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Continuous convergence and the Hahn-Banach problem

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In this note, a positive solution is given to the Hahn-Banach problem for an important class of convergence vector spaces. As well, a topological vector space characterization is obtained for fully complete and B_r-complete spaces.

1. Introduction and preliminaries

Let E be a Hausdorff locally convex topological vector space over $\mathbb{R}$ and $L_c(E)$ its dual space equipped with the continuous convergence structure. In this note, we examine the relationship between the Hahn-Banach extension property for closed subspaces of the convergence vector space $L_c(E)$ and completeness conditions on E. In particular, every closed subspace of $L_c(E)$ has the extension property if and only if E is fully complete. This relationship allows us to formulate the following topological vector space characterization of fully complete and B_r-complete spaces: E is fully complete $(B_r\text{-complete})$ if and only if every image of E by a (one-to-one) continuous and nearly open mapping is complete.

Throughout, all spaces are assumed to be Hausdorff. As well, mappings are always assumed to be linear. If E is a locally convex topological vector space, $L(E)$ will denote its dual space and $L_c(E)$ its c-dual; that is, the resulting convergence vector space when $L(E)$ carries the

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continuous convergence structure. For definitions and properties of convergence structures in general and the continuous convergence structure in particular, we refer to [1].

2. The Hahn-Banach problem for $L_c(E)$

A subspace V of a convergence vector space W is said to have the *Hahn-Banach extension property* if every continuous functional $\varphi : V \to \mathbb{R}$ has a continuous extension $\tilde{\varphi}$ to W. Extension theorems in general convergence vector spaces are rather difficult to obtain (see [7]). For c-dual spaces, Proposition III.3 and Theorem III.4 of [8] show that, even for a Fréchet space E, it is possible to find a subspace V of $L_c(E)$ and a continuous functional φ on V which cannot be extended to the adherence of V. In the light of this situation, we restrict our attention to the closed subspaces of $L_c(E)$.

Let E and F be locally convex topological vector spaces with 0 neighbourhood filters $\mathcal{U}$ and $\mathcal{V}$ respectively, and let $f : E \to F$ be a continuous mapping. We recall that f is called *open* (*nearly open*) if the filter $f(\mathcal{U})$ $(\overline{f(\mathcal{U})})$ is coarser than $\mathcal{V}$.

LEMMA 2.1. *Let E and F be locally convex topological vector spaces and $f : E \to F$ a continuous mapping. Then the adjoint $f^* : L_c(F) \to L_c(E)$ is an embedding if and only if f is nearly open.*

Proof. Suppose f is nearly open. Then $f(E)$ is dense in F and, since f is continuous, f^* is a continuous injection. $L_c(E)$ and $L_c(F)$ are locally compact convergence vector spaces ([2], Satz 5) and thus each is the inductive limit in the category of convergence spaces of its compact sets ([2], Lemma 1). Therefore, f^* is an embedding if and only if the images under f^* of the compact sets of $L_c(F)$ are precisely the compact sets of $L_c(E)$ contained in the image of f^*. However, the relatively compact sets of $L_c(E)$ and $L_c(F)$ are the equicontinuous subsets of $L(E)$ and $L(F)$ respectively ([4], Theorem 7). Thus, if E_E and E_F denote the collections of equicontinuous subsets of $L(E)$ and $L(F)$ respectively, f^* is an embedding if

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$$f^*(E_F) = E_E \cap f^*(L(F)) .$$

The inclusion $f^*(E_F) \subset E_E \cap f^*(L(F))$ follows immediately from the continuity of f^*. The reverse inclusion holds if and only if f is nearly open. (See for example, [5], 5.11.)

Assume, now, that f^* is an embedding. Then $E_E \cap f^*(L(F)) \subset f^*(E_F)$ and so f is nearly open.

Since the collection of polars $\{U^0 : U \in \mathcal{U}\}$ is cofinal in the system of compact subsets of $L_c(E)$, and since the continuous convergence structure agrees with the weak* topology $\sigma(L(E), E)$ on each U^0 ([4], Lemma 1), we have

$$L_c(E) = \operatorname*{ind}_{U \in \mathcal{U}} \left(U^0, \sigma(L(E), E)\right) ,$$

the inductive limit in the category of convergence spaces of the polars carrying the $\sigma(L(E), E)$ topology. A set $A \subset L(E)$ is closed in $L_c(E)$ or c-closed if and only if, for every U in $\mathcal{U}$, $A \cap U^0$ is $\sigma(L(E), E)$-closed. We recall that a locally convex topological vector space E is *fully complete* if every c-closed subspace of $L(E)$ is $\sigma(L(E), E)$-closed, and B_r*-complete* if every c-closed and $\sigma(L(E), E)$-dense subspace of $L(E)$ coincides with $L(E)$.

THEOREM 2.2. *Let E be a complete locally convex topological vector space. Then the following are equivalent:*

(1) *E is fully complete;*

(2) *every image of E by a continuous and nearly open mapping is complete;*

(3) *every closed subspace of $L_c(E)$ has the Hahn-Banach extension property.*

Proof. We show (1) ⇒ (2) ⇒ (3) ⇒ (1).

(1) ⇒ (2). This follows from the open mapping theorem for fully complete spaces and the fact that quotients of fully complete spaces are

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themselves fully complete and thus complete.

(2) ⇒ (3). Assume that E satisfies (2). Let V be a closed subspace of $L_c(E)$ and $e : V \to L_c(E)$ be the natural embedding. V is locally compact and is itself an $L_c(F)$ for some locally convex topological vector space F ([2], Satz 5). Then the adjoint of e, $e^* : L_cL_c(E) \to L_cL_c(F)$ is a continuous mapping. But the c-bidual $L_cL_c(F)$ of F can be identified with the completion $\tilde{F}$ of F ([3], Satz 8). Moreover, $L_c(F) = L_c(\tilde{F})$ ([3], Satz 7). Since $e = e^{**}$, by Lemma 2.1, $e^* : E \to \tilde{F}$ must be nearly open. Thus $e^*(E)$ is dense in $\tilde{F}$. Since E satisfies (2), however, $e^*(E)$ is complete, so that $e^* : L_cL_c(E) \to L_cL_c(F)$ is surjective. Thus $V = L_c(F)$ has the Hahn-Banach extension property in $L_c(E)$.

(3) ⇒ (1). Assume that $L_c(E)$ satisfies (3). Let V be a closed subspace of $L_c(E)$ and $x_0 \in L(E)\setminus V$. Consider the subspace $V' = V + \mathbb{R}x_0$ of $L_c(E)$. Since V is a closed hyperplane of $V + \mathbb{R}x_0$, V is a direct summand of V'. The proof of this is the same as for locally convex topological vector spaces (see [9], p. 96, Corollary) and requires only the fact, proved in [6], that a finite dimensional vector space has only one Hausdorff convergence vector space structure. Thus $V' = V \oplus \mathbb{R}x_0$ and V' is a complete ([2], Satz 5, Korollar) and hence closed subspace of $L_c(E)$.

Since V is a direct summand of V', the functional φ which is 0 on V and 1 at x_0 is continuous on V'. Since $L_c(E)$ satisfies (3), V' has the Hahn-Banach extension property, so that φ has a continuous extention $\tilde{\varphi}$ to $L_c(E)$. Hence $\tilde{\varphi}$ is a continuous functional on $L_c(E)$ which separates V and x_0. But

$$LL_c(E) = L\big(L(E),\ \sigma(L(E),\ E)\big) = E ,$$

so that $\tilde{\varphi}$ is a continuous functional on $\big(L(E),\ \sigma(L(E),\ E)\big)$ which separates V and x_0. Hence V is $\sigma(L(E),\ E)$-closed and E is fully complete.

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COROLLARY 2.3. *Let* E *be a complete locally convex topological vector space. Then the following are equivalent:*

(1) E *is* B_r*-complete;*

(2) *every image of* E *by a one-to-one, continuous, and nearly open mapping is complete;*

(3) *every* c*-closed and* $\sigma(L(E), E)$*-dense subspace of* $L_c(E)$ *has the Hahn-Banach extension property.*

Proof. The proof is similar to that of Theorem 2.2 and uses the fact that a subspace $L_c(F)$ of $L_c(E)$ is $\sigma(L(E), E)$-dense in $L(E)$ if and only if the adjoint $e^* : E \to L_cL_c(F)$ of the embedding e is one-to-one.

COROLLARY 2.4. *Let* E *be a complete locally convex topological vector space. Then the following are equivalent:*

(1) E *is quotient complete, that is, every image of* E *by a continuous and open mapping is complete;*

(2) *every* $\sigma(L(E), E)$*-closed subspace of* $L_c(E)$ *has the Hahn-Banach extension property.*

Proof. The proof is similar to that of Theorem 2.2 and uses the fact that a subspace $L_c(F)$ of $L_c(E)$ is $\sigma(L(E), E)$-closed in $L(E)$ if and only if the adjoint $e^* : E \to L_cL_c(F)$ of e is an open mapping onto its range.

In order to simplify the study of which subspaces inherit the Hahn-Banach extension property, we introduce the following notation. A c-dual $L_c(E)$ of a complete locally convex topological vector space E is called an

HB1 space if every closed subspace of $L_c(E)$ has the Hahn-Banach extension property,

HB2 space if every c-closed and $\sigma(L(E), E)$-dense subspace of $L_c(E)$ has the Hahn-Banach extension property,

HB3 space if every $\sigma(L(E), E)$-closed subspace of $L_c(E)$ has the

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Hahn-Banach extension property.

The equivalences given in Theorem 2.2 and Corollaries 2.3 and 2.4 together with well-known properties of locally convex topological vector spaces easily yield the following results: every closed subspace of an HB1 space is itself an HB1 space. Every c-closed and $\sigma(L(E), E)$-dense subspace of an HB2 space is itself an HB2 space. Finally, every $\sigma(L(E), E)$-closed subspace of an HB3 space is itself an HB3 space.

On the other hand, it is unknown whether every closed subspace of an HB2 or HB3 space must itself be an HB2 or HB3 space. In fact, these questions are equivalent to the long standing problems of whether every B_r-complete space is fully complete and whether every quotient complete space is fully complete.

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