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Lorentz-Schatten Classes and Pointwise Domination of Matrices

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Abstract. We investigate pointwise domination property in operator spaces generated by Lorentz sequence spaces.

0 Introduction

Let H be the (real or complex) Hilbert space $L_2(\Omega, \mathcal{F}, \mu)$ over an arbitrary σ-finite measure space. Given two (bounded linear) operators $A, B \colon H \to H$ we say that A is pointwise dominated by B, if for all $f \in H$ the inequality

$$|Af(x)| \le B|f|(x), \mu \text{ a.e.}$$

holds (see [10, p. 36]). Throughout this note we will write $|A| \le B$ to mean that A is pointwise dominated by B. Let us give two typical examples: If $H = \ell_2$ and $A = (a_{ij})$, $B = (b_{ij})$ are two matrix operators in H, then

$$|A| \le B \text{ if and only if } |a_{ij}| \le b_{ij} \text{ for all } i, j \in \mathbf{N}.$$

If $H = L_2[0,1]$ and $K \colon [0,1] \times [0,1] \to \mathbf{R}$ (or $\mathbf{C}$) is a measurable kernel, then $|T_K| \le T_{|K|}$. Here T_K stands for the integral operator with kernel K,

$$T_K f(x) = \int_0^1 K(x,y) f(y)\, dy,$$

and $T_{|K|}$ the one with kernel $|K|$.

Although pointwise domination is not stable under arbitrary orthogonal (resp. unitary) transformations, it has some stability properties. In particular, if $|A| \le B$, then $|A^*A| \le B^*B$ and $|A \otimes A| \le B \otimes B$. This fact will be useful for our later considerations.

In his lecture notes [10], Barry Simon studied pointwise domination in connection with Schatten classes S_p. We will work here in the more general context of Lorentz-Schatten classes $S_{p,q}$. Let us recall their definition.

Let H be any Hilbert space and let $T \colon H \to H$ be an operator. The singular numbers of T are

$$s_n(T) = \inf\{\|T - L\| : \operatorname{rank} L < n\} \quad \text{n} \in \mathbf{N}.$$

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The operator T is said to belong to $S_{p,q}$, $0 < p < \infty$, $0 < q \leq \infty$, if

$$\|T\|_{p,q} = \begin{cases} \left(\sum_{n=1}^{\infty} \left(n^{1/p-1/q} s_n(T)\right)^q\right)^{1/q}, & q < \infty \\ \sup_{n \in \mathbf{N}} \{n^{1/p} s_n(T)\}, & q = \infty \end{cases}$$

is finite. The Schatten-Lorentz classes $S_{p,q}$ are quasi-Banach spaces endowed with the quasi-norms $\|\cdot\|_{p,q}$. For $p = q$, we recover the Schatten classes $(S_p, \|\cdot\|_p) = (S_{p,p}, \|\cdot\|_{p,p})$. For more information on these spaces, we refer to the monographs [6], [7] and [9].

Simon in [10] asked for which $0 < p < \infty$ the following holds

$$(*) \qquad |A| \leq B \text{ implies } \|A\|_p \leq \|B\|_p.$$

This is clearly true for $p = 2$. From this case, he derived that (*) is valid for every even integer (see [10, Thm. 2.13]). On the other hand, he gave an example showing that (*) fails for $0 < p \leq 1$. Subsequently, Peller [8] showed that (*) fails whenever p is not an even integer. He derived it by combining his results on Hankel operators with an example by Boas [1] on Fourier coefficients and comparison of L_p-norms. Later on, property (*) has been studied by several authors (see, *e.g.*, [11] and [5]) in the finite-dimensional setting, and also in the context of operator spaces generated by Orlicz sequence spaces (see [4]).

We investigate next domination property for the classes $S_{p,q}$.

1 Pointwise Domination and Lorentz-Schatten Classes

Our aim is to determine those classes $S_{p,q}$ having the following domination property:

$$\text{(DP)} \qquad \left.\begin{array}{r} |A| \leq B \\ B \in S_{p,q} \end{array}\right\} \text{ implies } A \in S_{p,q}.$$

The underlying Hilbert space will be $H = \ell_2$, so that the operators A, B can be always regarded as infinite matrices with entries (a_{ij}), (b_{ij}), respectively. Condition $|A| \leq B$ reads then $|a_{ij}| \leq b_{ij}$ for all $i, j \in \mathbf{N}$.

The next result shows an alternative statement to condition (DP).

Lemma 1 *The following are equivalent:*

(i) $S_{p,q}$ *has* (DP)*.*
(ii) There is a constant $C \geq 1$ *such that* $\|A\|_{p,q} \leq C\|B\|_{p,q}$ *whenever* $|A| \leq B$*.*

Proof The implication (ii) $\Rightarrow$ (i) is obvious. Now assume that (ii) fails. Then there are matrices A_n and B_n, $n \in \mathbf{N}$, such that

$$|A_n| \leq B_n, \quad \|A_n\|_{p,q} \geq n \quad \text{and} \quad \|B_n\|_{p,q} \leq 2^{-n}.$$

According to [6, Thm. III.5.2], without loss of generality we may assume that all these matrices are finite. For the block-diagonal matrices

$$A = \sum_{n=1}^{\infty} A_n, B = \sum_{n=1}^{\infty} B_n$$

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we have $|A| \leq B$ and $\|A\|_{p,q} \geq \|A_n\|_{p,q} \geq n$ for all $n \in \mathbf{N}$, hence $A \notin S_{p,q}$. However, since $\|\cdot\|_{p,q}$ is equivalent to an r-norm, for some $0 < r \leq 1$, we get

$$\|B\|_{p,q}^r \leq c \sum_{n=1}^{\infty} \|B_n\|_{p,q}^r \leq c \sum_{n=1}^{\infty} 2^{-nr} < \infty.$$

That is to say, $|A| \leq B$, $B \in S_{p,q}$ but $A \notin S_{p,q}$. This shows that (i) also fails and completes the proof. ■

Remark If $p = q$ we can take $C = 1$ in statement (ii). This follows by using either Simon's tensor argument (see [11]) or Pietsch's approach to tensor stability of operator ideals (see [9]). Both arguments rely on the fact that S_p is tensor stable, *i.e.*,

$$\|A \otimes A\|_p = \|A\|_p^2 \text{ for every } A \in S_p.$$

Indeed, let $0 < p < \infty$ and let C_p be the smallest possible constant in statement (ii). Suppose that $|A| \leq B$. Then $|A \otimes A| \leq B \otimes B$ as well. Whence $\|A\|_p^2 = \|A \otimes A\|_p \leq C_p \|B \otimes B\|_p = C_p \|B\|_p^2$. By definition of C_p, we get $C_p \geq C_p^2$, which yields $C_p = 1$.

Consequently, in the case of Schatten classes S_p, property (DP) is equivalent to Simon's condition (*) mentioned in the Introduction. In the setting of Lorentz-Schatten classes it seems however more natural to work with (DP) (*i.e.*, allowing in (ii) a constant $C \geq 1$) because $S_{p,q}$, $q \neq p$, is not tensor stable.

We are now ready for establishing the results on domination property and Lorentz-Schatten classes.

Theorem 1 *The space $S_{p,q}$ fails* (DP) *in each of the following cases:*

(i) $0 < p < 2$ *and* $0 < q \leq \infty$.
(ii) $p = 2$ *and* $0 < q < 2$.
(iii) $p > 2$, *p not an even integer, and* $0 < q \leq \infty$.

Proof In each of the cases we will find matrices A_n and B_n with $|A_n| \leq B_n$ and $\lim_{n\to\infty} \|B_n\|_{p,q}/\|A_n\|_{p,q} = 0$. Then Lemma 1 will show the assertion.

(i) This is implicitely contained in [2]. Consider the Walsh matrices A_n, inductively defined as

$$A_0 = (1), \quad A_{n+1} = \begin{pmatrix} A_n & A_n \\ A_n & -A_n \end{pmatrix}.$$

A_n is a $2^n \times 2^n$-matrix with all entries being +1 or -1. Moreover $A_n^* A_n = 2^n I_n$, where I_n is the identity map in $\ell_2^{2^n}$. Therefore $s_k(A_n) = 2^{n/2}$ for $1 \leq k \leq 2^n$, and hence $\|A_n\|_{p,q} \cong 2^{n(1/p+1/2)}$. If B_n is the $2^n \times 2^n$-matrix with all entries being +1, then rank $B_n = 1$, therefore $\|B_n\|_{p,q} = \|B_n\|_\infty = 2^n$. Clearly $|A_n| \leq B_n$ and $\lim_{n\to\infty} \|B_n\|_{p,q}/\|A_n\|_{p,q} = 0$, for any $0 < p < 2$ and any $0 < q \leq \infty$.

(ii) In this situation the result can be derived from the example of [3]. Given any sequence $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$ with $\sum_{n=1}^\infty \alpha_n^2 = 1$, choose disjoint intervals $I_n \subseteq [0,1]$ of

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length $|I_n| = \alpha_n^2$, and let $\nu = \sum_{n=1}^{\infty} n\chi_{I_n}$, where χ_I is the characteristic function of the interval I. For the integral operator on $L_2[0,1]$ with kernel

$$K(x,y) = e^{2\pi i(x-y)\nu(y)}$$

one has $s_n(T_k) = \alpha_n$. Starting with a sequence $(\alpha_n) \in \ell_2 \backslash \ell_{2,q}$ we obtain $T_K \notin S_{2,q}$. On the other hand, since $|K(x,y)| = 1$ for all $x, y \in [0,1]$, the integral operator $T_{|K|}$ has rank one and clearly belongs to $S_{2,q}$. Moreover,

$$\|T_{|K|}\|_{2,q} = \|T_{|K|}\| = 1.$$

In order to obtain from these operators the desired matrices, we recall a well-known result on Schatten classes, which says that $\|T\|_{p,q} = \lim_{n\to\infty} \|P_n T P_n\|_{p,q}$ for every $T \in S_{p,q}$ and every sequence of monotonically increasing finite-dimensional orthogonal projections P_n tending strongly to the identity operator (see, *e.g.*, [6, Thm. III.5.2]). We shall apply this result with P_n being the orthogonal projection onto $H_n = \operatorname{span}\{\chi_j : 1 \le j \le 2^n\}$, where χ_j stands for the characteristic function of the interval $I_j = \left(\frac{j-1}{2^n}, \frac{j}{2^n}\right)$. Clearly the P_n's are monotonically increasing. Moreover, since the Haar system $\{h_j : j \in \mathbf{N}\}$ is an orthonormal basis in $L_2[0,1]$ and $H_n = \operatorname{span}\{h_j : 1 \le j \le 2^n\}$, the P_n's tend strongly to the identity operator. Whence $\lim_{n\to\infty} \|P_n T_k P_n\|_{2,q} = \infty$. On the other hand, for any $n \in \mathbf{N}$, $P_n T_{|K|} P_n = T_{|K|}$ so $\|P_n T_{|K|} P_n\|_{2,q} = 1$. The desired matrices will be the matrix representations $A_n = (a_{j\ell})$, $B_n = (b_{j\ell})$ of the operators $P_n T_K P_n$, $P_n T_{|K|} P_n$, respectively, with respect to the orthonormal basis $\{2^{n/2}\chi_1, \dots, 2^{n/2}\chi_{2^n}\}$ of H_n. Indeed, we have pointwise domination $|A_n| \le B_n$ because

$$|a_{j\ell}| = \left| 2^n \int_{I_j} \int_{I_\ell} K(x,y)\,dx\,dy \right| \le 2^{-n} = b_{j\ell},$$

while

$$\|A_n\|_{2,q} = \|P_n T_K P_n\|_{2,q} \to \infty \text{ as } n \to \infty \quad \text{and}$$
$$\|B_n\|_{2,q} = \|P_n T_{|K|} P_n\|_{2,q} \to 1 \text{ as } n \to \infty.$$

(iii) This last case follows from [4] where for any $p > 2$, p not an even integer, finite matrices A and B have been constructed with $|A| \le B$ and $\|A\|_p > 1 > \|B\|_p$. By a continuity argument there is $\epsilon > 0$ such that even

$$a = \|A\|_{p+\epsilon} > 1 > \|B\|_{p-\epsilon} = b.$$

Put

$$A_n = \underbrace{A \otimes \cdots \otimes A}_{n \text{ times}}, \quad B_n = \underbrace{B \otimes \cdots \otimes B}_{n \text{ times}}.$$

Then we still have pointwise domination $|A_n| \le B_n$ and $\|A_n\|_{p+\varepsilon} = a^n$, $\|B_n\|_{p-\epsilon} = b^n$. Repeating now the construction of Lemma 1 we get for $\mathbf{A} = A \oplus A_2 \oplus \cdots \oplus A_n \oplus \cdots$,

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$\mathbf{B} = B \oplus B_2 \oplus \cdots \oplus B_n \oplus \cdots$ that $|\mathbf{A}| \leq \mathbf{B}$, $\mathbf{A} \notin S_{p+\epsilon}$ and $\mathbf{B} \in S_{p-\epsilon}$. Whence $\mathbf{A} \notin S_{p,q}$ while $\mathbf{B} \in S_{p,q}$.

The proof is complete. ■

The next theorem is the main result of this note and refers to the cases $p = 2 < q \leq \infty$ and $p = 4 < q \leq \infty$.

Theorem 2 *The Lorentz-Schatten classes $S_{2,q}$, $2 < q \leq \infty$, and $S_{4,q}$, $4 < q \leq \infty$, fail* (DP)*.*

Proof Given any matrix M, we have that $M^*M \in S_{p,q}$ if and only if $M \in S_{2p,2q}$. Moreover, $|A| \leq B$ implies $|A^*A| \leq B^*B$. Hence it suffices to establish the result for $S_{4,q}$.

With this aim, let us consider the matrices

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

This is one of the examples considered in [5]. Their singular numbers are

$$s_1(A) = s_2(A) = \sqrt{3},\ s_3(A) = 0; \quad s_1(B) = 2,\ s_2(B) = s_3(B) = 1.$$

Let $A_n = \underbrace{A \otimes \cdots \otimes A}_{n \text{ times}}$, $B_n = \underbrace{B \otimes \cdots \otimes B}_{n \text{ times}}$. Our first goal is to estimate the norm of these matrices in $S_{4,\infty}$. For the singular numbers of A_n we have

$$s_k(A_n) = \begin{cases} 3^{n/2} & \text{for } 1 \leq k \leq 2^n \\ 0 & \text{for } k > 2^n \end{cases}$$

whence $\|A_n\|_{4,\infty} = 3^{n/2} 2^{n/4} = 18^{n/4}$. The singular numbers of B_n are all possible products $\prod_{j=1}^n s_{k_j}(B)$, $1 \leq k_j \leq 3$. We have to rearrange all these numbers in non-increasing order. Assume that exactly j of these factors are 1 and the remaining $n - j$ are 2. This happens $\binom{n}{j} 2^j$ times. Setting $N_{-1} = 0$ and $N_\ell = \sum_{j=0}^{\ell} \binom{n}{j} 2^j$ for $\ell = 0, 1, \ldots, n$, we get

$$s_k(B_n) = \begin{cases} 2^{n-\ell} & \text{if } N_{\ell-1} < k \leq N_\ell,\ 0 \leq \ell \leq n \\ 0 & \text{if } \ell \geq N_n = 3^n. \end{cases}$$

Consequently

$$\|B_n\|_{4,\infty}^4 = \max_{0 \leq \ell \leq n} N_\ell 2^{4(n-\ell)}.$$

For $\frac{n}{2} < \ell \leq n$, since $N_\ell \leq 3^n$, we get

$$\max_{n/2 < \ell \leq n} N_\ell 2^{4(n-\ell)} \leq 3^n 2^{2n} = 12^n.$$

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For $0 \le \ell \le \frac{n}{2}$, we estimate N_ℓ by

$$N_\ell \le \binom{n}{\ell} \sum_{j=0}^{\ell} 2^j \le 2^{\ell+1} \binom{n}{\ell}.$$

Hence

$$\max_{0 \le \ell \le n/2} N_\ell 2^{4(n-\ell)} \le 2^{4n+1} \max_{0 \le \ell \le n} \binom{n}{\ell} 2^{-3\ell}.$$

It is easily checked that the last maximum is attained at $\ell_0 = \left[\frac{n+1}{9}\right]$. Using Stirling's formula

$$\lim_{N \to \infty} \frac{\left(\frac{N}{e}\right)^N \sqrt{2\pi N}}{N!} = 1$$

we obtain, with some absolute constant,

$$\binom{n}{\ell_0} 2^{-3\ell_0} \le c \left(\frac{9}{8}\right)^n n^{-1/2}.$$

So

$$\max_{0 \le \ell \le n/2} N_\ell 2^{4(n-\ell)} \le 2c 18^n n^{-1/2}.$$

Altogether we derive

$$\|B_n\|_{4,\infty} \le c_1 18^{n/4} n^{-1/8}.$$

By Hölder's inequality, since $\|B_n\|_4 = \|B\|_4^n = 18^{n/4}$, we finally get, for $4 < q < \infty$

$$\|B_n\|_{4,q} \le \|B_n\|_4^{4/q} \|B_n\|_{4,\infty}^{1-4/q} \le c_2 18^{n/4} n^{-\alpha}$$

where $\alpha = \frac{1}{8} - \frac{1}{2q} > 0$. This implies

$$\lim_{n \to \infty} \frac{\|B_n\|_{4,q}}{\|A\|_{4,\infty}} = 0$$

whenever $4 < q \le \infty$. Since $\|A_n\|_{4,\infty} \le c\|A_n\|_{4,q}$, it follows from Lemma 1 that $S_{4,q}$ fails (DP).

The proof is complete. ■

Remark The proof shows that even the implication

$$\left.\begin{array}{l} |A| \le B \\ B \in S_{4,q} \end{array}\right\} \Rightarrow A \in S_{4,\infty}$$

fails for any $4 < q \le \infty$.

Whether or not $S_{p,q}$ fails (DP) for the cases not covered by Theorems 1 and 2 remains open.

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