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# Necessary and Sufficient Conditions for the Cohen–Macaulayness of Blowup Algebras

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**Abstract.** In this paper we provide a complete characterization for when the Rees algebra and the associated graded ring of a perfect Gorenstein ideal of grade three are Cohen–Macaulay. We also treat the case of second analytic deviation one ideals satisfying some mild assumptions. In another set of results we give criteria for an ideal to be of linear type. Finally, we describe the equations defining the Rees algebras of certain Northcott ideals.

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**Key words:** Rees algebra, associated graded ring, Cohen–Macaulayness, reduction number.

## Introduction

In this paper we investigate the Rees algebra $\mathcal{R} = R[It]$ ($t$ a variable) as well as the associated graded ring $\mathcal{G} = \mathcal{R}/I\mathcal{R}$ of an ideal $I$ in a Noetherian ring $R$. Both algebras play a crucial role in the birational study of algebraic varieties in that $\mathrm{Proj}(\mathcal{R})$ is the blowup of $\mathrm{Spec}(R)$ along $V(I)$, with $\mathrm{Proj}(\mathcal{G})$ corresponding to the exceptional fiber. Although blowing up is a fundamental operation, an explicit understanding of this process remains an open problem. In this context the Cohen–Macaulay property of the *blowup algebras* $\mathcal{R}$ and $\mathcal{G}$ is of central importance, in part because it helps to describe these algebras in terms of generators and relations. Recently numerous authors have discovered classes of ideals with Cohen–Macaulay blowup algebras. In the present work we wish to supply necessary *and* sufficient conditions for $\mathcal{R}$ or $\mathcal{G}$ to be Cohen–Macaulay. The emphasis here is on establishing the necessity of assumptions on the reduction number of $I$ that were known to imply Cohen–Macaulayness. As a second goal we wish to describe the equations defining certain Rees algebras.

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Usually one investigates the Cohen–Macaulayness of $\mathcal{R}$ and $\mathcal{G}$ by passing to a minimal reduction of $I$. Recall that an ideal $J \subset I$ is a *reduction* of $I$ if the extension of Rees algebras $\mathcal{R}(J) \hookrightarrow \mathcal{R}(I)$ is module finite, or equivalently, if $I^{r+1} = JI^r$ for some $r \geqslant 0$ ([31]). The least such $r$ is denoted by $r_J(I)$. Now assume that $(R, \mathfrak{m})$ is local with infinite residue field $k$. A reduction of $I$ is *minimal* if it is minimal with respect to inclusion, and the *reduction number* $r(I)$ of $I$ is defined as $\min\{r_J(I)\}$ where $J$ ranges over all minimal reductions of $I$. Finally, the *analytic spread* $\ell(I)$ of $I$ is the Krull dimension of the special fiber ring $\mathcal{R} \otimes_R k$, or equivalently, the minimal number of generators $\mu(J)$ of any minimal reduction $J$ of $I$ ([31]). Thus, $\ell(I)$ indicates the size of a minimal reduction and $r(I)$ measures how closely the two ideals are related. In this vein one may expect $\mathcal{R}$ and $\mathcal{G}$ to have good depth properties provided $r(I)$ is small.

Before describing one of the known sufficient conditions for the Cohen–Macaulayness of blowup algebras, we recall that $I$ satisfies $G_s$, $s$ an integer, if $\mu(I_\mathfrak{p}) \leqslant \dim R_\mathfrak{p}$ for every $\mathfrak{p} \in V(I)$ with $\dim R_\mathfrak{p} \leqslant s - 1$, and that $I$ is $G_\infty$ if $G_s$ holds for every $s$. Furthermore suppose $R$ is Gorenstein and write $g =$ grade $I$, $\ell = \ell(I)$, and $n = \mu(I)$. Now assuming $I$ satisfies $G_\ell$ and depth $R/I^j \geqslant \dim R/I - j + 1$ for $1 \leqslant j \leqslant \ell - g + 1$, it was shown in [22] that if $r(I) \leqslant \ell - g + 1$, then $\mathcal{G}$ is Cohen–Macaulay, and so is $\mathcal{R}$ in case $g \geqslant 2$ (for other results, see [11] for instance). It has been an open problem for some time as to what extent the converse of this statement holds. In other words, it remains to investigate under which circumstances the Cohen–Macaulayness of $\mathcal{R}$ or $\mathcal{G}$ forces the reduction number of $I$ to be at most the *expected* one, namely $\ell - g + 1$ (which, in the above setting, is the smallest positive value the reduction number can take, [22]). One knows, quite generally, that if $\mathcal{R}$ is Cohen–Macaulay then $r(I) \leqslant \ell - 1$ ([3, 20, 23, 34]), which yields the expected bound $r(I) \leqslant \ell - g + 1$ for $g = 2$. But, even for perfect Gorenstein ideals of grade 3 satisfying $G_\ell$, the stronger estimate was established only in special cases ([1, 21, 29, 30, 37]). Now we are able to treat this class of ideals in general, giving a complete characterization for when the blowup algebras of grade 3 perfect Gorenstein ideals satisfying $G_\ell$ are Cohen–Macaulay. As it turns out, $\mathcal{R}$ is Cohen–Macaulay if and only if $\mathcal{G}$ is Cohen–Macaulay if and only if the reduction number of $I$ is at most the expected one if and only if either $n = \ell$, or else, $n = \ell + 1$ and $I$ satisfies the row condition (Theorem 3.1). Here we say that an ideal $I$ satisfies the *row condition* if for some minimal presentation matrix $\varphi$ of $I$, the ideal $I_1(\varphi)$ is generated by the entries of a single row of $\varphi$ ([2]). The last equivalence in our theorem was known before ([34], [37]), and so was the fact that the reduction number satisfies the expected bound if $\mathcal{G}$ is Cohen–Macaulay and $n \leqslant \ell + 1$ ([21]). Thus, what remained to be shown was that the Cohen–Macaulayness of $\mathcal{G}$ imposes a severe restriction on the number of generators of a grade 3 perfect Gorenstein ideal, namely $n \leqslant \ell + 1$!

In another result of this paper, we give necessary and sufficient conditions for the blowup algebras of an ideal of arbitrary grade to be Cohen–Macaulay *assuming* that $n \leqslant \ell + 1$. Such ideals are said to have *second analytic deviation* (at most) *one*, the second analytic deviation of an ideal being the difference $n - \ell$ ([38]). More specifically, we prove: Assume that $n = \ell + 1 \geqslant 2$, that $I$ satisfies $G_\ell$, that depth $R/I^j \geqslant \dim R/I - j + 1$ and $S_j(I) \cong I^j$ for $1 \leqslant j \leqslant \ell - g + 1$, and that $I \otimes_R k$ does not embed into $I_1(\varphi) \otimes_R k$ for a minimal presentation matrix $\varphi$ of $I$; then $\mathcal{R}$ is Cohen–Macaulay (in case $g \geqslant 2$) if and only if $\mathcal{G}$ is Cohen–Macaulay if and only if the reduction number of $I$ is the expected one if and only if $I$ satisfies the row condition (Theorem 2.1). The last equivalence being a general and known fact about ideals of second analytic deviation one ([37]), we had to prove that the Cohen–Macaulayness of $\mathcal{G}$ forces $r(I) = \ell - g + 1$. The latter conclusion fails without the assumption $I \otimes_R k \not\hookrightarrow I_1(\varphi) \otimes_R k$, i.e., without requiring that a minimal generating set of $I$ cannot be extended to a minimal generating set of $I_1(\varphi)$. This weak condition, hardly ever violated by perfect noncomplete intersections, replaces the stronger assumption $I \subset (I_1(\varphi))^2$ under which the implication had been proved in [21]. As to the other assumptions in our theorem, notice that an ideal $I$ satisfies depth $R/I^j \geqslant \dim R/I - j + 1$ and $S_j(I) \cong I^j$ for $1 \leqslant j \leqslant s - g + 1$, if $I$ is $G_s$ and *strongly Cohen–Macaulay*, i.e., has Cohen–Macaulay Koszul homology $H_\bullet(I)$. The strong Cohen–Macaulay property, in turn, is a consequence of $I$ being *licci*, i.e., in the linkage class of a complete intersection ([14]). Standard examples of licci ideals include perfect ideals of grade 2 ([4], [10]), as well as perfect Gorenstein ideals of grade 3 ([40]). Combining these facts we obtain, for instance, a complete characterization for when Northcott ideals and perfect almost complete intersections of grade 3 have Cohen–Macaulay blowup algebras (Corollaries 2.5 and 2.6).

In another set of results we give criteria for an ideal to have reduction number zero (Theorems 4.2, 4.4 and their corollaries). These results all deal with the natural exact sequence $0 \to \mathcal{A} \to S(I) \to \mathcal{R} \to 0$ relating the symmetric algebra $S(I)$ and the Rees algebra $\mathcal{R}$ of $I$. Under suitable assumptions, which are automatically satisfied in case $I$ is licci, we show that if $\mathcal{G}$ is Cohen–Macaulay and $\mathcal{A}_i = 0$ for some $i \geqslant \ell - g + 2$, then $I$ is of *linear type*, i.e., $\mathcal{A} = 0$ (Theorem 4.4 and Corollary 4.5). Thus, in case $I$ is a licci ideal satisfying $G_\ell$, not of linear type, and $\mathcal{G}$ is Cohen–Macaulay, then $\mathcal{A}_i \neq 0$ if and only if $i \geqslant \ell - g + 2$.

Our results also provide classes of ideals in local Gorenstein rings for which the Cohen–Macaulay properties of $\mathcal{R}$ and of $\mathcal{G}$ are equivalent. In general, one only has the implication that the Cohen–Macaulayness passes from $\mathcal{R}$ to $\mathcal{G}$ (if $g > 0$, [15]), whereas the converse requires an assumption on the ambient ring, such as regularity ([28]).

Besides the Cohen–Macaulay property we are also interested in the defining equations of blowup algebras: We wish to give an explicit description of an ideal $Q$ in a polynomial ring $R[T_1, \ldots, T_n]$ so that $\mathcal{R} \cong R[T_1, \ldots, T_n]/Q$. In this context we reasonably restrict ourselves to considering ideals $I$ that satisfy a structure theorem and whose Rees algebra is Cohen–Macaulay. On the other hand, an explicit presentation of the symmetric algebra being known, it suffices to describe a generating set of the ideal $\mathcal{A}$. Vasconcelos was the first to address this question

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systematically ([39]). Later, the problem was solved for large classes of perfect ideals of grade 2 ([29]) and for perfect Gorenstein ideals of grade 3 and second analytic deviation one ([21]). Actually, in the light of Theorem 3.1, the latter class of ideals is the full class of perfect Gorenstein ideals of grade 3 (with Cohen–Macaulay Rees algebra). In this paper, we give an explicit presentation of $\mathcal{R}$ in the case of Northcott ideals (Theorem 5.4). Recall that a Northcott ideal is an ideal $I$ linked to a complete intersection in one step, which means that $I = J{:}K$ for complete intersection ideals $J \subset K$. For our theorem we need an assumption on $I$ that is slightly stronger than the one corresponding to the Cohen–Macaulayness of $\mathcal{R}$, but does not impose any restriction if $K$ is prime for instance.

## 1. Fundamental Exact Sequences

In this section we are going to prove several technical results that will be used throughout the paper. They are all derived in one sense or another from the exact sequence $0 \to \mathcal{A} \to S(I) \to \mathcal{R} \to 0$. Assuming that the associated graded ring $\mathcal{G}$ of $I$ is Cohen–Macaulay and that some other assumptions are satisfied, we will estimate the annihilator of the first nonvanishing component of $\mathcal{A}$ (Proposition 1.3), and we will show that this component fits into an exact sequence involving the symmetric algebra $S(I/J)$ and the canonical module $\omega_{R/J:I^2}$, where $J$ is a minimal reduction of $I$ (Proposition 1.5). Our approach is based in part on studying the Koszul homology $H_\bullet(T_1, \ldots, T_s)$ of suitable elements with values in $\mathcal{G}$ and in $\mathcal{R}$, respectively.

LEMMA 1.1. *Let $R$ be a Noetherian ring, let $s$ be an integer, and let $I$ be an $R$-ideal of height $g$ satisfying $G_s$ with $\mathcal{G}$ Cohen–Macaulay. Let $J = (a_1, \ldots, a_s)$ be a reduction of $I$ so that $\operatorname{ht} J{:}I \geqslant s$. Consider the $R$-algebra map from the polynomial ring $B = R[T_1, \ldots, T_s]$ to $S(I)$ sending $T_i$ to $a_i \in J \subset S_1(I)$, thus defining a graded $B$-module structure on $\mathcal{R}$ and $\mathcal{G}$. Then for every $i \geqslant 1$:*

(a) *$H_i(T_1, \ldots, T_s; \mathcal{G})$ is concentrated in degrees $\leqslant s - g$;*
(b) *$H_i(T_1, \ldots, T_s; \mathcal{R})$ is concentrated in degrees $\leqslant \max\{s - g, i\}$.*

*Proof.* Part (a) follows by the same arguments as in the proof of [34, 3.2]. To show (b) notice that $H_i(T_1, \ldots, T_s; \mathcal{R})$ is annihilated by some power of the $\mathcal{R}$-ideal $\mathcal{R}_+ \subset \sqrt{(T_1, \ldots, T_s)\mathcal{R}}$ since $J$ is a reduction of $I$. Hence $H_i(T_1, \ldots, T_s; \mathcal{R})$ is concentrated in finitely many degrees. As $H_i(T_1, \ldots, T_s; R)$ is concentrated in degree $i$ and $H_i(T_1, \ldots, T_s; \mathcal{G})$ is concentrated in degrees at most $s - g$, the conclusion now follows from the exact sequences

$$0 \to \mathcal{R}_+ \to \mathcal{R} \to R \to 0$$

$$0 \to \mathcal{R}_+(1) \to \mathcal{R} \to \mathcal{G} \to 0$$

(see also [35, 3.4(i)]). □

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From Lemma 1.1, one easily deduces two previously known facts. The first one (Lemma 1.2(a)) is an exact sequence that has essentially been shown in [21, the proof of 2.5], where methods from [1], [2] were used. The second one (Lemma 1.2(b)) deals with intersection properties of ideal powers and can be found in [2, the proof of 5.2].

LEMMA 1.2. *With the assumptions of Lemma* 1.1 *and with the* $B$*-module structure on* $R$ *given by the identification* $B/B_+ \cong R$*, the following hold:*

(a) *The sequence*

$$0 \to [\mathcal{A} \otimes_B R]_{\geqslant s-g+1} \to [S(I/J)]_{\geqslant s-g+1} \to [\mathcal{R}/Jt\mathcal{R}]_{\geqslant s-g+1} \to 0$$

*is exact*;

(b) $JI^{s-g} \cap I^i = JI^{i-1}$ *for every* $i \geqslant s-g+1$.

*Proof.* (a) Tensor the exact sequence

$$C_\bullet \colon 0 \to \mathcal{A} \to S(I) \to \mathcal{R} \to 0$$

with $\_\otimes_B R$, and notice that $\operatorname{Tor}_1^B(\mathcal{R}, R) \cong H_1(T_1, \ldots, T_s; \mathcal{R})$. Now Lemma 1.1(b) yields the exactness of $[C_\bullet \otimes_B R]_{\geqslant \max\{s-g+1,2\}}$. On the other hand, $[C_\bullet \otimes_B R]_1$ is trivially exact.

(b) We induct on $i \geqslant s-g+1$, the case $i = s-g+1$ being trivial. Now let $i \geqslant s-g+2$. By induction hypothesis, $JI^{s-g} \cap I^i = (JI^{s-g} \cap I^{i-1}) \cap I^i = JI^{i-2} \cap I^i$, which reduces us to showing that $JI^{i-2} \cap I^i \subset JI^{i-1}$. To this end let $x \in JI^{i-2} \cap I^i$. Write $x = \sum_{j=1}^s b_j a_j$ with $b_j \in I^{i-2}$ and set $b_j^* = b_j + I^{i-1} \in \mathcal{G}_{i-2}$. By $\mathcal{Z}_\bullet$ and $\mathcal{B}_\bullet$ we denote cycles and boundaries of the Koszul complex $K_\bullet(T_1, \ldots, T_s; \mathcal{G}) = \bigwedge^{\bullet} \mathcal{G}e_1 \oplus \cdots \oplus \mathcal{G}e_s$. Since $x \in I^i$, we have $\sum_{j=1}^s b_j^* T_j = 0$ in $\mathcal{G}$ and so $\sum_{j=1}^s b_j^* e_j \in [\mathcal{Z}_1]_{i-1}$. As $i \geqslant s-g+2$, Lemma 1.1.a yields $\sum_{j=1}^s b_j^* e_j \in [\mathcal{B}_1]_{i-1}$, and thus

$$[b_1^*, \ldots, b_s^*] = [T_1, \ldots, T_s] \cdot \varphi$$

for some alternating $s$ by $s$ matrix with homogeneous entries of degree $i-3$ in $\mathcal{G}$. Hence there exists an alternating $s$ by $s$ matrix $\Phi$ with entries in $R$ and elements $c_j \in I^{i-1}$ such that

$$[b_1, \ldots, b_s] = [a_1, \ldots, a_s] \cdot \Phi + [c_1, \ldots, c_s].$$

Multiplying by the column vector $[a_1, \ldots, a_s]^{\mathrm{t}}$ from the right and using the fact that $\Phi$ is alternating we conclude that $x = \sum_{j=1}^s b_j a_j = \sum_{j=1}^s c_j a_j \in JI^{i-1}$. □

The next proposition shows that a certain Fitting ideal of $I$ annihilates the first nontrivial component of $\mathcal{A}$. Annihilators of components of $\mathcal{A}$ have been investigated before, but only in the context of second analytic deviation one ([38, 2.5], [1, 2.1], [34, 4.2], [21, 3.1]).

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PROPOSITION 1.3. *Let $R$ be a Noetherian local ring and let $I$ be an $R$-ideal of height $g$ and analytic spread $\ell$. Assume that $I$ satisfies $G_\ell$, that $S_j(I) \cong I^j$ whenever $1 \leqslant j \leqslant \ell - g + 1$ and that $\mathcal{G}$ is Cohen–Macaulay. Then* $\operatorname{Fitt}_\ell(I) \subset \operatorname{Ann}(\mathcal{A}_{\ell-g+2})$.

*Proof.* Let $f_1, \ldots, f_n$ be a generating sequence of $I$, let $X$ be an $n$ by $n$ matrix of indeterminates, and write $[a_1, \ldots, a_n] = [f_1, \ldots, f_n] \cdot X$. We may replace $R$ by $R(X)$. Since

$$\begin{aligned}\operatorname{Fitt}_\ell(I) &= \sum_{1 \leqslant i_1 < \cdots < i_\ell \leqslant n} \operatorname{Fitt}_0(I/(a_{i_1}, \ldots, a_{i_\ell})) \\ &\subset \sum_{1 \leqslant i_1 < \cdots < i_\ell \leqslant n} \operatorname{Ann}(I/(a_{i_1}, \ldots, a_{i_\ell})),\end{aligned}$$

it suffices to show that $J\colon I$ annihilates $\mathcal{A}_{\ell-g+2}$ for any $J = (a_{i_1}, \ldots, a_{i_\ell})$. By the general choice of $a_1, \ldots, a_n$, the ideal $J$ is a reduction of $I$ and $\operatorname{ht} J\colon I \geqslant \ell$ because $I$ satisfies $G_\ell$ (see [19, the proof of 3.2]). We use the notation of Lemma 1.2 with $s = \ell$. Since $\mathcal{A}_i = 0$ for $i \leqslant \ell - g + 1$, $\mathcal{A}_{\ell-g+2} = [\mathcal{A} \otimes_B R]_{\ell-g+2}$. On the other hand, by the same lemma, $[\mathcal{A} \otimes_B R]_{\ell-g+2}$ embeds into $S_{\ell-g+2}(I/J)$, which is annihilated by $J\colon I$. □

To make use of the full strength of Proposition 1.3, we need a result about the canonical module of rings defined by certain residual intersections. For the proof and for future reference we recall that a proper ideal $K$ in a Noetherian ring $R$ is an *$s$-residual intersection* of an $R$-ideal $I$, if $\operatorname{ht} K \geqslant s \geqslant \operatorname{ht} I$ and $K = J\colon I$ for some $s$-generated $R$-ideal $J \subset I$. An $s$-residual intersection is *geometric* in case $\operatorname{ht} I + K \geqslant s + 1$.

LEMMA 1.4. *Let $R$ be a local Gorenstein ring of dimension $d$, let $I$ be an $R$-ideal of grade $g$ satisfying $G_d$, and assume that* depth $R/I^j \geqslant d - g - j + 1$ *whenever $1 \leqslant j \leqslant d - g$. Further, let $J$ be an $R$-ideal contained in $I$ so that $\mu(J) \leqslant d \leqslant$* $\operatorname{ht} J\colon I$. *If for some $i \geqslant d - g + 1$, $JI^{d-g} \cap I^i = JI^{i-1}$, then $\omega_{R/J\colon I^{i-d+g}} \cong I^i/JI^{i-1}$ (using the convention that the canonical module of the zero ring is zero).*

*Proof.* First notice that $(J\colon I^{i-d+g})I^i = (J\colon I^{i-d+g})I^{i-d+g}I^{d-g} \subset JI^{d-g} \cap I^i = JI^{i-1}$, by our assumption. Therefore $M = I^i/JI^{i-1}$ is a module over $R/J\colon I^{i-d+g}$. In particular, if $J\colon I^{i-d+g} = R$ then $M = 0$. Thus we may assume that $J\colon I^{i-d+g} \neq R$. Since $K = J\colon I \subset J\colon I^{i-d+g}$, it follows that $R/J\colon I^{i-d+g}$ is Artinian and that $K \neq R$ is a $d$-residual intersection of $I$. Hence, $\omega = \omega_{R/K} = I^{d-g+1}/JI^{d-g}$ by [36, 2.9(b)]. Now again by our assumption, $M = I^i/JI^{i-1} = I^i/(JI^{d-g} \cap I^i) = I^{i-d+g-1}\,(I^{d-g+1}/JI^{d-g}) \cong I^{i-d+g-1}\omega$. In particular, $M$ is a submodule of $\omega$, hence $\operatorname{type}(M) \leqslant 1$. Thus it suffices to show that $M$ is faithful over the Artinian ring $R/J\colon I^{i-d+g}$ (see, for instance, [6, 3.2.12(e)]). Indeed, $0:_R M =$

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$0:_R (I^{i-d+g-1}\omega) = (0:_R \omega):_R I^{i-d+g-1} = K:_R I^{i-d+g-1} = (J:_R I):_R I^{i-d+g-1} = J:_R I^{i-d+g}$. □

When combined with Lemma 1.2, Lemma 1.4 leads to the two fundamental exact sequences we will need later.

PROPOSITION 1.5. *Let $R$ be a local Gorenstein ring of dimension $d$ with infinite residue field and let $I$ be an $R$-ideal of grade $g$ satisfying $G_d$ with $\mathcal{G}$ Cohen–Macaulay. Assume that* depth $R/I^j \geqslant d-g-j+1$ *for* $1 \leqslant j \leqslant d-g$ *and* $S_j(I) \cong I^j$ *for* $1 \leqslant j \leqslant d-g+1$. *Let $J$ be a minimal reduction of $I$ and set $K = J:I$.*

(a) *There is an exact sequence*

$$0 \to \mathcal{A}_{d-g+2} \to S_{d-g+2}(I/J) \to \omega_{R/K:I} \to 0.$$

(b) *If $d > 0$ there is an exact sequence*

$$0 \to R/K:I \to \mathrm{Ext}^d_R(S_{d-g+2}(I/J), R) \to \mathrm{Ext}^d_R(S_{d-g+2}(I), R) \to 0.$$

*Proof.* Notice that ht $K \geqslant d$ since $I$ is of linear type locally in codimension $d-1$ (see [36, 2.9(a) and 1.11]). We may use the notation of Lemma 1.2 with $s = d$. Since $\mathcal{A}_i = 0$ for $i \leqslant d-g+1$ we have $\mathcal{A}_{d-g+2} = [\mathcal{A} \otimes_B R]_{d-g+2}$. By Lemma 1.2(b), $JI^{d-g} \cap I^{d-g+2} = JI^{d-g+1}$, hence $\omega_{R/K:I} \cong \omega_{R/J:I^2} \cong I^{d-g+2}/JI^{d-g+1} = [\mathcal{R}/Jt\mathcal{R}]_{d-g+2}$ by Lemma 1.4. Now Lemma 1.2(a) yields (a).

If $d > 0$ then $\mathrm{Ext}^d_R(I^{d-g+2}, R) = 0$, therefore the exact sequence defining $\mathcal{A}_{d-g+2}$ induces an isomorphism $\mathrm{Ext}^d_R(\mathcal{A}_{d-g+2}, R) \cong \mathrm{Ext}^d_R(S_{d-g+2}(I), R)$. On the other hand the modules occurring in (a) are annihilated by $K$ and, hence, have finite length. Now applying $\mathrm{Ext}^d_R(\_, R)$ to this sequence we obtain (b). □

Notice that if in the above proposition, $I$ has second analytic deviation at most one, then $I/J$ is cyclic and therefore $S_{d-g+2}(I/J) \cong R/K$.

## 2. Second Analytic Deviation One Ideals

Now we are ready for our first main result. To prove it, we compute annihilators along the exact sequence of Proposition 1.5(a) and invoke Proposition 1.3.

THEOREM 2.1. *Let $R$ be a local Gorenstein ring of dimension $d$ with infinite residue field $k$ and let $I$ be an $R$-ideal of grade $g$, analytic spread $\ell$, minimally generated by $n = \ell + 1 \geqslant 2$ elements. Suppose that $I$ satisfies $G_\ell$ and that* depth $R/I^j \geqslant d-g-j+1$ *and* $S_j(I) \cong I^j$ *whenever* $1 \leqslant j \leqslant \ell-g+1$. *Let $\varphi$ be a matrix with $n$ rows presenting $I$. Further assume that $I \otimes_R k$ does not embed into $I_1(\varphi) \otimes_R k$ (i.e., that a minimal generating set of $I$ cannot be extended to a minimal generating set of $I_1(\varphi)$). The following are equivalent:*

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(a) *After elementary row operations,* $I_1(\varphi)$ *is generated by the last row of* $\varphi$;
(b) *The reduction number of* $I$ *is* $\ell - g + 1$;
(c) $\mathcal{G}$ *is Cohen–Macaulay.*
*Furthermore, if* $g \geqslant 2$ *the above conditions are equivalent to*
(d) $\mathcal{R}$ *is Cohen–Macaulay.*

*Proof.* The equivalence of (a) and (b) is proved in [37, 5.1], while the fact that these imply (c) and (d) is given by [22, 3.1 and 3.4]. Furthermore (c) follows from (d) by [15, Prop. 1.1]. We are left to prove that (c) implies (b).

Let $\mathfrak{m}$ be the maximal ideal of $R$. By the Cohen–Macaulayness of $\mathcal{G}$ one has grade $\mathfrak{m}\mathcal{G} = d - \ell$. Since furthermore $I$ satisfies $G_\ell$, there exists an $R$-regular sequence $\mathbf{x} = x_1, \ldots, x_{d-\ell}$ so that $\mathbf{x}$ is regular on $\mathcal{G}$ and the image of $I$ in $R/(\mathbf{x})$ still satisfies $G_\ell$. Our assumptions and conclusions are preserved if we replace $R$ by $R/(\mathbf{x})$. Thus we may assume that $\ell = d$.

By assumption there exists an $\alpha \in \mathfrak{m}I_1(\varphi)$ that is part of a minimal generating set of $I$. As $n = \ell + 1$, using a general position argument one may find a minimal reduction $J$ of $I$ so that $I = (J, \alpha)$. Let $K = J : I$. Notice that ht $K \geqslant \ell$ since $I$ is of linear type locally in codimension $\ell - 1$ ([36, 2.9(a) and 1.11]). Therefore $R/K$ is an Artinian Gorenstein ring because $K$ is a residual intersection of height $\ell = d$ and $I/J$ is cyclic ([36, 2.9(b)]). Combining Proposition 1.3 and Proposition 1.5(a) we see that $I_1(\varphi)(K : I) \subset \mathrm{Ann}(S_{\ell-g+2}(I/J))$. Since $I/J \cong R/K$ we obtain $I_1(\varphi)(K : I) \subset K$ or, equivalently, $I_1(\varphi) \subset K : (K : I)$. As $R/K$ is an Artinian Gorenstein ring it follows that

$$I_1(\varphi) \subset I + K = (K, \alpha) \subset K + \mathfrak{m}I_1(\varphi).$$

Hence $I_1(\varphi) = K$, which gives (a), and the equivalence of (a) with (b) yields the conclusion. □

In the above theorem we assume that there exists an element contained in $\mathfrak{m}I_1(\varphi)$ which is part of a minimal generating set of $I$, a condition that is certainly satisfied if $I \subset \mathfrak{m}I_1(\varphi)$. Generalizing [1], M. Johnson had shown a result similar to Theorem 2.1 requiring the stronger condition $I \subset (I_1(\varphi))^2$ ([21, 3.6.b]). The above proof shows that one can weaken the assumption $I \otimes_R k \not\twoheadrightarrow I_1(\varphi) \otimes_R k$ even further by only requiring that $I \subset K + \mathfrak{m}I_1(\varphi)$, where $K$ is the ideal generated by the entries of a general row of $\varphi$. On the other hand, the theorem is no longer true without any such assumption, as can be seen by taking $I$ to be the maximal ideal of a local hypersurface ring of positive dimension and multiplicity at least 3.

We pause for a further discussion of the property $I \otimes_R k \not\twoheadrightarrow I_1(\varphi) \otimes_R k$. Although this condition is quite common, it may fail to hold even in the case of perfect non complete intersection ideals: For instance let $R = k[x_1, \ldots, x_4]$ be a polynomial ring over an infinite field and let $I$ be an $R$-ideal generated by five general quadrics; then $I$ has no linear relations by [13], and hence $I \otimes_R k \hookrightarrow I_1(\varphi) \otimes_R k$. In fact even the weaker condition $I \subset K + \mathfrak{m}I_1(\varphi)$ fails in this case, whereas we do not

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know of any licci non complete intersection ideal exhibiting this behaviour. On the other hand, there exist licci non complete intersections with $I \otimes_R k \hookrightarrow I_1(\varphi) \otimes_R k$: Examples are given by the ideals of [25, 5.7], which are grade 4 licci almost complete intersections (but not complete intersections). (We are grateful to Matthew Miller for pointing this out to us.) Our next results will illustrate how tight these counterexamples are.

REMARK 2.2. Let $R$ be a Noetherian local ring with residue field $k$, let $I$ be an $R$-ideal minimally presented by a matrix $\varphi$ with $n$ rows, and write $\Gamma_\bullet$ for the graded $k$-algebra $\operatorname{Tor}_\bullet^R(R/I, k)$. If for some $0 \neq e \in \Gamma_1$, $\dim_k e\Gamma_1 \leqslant n-2$, then $I \otimes_R k$ does not embed into $I_1(\varphi) \otimes_R k$.

*Proof.* Let $\mathfrak{m}$ denote the maximal ideal of $R$ and notice that $n \geqslant 2$. Consider the exact sequence $0 \to Z \to R^n = \bigoplus_{i=1}^n \mathrm{R}e_i \to I \to 0$, with $e_i$ mapping to $f_i$ in $I$. We may assume that $e$ is the image of $e_n$. By our assumption, the images in $\Gamma_2$ of the Koszul relations $\{f_n e_i - f_i e_n | 1 \leqslant i \leqslant n-1\}$ are linearly dependent over $k$. Hence for some $j$, $1 \leqslant j \leqslant n-1$, we have $f_n e_j - f_j e_n \in (\{f_n e_i - f_i e_n | 1 \leqslant i \leqslant n-1, i \neq j\}) + \mathfrak{m}Z$. Now, reading the coefficient of $e_j$, we conclude that $f_n \in \mathfrak{m} I_1(\varphi)$. □

PROPOSITION 2.3. *Let $R$ be a Noetherian local ring with residue field $k$, and let $I$ be a non complete intersection $R$-ideal minimally presented by $\varphi$. Assume either that $I$ is perfect of grade $\leqslant 3$, or else that $R$ is Gorenstein and $I$ is perfect Gorenstein of grade 4. Then $I \otimes_R k$ does not embed into $I_1(\varphi) \otimes_R k$.*

*Proof.* The assertion follows from Remark 2.2 and the classification of the algebras $\operatorname{Tor}_\bullet^R(R/I, k)$ as given in [5, 2.1] and in [26, 2.2], [24], respectively (we may assume that $k$ is algebraically closed). □

For our next observation we recall that two proper ideals $I$ and $K$ of a Noetherian ring $R$ are said to be (directly) *linked*, $I \sim K$, if $K = J : I$ and $I = J : K$ for some complete intersection $R$-ideal $J \subset I \cap K$.

PROPOSITION 2.4. *Let $R$ be a local Gorenstein ring with infinite residue field $k$ and let $I$ be a Cohen–Macaulay $R$-ideal minimally presented by $\varphi$. If there exists an $R$-ideal $H$ doubly linked to $I$, $I \sim K \sim H$, so that* $\operatorname{type}(R/H) < \operatorname{type}(R/I)$, *then $I \otimes_R k$ does not embed into $I_1(\varphi) \otimes_R k$.*

*Proof.* First notice that $\mu(I) > g = \operatorname{grade} I > 0$. Let $J$ and $L$ be complete intersection ideals defining the links $I \sim K$ and $K \sim H$, respectively. By [18, 2.5] we may pass to a new double link so that $J \otimes_R k \hookrightarrow I \otimes_R k$. Now $K = J : I \subset I_1(\varphi)$, which yields a commutative diagram

$$\begin{array}{ccc} J \otimes_R k & \hookrightarrow & I \otimes_R k \\ \downarrow & & \downarrow \\ K \otimes_R k & \longrightarrow & I_1(\varphi) \otimes_R k. \end{array}$$

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Suppose that $I \otimes_R k \hookrightarrow I_1(\varphi) \otimes_R k$, then $J \otimes_R k \hookrightarrow K \otimes_R k$. Thus $\mathrm{type}(R/I) = \mu(K/J) = \mu(K) - g \leqslant \mu(K/L) = \mathrm{type}(R/H)$, which is a contradiction. □

We now return to the study of blowup algebras. First recall that an ideal $I$ in a Noetherian local ring is called *equimultiple* if $\ell(I) = \mathrm{ht}\, I$. Second, an ideal $I$ of a Noetherian ring $R$ is called a *Northcott ideal* if it is directly linked to a complete intersection, or equivalently, if $I = J : K$ where $J \subset K$ are $R$-ideals generated by regular sequences $\mathbf{a} = a_1, \ldots, a_g$ and $\mathbf{x} = x_1, \ldots, x_g$, respectively. Writing $\Phi$ for a $g$ by $g$ matrix with $\mathbf{a} = \mathbf{x} \cdot \Phi$, one easily sees that $I = I_1(\mathbf{x} \cdot \Phi) + I_g(\Phi)$.

COROLLARY 2.5. *Let $R$ be a local Gorenstein ring with infinite residue field and let $I$ be a Northcott ideal of grade $g$ that is not a complete intersection. The following are equivalent:*

(a) *For some* $\mathbf{x}$ *and* $\Phi$, $I_{g-1}(\Phi) \subset (\mathbf{x})$;
(b) *$I$ is equimultiple with reduction number* 1;
(c) *$I$ is equimultiple and $\mathcal{R}$ is Cohen–Macaulay* (*respectively $\mathcal{G}$ is Cohen–Macaulay*).

*Proof.* First notice that $g \geqslant 2$. Let $k$ denote the residue field of $R$ and let $\varphi$ be a matrix presenting $I$ with respect to the minimal generators $a_1, \ldots, a_g, \det(\Phi)$. Now $I_1(\varphi) = (\mathbf{x}) + I_{g-1}(\Phi)$. Also, $I_1(\varphi)$ is independent of the chosen presentation matrix.

First assume that (a) holds. Then $I_1(\varphi) = (\mathbf{x}) = J : I$, hence by [32, 3.6], $I^2 = JI$, which gives (b). Next, the equivalence of (b) and (c) follows from Theorem 2.1. Indeed, the assumption $I \otimes_R k \not\hookrightarrow I_1(\varphi) \otimes_R k$ of the theorem is satisfied by Proposition 2.4, because $R/I$ is not Gorenstein, but $I$ is doubly linked to a complete intersection (alternatively, one can use a direct computation). Finally, to see that (b) implies (a) notice that $I$ is $g$-balanced in the sense of [37, 3.1] since $I$ is equimultiple of reduction number 1 ([37, 2.6]). Thus, after adjoining a finite set $X$ of indeterminates, $I_1(\varphi)R(X)$ is a first universal link $L^1(I)$ of $I$ as in [17, 2.12(b)]. However, $I$ being directly linked to a complete intersection, $L^1(I)$ has to be a complete intersection ([17, 2.14(b) and 2.3(b)]), which forces $I_1(\varphi)$ to be one as well. So let $I_1(\varphi) = (x_1, \ldots, x_g)$. The $g$-balancedness also implies that $I_1(\varphi) = J : I$ for some $g$ generated ideal $J = (\mathbf{a}) = (a_1, \ldots, a_g)$ ([37, 3.6(c)]), which is necessarily a complete intersection since $I \subset I_1(\varphi)$. Thus taking $\Phi$ to be a $g$ by $g$ matrix with $\mathbf{a} = \mathbf{x} \cdot \Phi$ we obtain (a), because $(\mathbf{x}) = I_1(\varphi) = (\mathbf{x}) + I_{g-1}(\Phi)$. □

COROLLARY 2.6. *Let $R$ be a local Gorenstein ring with infinite residue field and let $I$ be a perfect almost complete intersection $R$-ideal of grade* 3 *that is not a complete intersection. Let $\varphi$ be a matrix with four rows presenting $I$. The following are equivalent:*

(a) *After elementary row operations, $I_1(\varphi)$ is generated by the last row of $\varphi$;*
(b) *$I$ is equimultiple with reduction number* 1;

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(c) *$I$ is equimultiple and $\mathcal{R}$ is Cohen–Macaulay* (*respectively $\mathcal{G}$ is Cohen–Macaulay*).

*Proof.* The assertion follows from [37, 5.1], Proposition 2.3 (or Proposition 2.4 via [7, 5.3]) and Theorem 2.1. □

## 3. Perfect Gorenstein Ideals of Grade Three

Now we are able to turn to our main result about grade three Gorenstein ideals. For its proof we compare minimal numbers of generators along the exact sequence of Proposition 1.5(b), using the resolutions worked out in [27].

THEOREM 3.1. *Let $R$ be a local Gorenstein ring with infinite residue field, let $I$ be a perfect Gorenstein $R$-ideal of grade 3, set $\ell = \ell(I)$, $n = \mu(I)$, $r = r(I)$, and assume that $I$ satisfies $G_\ell$. The following are equivalent:*

(a) *$\mathcal{R}$ is Cohen–Macaulay*;
(b) *$\mathcal{G}$ is Cohen–Macaulay*;
(c) $r \leqslant \ell - 2$;
(d) *either $n = \ell$ and $r = 0$, or $n = \ell + 1$ and $r = \ell - 2$*;
(e) *either $n = \ell$, or $n = \ell + 1$ and $I$ can be presented by an alternating $n$ by $n$ matrix $\varphi$ with $I_1(\varphi)$ generated by the entries of the last row of $\varphi$.*

*Proof.* Clearly (a) implies (b) ([15, Prop. 1.1]). In this proof we will show that if $\mathcal{G}$ is Cohen–Macaulay then $n \leqslant \ell + 1$. Combining this with Proposition 2.3 (or [7, 2.1]) and Theorem 2.1 (or [21, 3.8]) one sees that (d) follows from (b), if $n \neq \ell$. On the other hand, if $n = \ell$ then $r = 0$ by [12, 9.1]. Furthermore, (d) is equivalent to (e) by [7, 2.1] and [34, 4.10], and obviously implies (c). Finally, (a) is a consequence of (c) by [22, 3.4].

Now it remains to show that $n \leqslant \ell + 1$ if $\mathcal{G}$ is Cohen–Macaulay. So suppose that $n > \ell + 1$. As in the proof of Theorem 2.1 one reduces to the case where $\ell = d = \dim R$. Now $d - \operatorname{grade} I + 2 = d - 1$ and $I$ satisfies $G_d$. The complexes defined in [27, 2.15] with $F = 0$ (see also [27, 4.7]) yield a free $R$-resolution $\mathcal{F}_\bullet$ of $S_{d-1}(I)$ by [27, 4.13(b), 4.13(d)ii, and 8.3(c)i]. By [27, 2.15(c)], $\mathcal{F}_\bullet$ has length at most $d$ and $\mathcal{F}_d$ has rank at most one. Hence $\operatorname{Ext}^d_R(S_{d-1}(I), R)$ is cyclic. Now Proposition 1.5(b) implies that $\mu(\operatorname{Ext}^d_R(S_{d-1}(I/J), R)) \leqslant 2$.

On the other hand, the complexes of [27, 2.15] with $F = R^d$ yield a resolution $(\mathcal{H}_\bullet, \partial_\bullet)$ of length $d$ of the module $S_{d-1}(I/J)$ by [27, 4.13(b), 4.13(d)ii, and 8.3(b)i]. The last map in this resolution is of the form

$$0 \to \mathcal{H}_d = \bigoplus \bigwedge^s F \xrightarrow{\partial_d} \mathcal{H}_{d-1},$$

with $s$ ranging over all integers so that $s - d$ is even and $0 \leqslant s \leqslant d$ ([27, 2.15(c)]). Let $\mathfrak{m}$ denote the maximal ideal of $R$, let $\varphi$ be an alternating $n$ by $n$ matrix with

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entries in $\mathfrak{m}$ presenting $I$, let $f_1, \dots, f_n$ be the signed $n-1$ by $n-1$ Pfaffians of $\varphi$, let $a_1, \dots, a_d$ be elements generating $J$, and let $\psi$ be an $n$ by $d$ matrix with entries in $R$ so that $[a_1, \dots, a_d] = [f_1, \dots, f_n] \cdot \psi$. Finally, consider the alternating $n+d$ by $n+d$ matrix

$$\Theta = \left(\begin{array}{c|c} \varphi & \psi \\ \hline -\psi^{\mathrm{t}} & 0 \end{array}\right).$$

First assume that $d$ is odd. Define $\mathfrak{a}$ to be the $R$-ideal generated by all $n-d+2$ by $n-d+2$ Pfaffians of $\Theta$ involving at most one of the last $d$ rows or columns of $\Theta$. Since $n \geqslant \ell + 2 = d + 2$ we have $\mathfrak{a} \subset \mathfrak{m}$. On the other hand, $F = \bigwedge^1 F$ is a direct summand of $\mathcal{H}_d$, and the description of the differential in [27, 2.15(f)] shows that $\partial_d(F) \subset \mathfrak{a}\mathcal{H}_{d-1} \subset \mathfrak{m}\mathcal{H}_{d-1}$. Thus $\mu(\mathrm{Ext}_R^d(S_{d-1}(I/J), R)) \geqslant \operatorname{rank} F = d > 2$, which yields a contradiction.

Next assume that $d$ is even, in which case we define $\mathfrak{a}$ to be the $R$-ideal generated by all $n-d+3$ by $n-d+3$ Pfaffians of $\Theta$ involving at most two of the last $d$ rows or columns of $\Theta$. As $n$ is odd, it follows that $n \geqslant d+3$ and therefore $\mathfrak{a} \subset \mathfrak{m}$. Now $\bigwedge^2 F$ is a direct summand of $\mathcal{H}_d$, and $\partial_d(\bigwedge^2 F) \subset \mathfrak{a}\mathcal{H}_{d-1} \subset \mathfrak{m}\mathcal{H}_{d-1}$ ([27, 2.15(f)]). Hence $\mu(\mathrm{Ext}_R^d(S_{d-1}(I/J), R)) \geqslant \operatorname{rank} \bigwedge^2 F = \binom{d}{2} > 2$, again yielding a contradiction. □

## 4. Ideals of Linear Type

In this section we are going to prove several criteria for an ideal to have reduction number zero. For this we need to recall the notion of a deformation: A pair $(\widetilde{R}, \widetilde{I})$ consisting of a Noetherian local ring $\widetilde{R}$ and an $\widetilde{R}$-ideal $\widetilde{I}$ is a *deformation* of a pair $(R, I)$ if there exists a sequence $\mathbf{z} \subset \widetilde{R}$ regular on $\widetilde{R}$ and on $\widetilde{R}/\widetilde{I}$ so that $R \cong \widetilde{R}/(\mathbf{z})$ and $I = \widetilde{I}R$. We show first a result relating syzygetic and intersection properties to the geometricity of residual intersections.

PROPOSITION 4.1. *Let $R$ be a local Gorenstein ring, let $I$ be an $R$-ideal of grade $g$, let $s$ be an integer and assume that $I$ satisfies $G_s$ and* $\operatorname{depth} R/I^j \geqslant \dim R/I - j + 1$ *whenever* $1 \leqslant j \leqslant s-g$. *Suppose that $(R, I)$ has a deformation $(\widetilde{R}, \widetilde{I})$ such that $\widetilde{I}$ satisfies $G_{s+1}$ and* $\operatorname{depth} \widetilde{R}/\widetilde{I}^j \geqslant \dim \widetilde{R}/\widetilde{I} - j + 1$ *whenever* $1 \leqslant j \leqslant s-g+1$. *Further, assume that there exists an $i \geqslant s-g+2$ such that $\mathcal{A}_i = 0$ and $JI^{s-g} \cap I^i = JI^{i-1}$ for some $s$-residual intersection $K = J : I$. Then $K$ is a geometric $s$-residual intersection of $I$; in particular $I$ satisfies $G_{s+1}$.*

*Proof.* Let $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} = s$. Suppose that $K = J : I \subset \mathfrak{p}$. Localizing $R$ at $\mathfrak{p}$ and $\widetilde{R}$ at the preimage of $\mathfrak{p}$, we may assume that $\dim R = s$. Since

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$\mathcal{A}_i = 0$ we have that $S_i(I/J) \cong I^i/JI^{i-1}$. On the other hand, since $JI^{s-g} \cap I^i = JI^{i-1}$, Lemma 1.4 yields $I^i/JI^{i-1} \cong \omega_{R/J:I^{i-s+g}} \cong \omega_{R/K:I^{i-s+g-1}}$. Therefore we obtain the equality of lengths, $\lambda(S_i(I/J)) = \lambda(R/K:I^{i-s+g-1})$. We will arrive at a contradiction once we have shown that $\lambda(S_i(I/J)) \geqslant \lambda(R/K)$, since then $K:I^{i-s+g-1} = K$, which is impossible because $I^{i-s+g-1} \neq R$ and $R/K \neq 0$ is Artinian.

To estimate the length of $S_i(I/J)$, let $\widetilde{f}_1, \dots, \widetilde{f}_n$ be a generating set of $\widetilde{I}$ mapping to a generating set $f_1, \dots, f_n$ of $I$. Let $a_1, \dots, a_s$ be generators of $J$ and write

$$[a_1, \dots, a_s] = [f_1, \dots, f_n] \cdot \psi$$

for some $n$ by $s$ matrix $\psi = (\psi_{ij})$ with entries in $R$. Let $X = (X_{ij})$ be a generic $n$ by $s$ matrix over $\widetilde{R}$, let $\widetilde{\mathfrak{m}}$ be the maximal ideal of $\widetilde{R}$, set $S = \widetilde{R}[\{X_{ij}\}]_{(\widetilde{\mathfrak{m}}, \{X_{ij}-\psi_{ij}\})}$, and

$$[\widetilde{a}_1, \dots, \widetilde{a}_s] = [\widetilde{f}_1, \dots, \widetilde{f}_n] \cdot X.$$

Further define the $S$-ideals $\widetilde{J} = (\widetilde{a}_1, \dots, \widetilde{a}_s)$ and $\widetilde{K} = \widetilde{J}: S\widetilde{I}$. Let $\pi: S \to R$ be the composition of the $\widetilde{R}$-algebra epimorphism sending $X$ to $\psi$ with the epimorphism $\widetilde{R} \to R$, and let $\mathbf{z}$ be the $S$-regular sequence generating $\ker \pi$.

First, notice that $\mathbf{z}$ is regular on $S/S\widetilde{I}$, hence $(S\widetilde{I}/\widetilde{J}) \otimes_S R \cong I/J$ and therefore $S_i(S\widetilde{I}/\widetilde{J}) \otimes_S R \cong S_i(I/J)$. Second, $\widetilde{K}$ is a geometric $s$-residual intersection of $S\widetilde{I}$ since $\widetilde{I}$ satisfies $G_{s+1}$ ([19, 3.2]), and $\widetilde{K}$ is unmixed of grade $s$ ([36, 2.9(a) and 1.7(a)]). Thus $S\widetilde{I}/\widetilde{J}$ has rank one as a module over $S/\widetilde{K}$, and hence the same holds true for $S_i(S\widetilde{I}/\widetilde{J})$. Third, $\mathbf{z}$ form a system of parameters on $S/\widetilde{K}$, because $\pi(\widetilde{K}) \subset K \subset \sqrt{\pi(\widetilde{K})}$ ([19, 4.1]) and hence $\operatorname{ht} \pi(\widetilde{K}) = s = \operatorname{ht} \widetilde{K}$. Furthermore $S/\widetilde{K}$ is Cohen–Macaulay ([36, 2.9(a)]).

Now comparing lengths and multiplicities, we obtain

$$\begin{aligned} \lambda(S_i(I/J)) &= \lambda(S_i(S\widetilde{I}/\widetilde{J}) \otimes_S R) \geqslant e(\mathbf{z}; S_i(S\widetilde{I}/\widetilde{J})) \\ &= e(\mathbf{z}; S/\widetilde{K}) \cdot \operatorname{rank}_{S/\widetilde{K}} S_i(S\widetilde{I}/\widetilde{J}) = \lambda(S/\widetilde{K} \otimes_S R) \cdot 1 \\ &= \lambda(R/\pi(\widetilde{K})) \geqslant \lambda(R/K) \end{aligned}$$

(cf. [6, 4.6.9 and 4.6.11]). □

THEOREM 4.2. *Let $R$ be a local Gorenstein ring with infinite residue field, let $I$ be an $R$-ideal of grade $g$ and analytic spread $\ell$, and assume that $I$ satisfies $G_\ell$ and* depth $R/I^j \geqslant \dim R/I - j + 1$ *whenever $1 \leqslant j \leqslant \ell - g$. Suppose that $(R, I)$ has a deformation $(\widetilde{R}, \widetilde{I})$ such that $\widetilde{I}$ satisfies $G_{\ell+1}$ and* depth $\widetilde{R}/\widetilde{I}^j \geqslant \dim \widetilde{R}/\widetilde{I} - j + 1$ *whenever $1 \leqslant j \leqslant \ell - g + 1$. The following are equivalent:*

(a) *$\mathcal{G}$ is Cohen–Macaulay and $\mathcal{A}_i = 0$ for some $i \geqslant \ell - g + 2$;*

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(b) *$JI^{\ell-g} \cap I^i = JI^{i-1}$ and $\mathcal{A}_i = 0$ for some $i \geqslant \ell - g + 2$ and some minimal reduction $J$ of $I$;*
(c) *$I$ satisfies $G_\infty$.*

*In either case $I$ is strongly Cohen–Macaulay and of linear type.*

*Proof.* First notice that $\operatorname{ht} J{:}I \geqslant \ell$ for every minimal reduction $J$ of $I$, since $I$ is of linear type locally in codimension $\ell - 1$ ([36, 2.9(a) and 1.11]). Now part (b) follows from (a) by Lemma 1.2(b).

Next, to show that (b) implies (c) it suffices to verify the equality $I = J$. So suppose that $I \neq J$. Then $J{:}I$ is an $\ell$-residual intersection ([36, 1.11]) with $\operatorname{ht} J{:}I = \ell$ ([36, 1.7(a)]). On the other hand by Proposition 4.1, $I$ satisfies $G_{\ell+1}$, which implies $\operatorname{ht} J{:}I \geqslant \ell + 1$ ([36, 1.11]).

Finally, if (c) holds, then $n = l$ by the above and hence $I$ is strongly Cohen–Macaulay by [36, 2.13]. Therefore $I$ is of linear type and $\mathcal{G}$ is Cohen–Macaulay by [12, 9.1], which yields (a). □

For the next result recall that an ideal $I$ is called *syzygetic* if $S_2(I) \cong I^2$ or, equivalently, $\mathcal{A}_2 = 0$.

COROLLARY 4.3. *Let $R$ be a local Gorenstein ring with infinite residue field and let $I$ be a syzygetic equimultiple $R$-ideal of grade $g$ that has a deformation satisfying $G_{g+1}$. The following are equivalent:*

(a) *$\mathcal{G}$ is Cohen–Macaulay;*
(b) *$J \cap I^2 = JI$ for some minimal reduction $J$ of $I$;*
(c) *$I$ is a complete intersection.*

*Proof.* To prove that (a) or (b) implies (c) it suffices to check that $I$ satisfies $G_{g+1}$ ([9]). For this one applies Theorem 4.2 to $I_\mathfrak{p}$ for $\mathfrak{p} \in V(I)$ with $\dim R_\mathfrak{p} = g$. □

THEOREM 4.4. *Let $R$ be a local Gorenstein ring, let $I$ be an $R$-ideal of grade $g$ and analytic spread $\ell$, and assume that $(R, I)$ has a deformation $(\widetilde{R}, \widetilde{I})$ such that $\widetilde{I}$ satisfies $G_{\ell+1}$ and the Koszul homology modules $H_j(\widetilde{I})$ are Cohen–Macaulay whenever $0 \leqslant j \leqslant \ell - g$. The following are equivalent:*

(a) *$\mathcal{G}$ is Cohen–Macaulay and $\mathcal{A}_i = 0$ for some $i \geqslant \ell - g + 2$;*
(b) *$I$ satisfies $G_\infty$.*

*In either case $I$ is strongly Cohen–Macaulay and of linear type.*

*Proof.* After a purely transcendental extension if needed we may assume that the residue field of $R$ is infinite. Also notice that $H_j(I)$ are Cohen–Macaulay modules whenever $0 \leqslant j \leqslant \ell - g$. Indeed, let $\mathbf{z} = z_1, \ldots, z_t$ be a sequence regular on $\widetilde{R}$ and $\widetilde{R}/\widetilde{I}$ that generates the kernel of the map $\widetilde{R} \to R$. For $0 \leqslant j \leqslant \ell - g$, either $H_j(\widetilde{I}) = 0$ or $\mathbf{z}$ is regular on $H_j(\widetilde{I})$ since the latter module is a maximal Cohen–Macaulay module over $\widetilde{R}/\widetilde{I}$. Now inducting on $t$ and using the long exact

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sequence of Koszul homology one sees that $H_j(\widetilde{I}) \otimes_{\widetilde{R}} R \cong H_j(I)$. Thus $H_j(I)$ is Cohen–Macaulay as well.

Now our assertions follow from Theorem 4.2 once we have shown that part (a) forces $I$ to satisfy $G_\ell$. To this end we prove by induction on $s$, $g \leqslant s \leqslant \ell$, that $I$ has $G_s$. The case $s = g$ being trivial, assume that $G_s$ holds for $g \leqslant s < \ell$. Let $\mathfrak{p} \in V(I)$ with dim $R_\mathfrak{p} = s$. Applying Theorem 4.2 to the ideal $I_\mathfrak{p}$, we conclude that $\mu(I_\mathfrak{p}) \leqslant s$. Thus $I$ satisfies $G_{s+1}$. □

COROLLARY 4.5. *Let $R$ be a local Gorenstein ring and let $I$ be a licci $R$-ideal of grade $g$ and analytic spread $\ell$. The following are equivalent:*

(a) *$\mathcal{G}$ is Cohen–Macaulay and $\mathcal{A}_i = 0$ for some $i \geqslant \ell - g + 2$;*
(b) *$I$ is of linear type.*

*Proof.* By [14, 1.14] and [19, the proof of 5.3], the assumptions of Theorem 4.4 are satisfied. Now the assertion follows from that theorem. □

## 5. Equations of Blowups of Northcott Ideals

Let $R$ be a Gorenstein ring. For $g \geqslant 2$ let $J$, $K$ be complete intersection $R$-ideals of grade $g$ with $J \subset K \subset \mathrm{Rad}(R)$, and consider the Northcott ideal $I = J : K$. In this section we wish to find the equations defining the Rees algebra $\mathcal{R}$ of $I$.

As before let $\mathbf{a} = a_1, \ldots, a_g$ and $\mathbf{x} = x_1, \ldots, x_g$ be $R$-regular sequences generating $J$ and $K$, respectively, let $\Phi = (\Phi_{ij})$ be a $g$ by $g$ matrix with entries in $R$ so that

$$[a_1, \ldots, a_g] = [x_1, \ldots, x_g] \cdot \Phi,$$

and write $\Delta = \det \Phi$. Then

$$I = (a_1, \ldots, a_g, \Delta),$$

and we obtain a presentation of $S(I)$ as an epimorphic image of the polynomial ring $R[T_1, \ldots, T_{g+1}]$ by mapping $T_i$ to $a_i$ for $1 \leqslant i \leqslant g$ and $T_{g+1}$ to $\Delta$. The kernel of this map is generated by the entries $l_1, \ldots, l_g$ of the product matrix

$$[T_1, \ldots, T_{g+1}] \cdot \left[ \begin{array}{c} \mathrm{adj}\, \Phi \\ \hline -x_1 \ \cdots \ -x_g \end{array} \right]$$

and the elements $a_i T_j - a_j T_i$, $1 \leqslant i < j \leqslant g$. Thus, to find the defining ideal of $\mathcal{R}$, it actually suffices to describe a generating set of the ideal $\mathcal{A}$ of the symmetric algebra $S(I)$ that fits into the exact sequence

$$0 \to \mathcal{A} \to S(I) \to \mathcal{R} \to 0.$$

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We will do this in Theorem 5.4. First however, we need to prove several lemmas.

LEMMA 5.1. *With the above assumptions,* $\operatorname{ht} I_t(\Phi) \geqslant g - t + 1$ *for* $1 \leqslant t \leqslant g$.

*Proof.* For any minimal prime $\mathfrak{q}$ of $I_t(\Phi)$ choose a maximal ideal $\mathfrak{m}$ of $R$ containing $\mathfrak{q}$ and observe that $K \subset \mathfrak{m}$. Thus we may replace $R$ by $R_{\mathfrak{m}}$ to assume that $(R, \mathfrak{m})$ is local. Let $\mathbf{T} = T_1, \ldots, T_g$ be variables. Modulo the ideal $(T_1 - x_1, \ldots, T_g - x_g)$, the local ring $R[\mathbf{T}]_{(\mathfrak{m},\mathbf{T})}/(\mathbf{T} \cdot \Phi)$ specializes to $R/J$. Thus $\dim R[\mathbf{T}]/(\mathbf{T} \cdot \Phi) = \dim R[\mathbf{T}]_{(\mathfrak{m},\mathbf{T})}/(\mathbf{T} \cdot \Phi) \leqslant \dim R/J + g = \dim R$. But $R[\mathbf{T}]/(\mathbf{T} \cdot \Phi)$ is the symmetric algebra of the $R$-module $M = \operatorname{coker} \Phi$, hence $\dim S(M) \leqslant \dim R$. Now [16, 2.6] shows that $\mu(M_{\mathfrak{p}}) \leqslant \dim R_{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Spec}(R)$, which yields the assertion. □

By '$-$' we will indicate images of elements of $R[T_1, \ldots, T_{g+1}]$ in $S(I)$.

PROPOSITION 5.2. *With the above assumptions,* $S(I)$ *is Cohen–Macaulay with* $\dim S(I) = \dim R + 1$, *and* $\overline{T}_{g+1}$ *is regular on* $S(I)$.

*Proof.* We may localize at any maximal ideal of $R$ to assume that $R$ is local. Now, $I$ being the unit ideal or a Cohen–Macaulay almost complete intersection, it follows that $S(I)$ is a Cohen–Macaulay ring of dimension $\dim R + 1$ ([12, 10.1] and [16, 2.6]). On the other hand, since by Lemma 5.1, $\Delta$ is a nonzero divisor in $R$, $M = I/(\Delta)$ is an $R$-module with $\mu(M_{\mathfrak{p}}) \leqslant \dim R_{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Spec}(R)$. Therefore $\dim S(M) = \dim R$ ([16, 2.6]). Hence $S(I)/(\overline{T}_{g+1}) \cong S(M)$ has dimension $\dim S(I) - 1$, which shows that $\overline{T}_{g+1}$ is a regular element. □

LEMMA 5.3. *In addition to the above assumptions, suppose that* $R$ *is local and let* $L$ *be the ideal* $(l_1, \ldots, l_g, T_{g+1}^{g-2})$ *in the polynomial ring* $R[T_1, \ldots, T_{g+1}]$. *Then* $\operatorname{depth} R[T_1, \ldots, T_{g+1}]/L \geqslant \dim R$.

*Proof.* We may assume that $g \geqslant 3$. In $R[T_1, \ldots, T_g]$ define the ideal

$$H = I_g\left(\left[\begin{array}{c} \Phi \\ \hline T_1 \;\cdots\; T_g \end{array}\right]\right).$$

We first claim that if $H$ is not the unit ideal it must be perfect of grade 2. Indeed, $H$ contains the ideal defining the symmetric algebra of the $R$-module $N = \operatorname{coker}(\operatorname{adj} \Phi)$. By Lemma 5.1, $N$ is a torsion $R$-module that is $g - 1$ generated locally in codimension one, and hence by [16, 2.6], $\dim S(N) \leqslant \dim R + g - 2$. Thus $\operatorname{ht} H \geqslant 2$, showing that $H$ is a perfect ideal of grade 2 unless it is the unit ideal.

Next, we consider the ideal $\mathcal{L} = (H, T_{g+1})$ of $R[T_1, \ldots, T_{g+1}]$. This ideal is either licci or else the unit ideal, since $H$ is. Furthermore $L \subset \mathcal{L} = (L, \Delta, T_{g+1})$. Our conclusion will follow once we have shown that

$$\operatorname{ht} L \colon \mathcal{L} \geqslant g + 1. \tag{1}$$

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Indeed, (1) yields $\operatorname{ht} L\colon \mathcal{L} \geqslant g+1 \geqslant \mu(L)$. Now if $\mathcal{L}$ is licci and $L \neq \mathcal{L}$, we may use [19, 5.3] to deduce that depth $R[T_1, \ldots, T_{g+1}]/L \geqslant \dim R[T_1, \ldots, T_{g+1}] - (g+1) = \dim R$. Otherwise the assertion is obvious.

To prove (1) we first show that for every $1 \leqslant i < j \leqslant g$, $\Delta(a_j T_i - a_i T_j)$ as well as $T_{g+1}(a_j T_i - a_i T_j)^2$ are in $L$. We may actually assume that $i = 1$, $j = 2$. Now

$$\begin{aligned}\Delta(a_2T_1 - a_1T_2) &= \Delta \cdot [T_1, \ldots, T_g] \cdot \begin{bmatrix} a_2 \\ -a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [T_1, \ldots, T_g] \cdot \Delta \cdot \begin{bmatrix} a_2 \\ -a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= [T_1, \ldots, T_g] \cdot \operatorname{adj} \Phi \cdot \Phi \cdot \begin{bmatrix} a_2 \\ -a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= [T_1, \ldots, T_{g+1}] \cdot \left[\begin{array}{c} \operatorname{adj} \Phi \\ \hline -x_1 \ \cdots \ -x_g \end{array}\right] \cdot \Phi \cdot \begin{bmatrix} a_2 \\ -a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\end{aligned}$$

where the last equality holds since $[x_1, \ldots, x_g] \cdot \Phi = [a_1, \ldots, a_g]$. The resulting row vector has all its entries in $L$, showing that $\Delta(a_2T_1 - a_1T_2) \in L$. Similarly

$$\begin{aligned}[T_1\Delta - T_{g+1}a_1, \ldots, T_g\Delta - T_{g+1}a_g] &= [T_1, \ldots, T_{g+1}] \cdot \left[\begin{array}{ccc} \Delta & & \\ & \ddots & \\ & & \Delta \\ \hline -a_1 & \ldots & -a_g \end{array}\right] \\ &= [T_1, \ldots, T_{g+1}] \cdot \left[\begin{array}{c} \operatorname{adj} \Phi \\ \hline -x_1 \ \cdots \ -x_g \end{array}\right] \cdot \Phi.\end{aligned}$$

Since this vector has all its entries in $L$, we conclude that $T_{g+1}a_i \in (L, \Delta)$. Now by the previous calculation, $T_{g+1}(a_2T_1 - a_1T_2)^2 \in L$. Hence

$$L + (\{(a_iT_j - a_jT_i)^2\}) \subset L\colon \mathcal{L}.$$

But $L + (\{a_iT_j - a_jT_i\})$ is the defining ideal of $S(I)/(\overline{T}_{g+1}^{g-2})$, which by Proposition 5.2 has height $g+1$. This shows (1). □

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In order to describe a generating set of the ideal $\mathcal{A}$ we need to assume that 2 is invertible in $R$. We also suppose that for some choice of $a_1, \ldots, a_g$,

$$a_{g-1}, a_g \in K^2. \tag{2}$$

This type of assumption came up in [8], where links of prime ideals were studied. It is closely related to the condition $I_{g-1}(\Phi) \subset K$ of Corollary 2.5 that essentially characterizes the Cohen–Macaulayness of $\mathcal{R}$. In fact, the present condition (2) implies the one of Corollary 2.5, and the converse holds in case $(R, \mathfrak{m})$ is local and $K = \mathfrak{m}$, for instance. Furthermore, (2) does not pose any restriction if $K$ happens to be a prime ideal; for in this case the failure of (2) implies that $I$ is generically a complete intersection and hence $\mathcal{A} = 0$ ([12, 9.1]).

Now assuming (2) we can write $\Phi_{ij} = \sum_{k=1}^{g} \alpha_{ijk} x_k$ for $g-1 \leqslant j \leqslant g$. Let $\widetilde{R}$ be the polynomial ring $R[X_1, \ldots, X_g]$ and let $\widetilde{\Phi} = (\widetilde{\Phi}_{ij})$ be the $g$ by $g$ matrix with entries in $\widetilde{R}$ so that $\widetilde{\Phi}_{ij} = \Phi_{ij}$ for $1 \leqslant j \leqslant g-2$ and $\widetilde{\Phi}_{ij} = \sum_{k=1}^{g} \alpha_{ijk} X_k$ for $g-1 \leqslant j \leqslant g$. Define

$$[\widetilde{l}_1, \ldots, \widetilde{l}_g] = [T_1, \ldots, T_{g+1}] \cdot \left[ \frac{\operatorname{adj} \widetilde{\Phi}}{-X_1 \ \cdots \ -X_g} \right].$$

We can write $\widetilde{l}_j = \widetilde{l}_{j(1)} + \widetilde{l}_{j(2)}$ where $\widetilde{l}_{j(d)}$ are homogeneous polynomials of degree $d$ in $X_1, \ldots, X_g$ with coefficients in $R[T_1, \ldots, T_{g+1}]$. Consider the $g$ by $g$ matrix

$$\widetilde{B} = (\widetilde{b}_{ij}) = \left( \frac{\partial \widetilde{l}_{j(1)}}{\partial X_i} + \frac{1}{2} \frac{\partial \widetilde{l}_{j(2)}}{\partial X_i} \right),$$

with entries in $\widetilde{R}[T_1, \ldots, T_{g+1}]$ and let $B$ be the image of $\widetilde{B}$ in $R[T_1, \ldots, T_{g+1}]$ as $X_i$ are mapped to $x_i$ for $1 \leqslant i \leqslant g$. Notice that $[l_1, \ldots, l_g] = [x_1, \ldots, x_g] \cdot B$.

THEOREM 5.4. *With the above assumptions* (*including* $1/2 \in R$ *and* (2)), $\mathcal{A}$ *is generated by* $\overline{\det B}/\overline{T}_{g+1}^{g-2}$.

*Proof.* First notice that by Proposition 5.2, $\overline{T}_{g+1}$ is a non zerodivisor on $S(I)$. The assertion of the theorem will follow once we have shown that $\overline{T}_{g+1}^{g-2}$ divides $\overline{\det B}$ in $S(I)$. To see this, notice that the equality $[l_1, \ldots, l_g] = [x_1, \ldots, x_g] \cdot B$ implies that $0 = x_1 \overline{\det B} = x_1 \overline{T}_{g+1}^{g-2} (\overline{\det B}/\overline{T}_{g+1}^{g-2})$ in $S(I)$. Now as $x_1$ and $\overline{T}_{g+1}$ are non zerodivisors on $S(I)/\mathcal{A} \cong \mathcal{R}$ ($\Delta$ being a non zerodivisor on $R$ by Lemma 5.1), we conclude that $\overline{\det B}/\overline{T}_{g+1}^{g-2}$ is contained in $\mathcal{A}$. To prove that this element generates $\mathcal{A}$, we may localize at any maximal ideal of $R$ to assume that $(R, \mathfrak{m})$ is local. We may also assume that $\mu(I) = g+1$ since otherwise $\mathcal{A} = 0$. Now by Corollary 2.5 for example, $\ell(I) = g$. The ideal $I$ is strongly Cohen–Macaulay and is presented by a matrix whose ideal of entries $K$ is generated by the entries of one row. Thus by [34, 4.10], $\mathcal{A}$ is generated by one homogeneous element of degree

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two in the graded ring $S(I)$. Now $\overline{\det B}/\overline{T}_{g+1}^{g-2}$ is a form of degree two contained in $\mathcal{A}$, but not lying in $\mathfrak{m}\mathcal{A}$ since the polynomial $(-1)^g \det B$ is monic in $T_{g+1}$. Thus $\overline{\det B}/\overline{T}_{g+1}^{g-2}$ generates $\mathcal{A}$.

To prove that $\overline{T}_{g+1}^{g-2}$ divides $\overline{\det B}$, we may assume that $g \geqslant 3$. If we write $\Phi = [\Phi_1|\Phi_2]$ where $\Phi_1$ has $g-2$ columns, then grade $I_{g-2}(\Phi_1) > 0$ since $\Delta$ is an $R$-regular element by Lemma 5.1. We now replace $R$ and $\Phi$ by $\widetilde{R} = R[X_1, \ldots, X_g]$ and $\widetilde{\Phi}$ as above, but we revert to our original notation except that we will still write $\widetilde{R} = R[X_1, \ldots, X_g]$. Notice the $R$-ideal $I_{g-2}(\Phi_1)$ is not contained in any minimal prime of the $\widetilde{R}[T_1, \ldots, T_{g+1}]$-ideal $(X_1, \ldots, X_g, T_{g+1})$. We will be done once we have shown that in $\widetilde{R}[T_1, \ldots, T_{g+1}]$, $\det B$ is contained in the ideal $L = (l_1, \ldots, l_g, T_{g+1}^{g-2})$.

It suffices to check this containment locally at every associated prime $\mathfrak{p}$ of the ideal $L$. Since $X_i \det B \in (l_1, \ldots, l_g)$ for $1 \leqslant i \leqslant g$, it follows that the assertion is clear if $(X_1, \ldots, X_g) \not\subset \mathfrak{p}$. Thus we may assume $(X_1, \ldots, X_g) \subset \mathfrak{p}$, and hence $(X_1, \ldots, X_g, T_{g+1}) \subset \mathfrak{p}$. Localizing at the contraction of $\mathfrak{p}$, we may further suppose that $(R, \mathfrak{m})$ is local with $\mathfrak{m} = \mathfrak{p} \cap R$. Since $\operatorname{ht} \mathfrak{p} \leqslant g+1$ by Lemma 5.3, it follows that $\mathfrak{p}$ is a minimal prime of $(X_1, \ldots, X_g, T_{g+1})$. Thus $I_{g-2}(\Phi_1) = R$ by the above, and $T_i \notin \mathfrak{p}$ for $1 \leqslant i \leqslant g$.

Hence there exist invertible $g$ by $g$ matrices $U$ with entries in $R$ and $V$ with entries in $\widetilde{R}$ so that $\det U = \det V = 1$ and

$$U \Phi V = \Phi' = \left[\begin{array}{ccc|cc} 1 & & & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ & & 1 & 0 & 0 \\ \hline 0 & \cdots & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\Phi''$}} \\ 0 & \cdots & 0 & & \end{array}\right],$$

where $\Phi''$ is a 2 by 2 matrix with linear entries in $\widetilde{R}$. Set

$$[X_1', \ldots, X_g'] = [X_1, \ldots, X_g] \cdot U^{-1},$$

$$[T_1', \ldots, T_g'] = [T_1, \ldots, T_g] \cdot V,$$

$$T_{g+1}' = T_{g+1},$$

$$[l_1', \ldots, l_g'] = [l_1, \ldots, l_g] \cdot U^{-1}.$$

Notice that

$$R[X_1', \ldots, X_g'] = R[X_1, \ldots, X_g], \qquad \widetilde{R}[T_1', \ldots, T_{g+1}'] = \widetilde{R}[T_1, \ldots, T_{g+1}],$$

and

$$[l_1', \ldots, l_g'] = [T_1', \ldots, T_{g+1}'] \cdot \left[\begin{array}{c} \operatorname{adj} \Phi' \\ \hline -X_1' \ \cdots \ -X_g' \end{array}\right].$$

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Furthermore, if we let $B'$ be the $n$ by $n$ matrix whose $(i, j)$-entry is

$$\frac{\partial l'_{j(1)}}{\partial X'_i} + \frac{1}{2}\frac{\partial l'_{j(2)}}{\partial X'_i},$$

then

$$\begin{aligned} B' &= \left(\frac{\partial l'_{j(1)}}{\partial X'_i}\right) + \frac{1}{2}\left(\frac{\partial l'_{j(2)}}{\partial X'_i}\right) \\ &= U \cdot \left(\frac{\partial l_{j(1)}}{\partial X_i}\right) \cdot U^{-1} + \frac{1}{2} U \cdot \left(\frac{\partial l_{j(2)}}{\partial X_i}\right) \cdot U^{-1} \\ &= U \cdot B \cdot U^{-1}, \end{aligned}$$

where the second equality holds because the entries of $U$ and $U^{-1}$ are in $R$. Now since $\det B' = \det B$ and since $(l'_1, \ldots, l'_g, (T'_{g+1})^{g-2}) = (l_1, \ldots, l_g, T^{g-2}_{g+1})$, we may replace $\Phi$ by $\Phi'$ and return to our original notation.

Write $T = \prod_{i=2}^{g-1} T_i$. As $T \notin \mathfrak{p}$ it suffices to show that $T \det B \in L$. This will follow once we prove that $T \det B$ is the determinant of a $g$ by $g$ matrix whose first $g - 2$ columns have entries in the ideal $(l_{g-1}, l_g, T_{g+1})$.

We have

$$[l_1, \ldots, l_g] = [T_1, \ldots, T_{g+1}] \cdot \left[\begin{array}{ccc|cc} \Delta & & & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ & & \Delta & 0 & 0 \\ \hline 0 & \cdots & 0 & \alpha & \beta \\ 0 & \cdots & 0 & \gamma & \delta \\ \hline -X_1 & & \cdots & & -X_g \end{array}\right],$$

where $\alpha = \sum_{i=1}^g \alpha_i X_i$, $\beta = \sum_{i=1}^g \beta_i X_i$, $\gamma = \sum_{i=1}^g \gamma_i X_i$ and $\delta = \sum_{i=1}^g \delta_i X_i$ are linear forms with coefficients in $R$ so that $\alpha\delta - \beta\gamma = \Delta$. Recursively, for $1 \leqslant j \leqslant g - 3$, we multiply the $j$th column of $B$ by $T_{j+1}$ and then subtract $T_j$ times the $(j+1)$st column from it. We also multiply the $(g-2)$nd column by $T_{g-1}$. It follows that $T \det B$ is the determinant of a $g$ by $g$ matrix, whose $i$th row has the following form modulo the ideal $(l_{g-1}, l_g, T_{g+1})$,

$$\left[0, \ldots\ldots, 0, \quad \frac{1}{2}\frac{\partial \Delta}{\partial X_i} T_{g-2} T_{g-1}, \quad \alpha_i T_{g-1} + \gamma_i T_g, \quad \beta_i T_{g-1} + \delta_i T_g\right].$$

Since modulo the ideal $(l_{g-1}, l_g, T_{g+1})$, $\alpha T_{g-1} + \gamma T_g \equiv 0$ and $\beta T_{g-1} + \delta T_g \equiv 0$, we see that

$$\frac{1}{2}\frac{\partial \Delta}{\partial X_i} T_{g-2} T_{g-1} \equiv \tfrac{1}{2} T_{g-2}\delta(\alpha_i T_{g-1} + \gamma_i T_g) - \tfrac{1}{2} T_{g-2}\gamma(\beta_i T_{g-1} + \delta_i T_g).$$

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Thus, adding a suitable linear combination of the last two columns to the $(g-2)$nd one, it follows that modulo $(l_{g-1}, l_g, T_{g+1})$, the matrix can be transformed into another one whose first $g-2$ columns have zero entries. □

Let $(R, \mathfrak{m})$ be a power series ring $k[[X_1, \dots, X_g]]$ in $g \geqslant 2$ variables over a field $k$ of characteristic zero. Consider an $R$-regular sequence $a_1, \dots, a_g$ and the Jacobian determinant $\delta = |\partial a_j / \partial X_i|$. For instance, one could choose $a_1, \dots, a_g$ to be the partial derivatives of a power series $f \in R$ defining an isolated singularity $R/(f)$, in which case $\delta$ is the Hessian of $f$. As before write $J = (a_1, \dots, a_g)$ and let $I$ be the Northcott ideal $J : \mathfrak{m}$. By [33, p. 187] one has $I = (a_1, \dots, a_g, \delta)$. If $a_{g-1}, a_g \subset \mathfrak{m}^2$, then $J$ is a reduction of $I$ with $r_J(I) = 1$ ([8, 2.1] or Theorem 5.4). From Theorem 5.4, one can actually obtain a quadratic equation of integrality of $\delta$ over $J$. The situation is particularly agreeable if $J$ is homogeneous and $g = 2$:

EXAMPLE 5.5. *Let $k[X, Y]$ be a polynomial ring over a field of characteristic zero, let $f, h$ be a regular sequence of forms of degrees $d_1 \geqslant 2$, $d_2 \geqslant 2$, and write*

$$\delta = \begin{vmatrix} f_x & h_x \\ f_y & h_y \end{vmatrix}.$$

*Then*

$$\begin{aligned} \delta^2 = {} & \frac{1}{d_2^2(d_2-1)^2}(h_{xy}^2 - h_{xx}h_{yy})f^2 + \frac{1}{d_1 d_2(d_1-1)(d_2-1)} \times \\ & \times (f_{xx}h_{yy} - 2f_{xy}h_{xy} + f_{yy}h_{xx})fh + \frac{1}{d_1^2(d_1-1)^2}(f_{xy}^2 - f_{xx}f_{yy})h^2. \end{aligned}$$

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