. Because of the asymmetry this gives a different self-similar sequence of graphs. Eigenvalues $\lambda$ and $\mu$ of Laplacians of successive members of the sequence are related by the same equation (2.1) when $\lambda \notin \{0, 1, 2\}$, but the two equations (2.9) and (2.10) for $\mu$ when $\lambda = 1$ are reversed. This has the effect that, in Theorem 4.1, the roles of $\alpha_i^{(n)}$ and $\beta_i^{(n)}$ are reversed.

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