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Connected Numbers and the Embedded Topology of Plane Curves

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Abstract. The splitting number of a plane irreducible curve for a Galois cover is effective in distinguishing the embedded topology of plane curves. In this paper, we define the connected number of a plane curve (possibly reducible) for a Galois cover, which is similar to the splitting number. By using the connected number, we distinguish the embedded topology of Artal arrangements of degree $b \geq 4$, where an Artal arrangement of degree b is a plane curve consisting of one smooth curve of degree b and three of its total inflectional tangents.

1 Introduction

In this paper, we investigate the embedded topology of plane curves in the complex projective plane $\mathbb{P}^2 := \mathbb{CP}^2$, which is an object in the intersection of algebraic geometry and geometrical topology. The first result of this study is due to O. Zariski [17]. He considered the following question:

> *Does an algebraic function z of x and y exist, possessing a preassigned curve f as branch curve?*

Zariski pointed out that this question can be reduced to finding the fundamental group of the complement of the given curve (the word *complement* is understood and often omitted for short). Moreover, he shows by the following example that the configuration of singularities of a plane curve can affect the fundamental group (hence the embedded topology) of the plane curve.

Example 1.1 ([17]) For a sextic curve (*i.e.,* of degree six) with six cusps on the projective plane $\mathbb{P}^2$, its fundamental group is the free product of two cyclic groups of orders two and three, respectively if the six cusps lie on a conic, and is the cyclic group of order six otherwise.

In [1], E. Artal suggested calling such pair of curves a Zariski pair. More precisely, a pair (C_1, C_2) of plane curves $C_1, C_2 \subset \mathbb{P}^2$ is a *Zariski pair* if C_1 and C_2 satisfy the following two conditions:

(a) C_1 and C_2 have the same combinatorics, *i.e.,* they are equisingular (see [2] for details), and
(b) C_1 and C_2 have different embedded topology, *i.e.,* the pairs $(\mathbb{P}^2, C_1)$ and $(\mathbb{P}^2, C_2)$ are not homeomorphic.

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A k-tuple $(C_1, \dots, C_k)$ of k plane curves $C_i \subset \mathbb{P}^2$ is called a *Zariski k-plet* if (C_i, C_j) is a Zariski pair for any $i \neq j$. Example 1.1 is the first example of Zariski pairs. E. Artal [1], M. Oka [11], H. Tokunaga [15,16], and others have discovered Zariski pairs by studying fundamental groups (or related invariants) of plane curves. They seem interested in the embedded topology, such as in knot and link theory, rather than the existence of algebraic functions as in the original question. As in the study of embedded topology, the following problem has arisen naturally.

Problem 1.2 Give a method for distinguishing the embedded topology of plane curves whose fundamental groups are isomorphic.

A pair (C_1, C_2) of plane curves is said to be π_1-*equivalent* if the fundamental groups of C_1 and C_2 are isomorphic. Several methods are known as partial answers of Problem 1.2. For example, there are methods using the theory of K3-surfaces [8], the braid monodromy [2], the splitting number [14], and the linking set [9]. The methods using the theory of K3-surfaces and the splitting number are based on techniques of algebraic geometry. On the other hand, the ones using the braid monodromy and the linking set are derived from invariants of geometrical topology. Recently, B. Guerville and the author gave a relation between the splitting number and the linking set for certain plane curves in [10].

In this paper we introduce a new invariant, called the *connected number*, which is a modification of the splitting number. The splitting number is derived from the studies of splitting curves by E. Artal, H. Tokunaga, and S. Bannai, where a curve $C \subset \mathbb{P}^2$ is called a *splitting curve* with respect to a double cover $\phi\colon X \to \mathbb{P}^2$ if there are two curves $C^+, C^- \subset X$ with no common components such that $\phi^* C = C^+ + C^-$ and $C^- = \iota^* C^+$ for the covering transformation $\iota\colon X \to X$ of ϕ. In [3], Artal and Tokunaga used splitting curves for double covers to establish the difference of the fundamental groups of plane curves. In [4], Bannai introduced the *splitting type* of a splitting curve with respect to a double cover. The splitting type gives a method for proving the difference of the embedded topology of plane curves without going through the fundamental groups. In [14], the author defined the *splitting number* of irreducible curves for Galois covers based on Bannai's idea, and by using the splitting number, he proved that the π_1-equivalent equisingular curves defined by Shimada in [13] provide π_1-equivalent Zariski k-plets. This result shows the importance of studying of splitting curves to distinguish the embedded topology of plane curves. In particular, it also shows that the connected number is not determined by the fundamental group (see Remark 2.2).

In this paper, we define the connected number of plane curves (not necessarily irreducible) for Galois covers, which is an invariant under certain homeomorphisms from $\mathbb{P}^2$ to itself (see Definition 2.1 and Proposition 2.3). The connected number is similar to the splitting number, but not its generalization (see Remark 2.2). The main results of this paper are Theorem 2.4 and Corollary 2.6, which enable us to compute the connected number for certain cases. Finally, we distinguish the embedded topology of Artal arrangements of degree $b \geq 4$ by the connected number; *i.e.,* we give Zariski k-plets of Artal arrangements of $b \geq 4$, where an Artal arrangement of degree b is a plane curve consisting of one smooth curve of degree b and three lines (the Artal arrangements of degree 3 were first studied by Artal [1]; see Section 4 for details).

2 Connected Number

In this section, we define the connected number of plane curves for Galois covers (Definition 2.1), and state the main results regarding the connected number for cyclic covers (Theorem 2.4 and Corollary 2.6).

Definition 2.1 Let Y be a smooth variety, and let $\phi: X \to Y$ be a Galois cover branched along B. Let $C \subset Y$ be an algebraic subset of Y such that not all irreducible components of C are contained in B, and $C \smallsetminus B$ is connected. We call the number of connected components of $\phi^{-1}(C \smallsetminus B)$ the *connected number* of C for ϕ, and denote it by $c_\phi(C)$.

Remark 2.2 There are three remarks about the connected number.

(i) It is obvious that the connected number $c_\phi(C)$ divides the degree of ϕ.

(ii) For an irreducible plane curve $C \subset \mathbb{P}^2$, we have $c_\phi(C) \le s_\phi(C)$, where $s_\phi(C)$ is the splitting number of C for ϕ. There exists a nodal irreducible plane curve C of degree $d = 6, 7$ with a simple contact conic Δ such that $s_\phi(C) = 2$ and $c_\phi(C) = 1$ for the double cover $\phi: X \to \mathbb{P}^2$ branched at Δ, where a simple contact conic Δ of C is a smooth curve of degree 2 which is tangent with C at just d smooth points of C (see [6]). If C is non-singular, then $c_\phi(C) = s_\phi(C)$.

(iii) Since the π_1-equivalent plane curves constructed by Shimada [13] are distinguished by the splitting numbers of non-singular curves for cyclic covers in [14], they are also distinguished by the connected numbers. In particular, the connected number is not determined by the fundamental group.

By [14, Proposition 1.3], we obtain the following proposition, which means that the connected number is invariant under homeomorphisms from the ambient space to itself that do not interchange any component of the algebraic subset and any one of the branch locus, and keep the Galois cover.

Proposition 2.3 *Let B_i ($i = 1, 2$) be reduced divisors on a smooth variety Y and let G be a finite group. For each $i = 1, 2$, let $\phi_i: X_i \to Y$ be the G-cover induced by a surjection $\theta_i: \pi_1(Y \smallsetminus B_i) \twoheadrightarrow G$. Let C_1 be an algebraic subset of Y such that not all irreducible components of C_1 are contained in B_1, and $C_1 \smallsetminus B_1$ is connected. Assume that there exists a homeomorphism $h: Y \to Y$ with $h(B_1) = B_2$ and an automorphism $\sigma: G \to G$ such that $\sigma \circ \theta_2 \circ h_* = \theta_1$. Then $c_{\phi_1}(C_1) = c_{\phi_2}(C_2)$, where $C_2 = h(C_1)$.*

For a plane curve $B \subset \mathbb{P}^2$ and a cyclic cover $\phi: X \to \mathbb{P}^2$ induced by a surjection $\theta: \pi_1(\mathbb{P}^2 \smallsetminus B) \twoheadrightarrow \mathbb{Z}/m\mathbb{Z}$, let $\mathcal{B}_\theta$ denote the divisor $\mathcal{B}_\theta := \sum_{i=1}^{m-1} i \cdot B_i$, where B_i is the sum of irreducible components of B whose meridians are mapped to $[i] \in \mathbb{Z}/m\mathbb{Z}$ by θ for each $i = 1, \dots, m-1$. In this case, the degree of $\mathcal{B}_\theta$ is divisible by m, by [12, Example 2.1], and we call $\phi: X \to \mathbb{P}^2$ the *cyclic cover of degree m branched at $\mathcal{B}_\theta$*.

Conversely, for a divisor $\mathcal{B} = \sum_{i=1}^{m-1} i \cdot B_i$ such that $B = \sum_{i=1}^{m-1} B_i$ is reduced, if the degree of $\mathcal{B}$ is divisible by m, then there exists a cyclic cover of degree m branched at $\mathcal{B}$ by [12, Example 2.1]. This cyclic cover $\phi_\mathcal{B}: X_\mathcal{B} \to \mathbb{P}^2$ branched at $\mathcal{B}$ is constructed as follows:

Let n be the quotient of $\deg \mathcal{B}$ by m, $\deg \mathcal{B} = mn$. For the invertible sheaf $\mathcal{O}(n)$ on $\mathbb{P}^2$, let $p_n: \mathcal{T}_n \to \mathbb{P}^2$ be the line bundle associated with $\mathcal{O}(n)$. Let $t_n \in \mathrm{H}^0(\mathcal{T}_n, p_n^* \mathcal{O}(n))$ be a tautological section. The variety $X_{\mathcal{B}}$ is constructed as the normalization of $X'_{\mathcal{B}}$ defined by $t_n^m = p_n^* F_{\mathcal{B}}$ in $\mathcal{T}_n$, where $F_{\mathcal{B}} \in \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(mn))$ is a global section defining $\mathcal{B}$ and $\phi_{\mathcal{B}}: X_{\mathcal{B}} \to \mathbb{P}^2$ is the composition of the normalization $X_{\mathcal{B}} \to X'_{\mathcal{B}}$ and $p_n: \mathcal{T}_n \to \mathbb{P}^2$. Note that a global section $F \in \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(n))$ of $\mathcal{O}(n)$ corresponds to a section $\mathbb{P}^2 \to \mathcal{T}_n$ of the line bundle p_n, and denote it by the same notation $F: \mathbb{P}^2 \to \mathcal{T}_n$. On an affine open subset $U \subset \mathbb{P}^2$, the section $F|_U: U \to \mathcal{T}_n|_U \cong U \times \mathbb{C}$ is given by $P \mapsto (P, F|_U(P))$, regarding $F|_U \in \Gamma(U, \mathcal{O}_{\mathbb{P}^2})$ as a polynomial. Since we have the morphism $p_{\mathrm{pow}}: \mathcal{T}_n \to \mathcal{T}_n^{\otimes \mu} \cong \mathcal{T}_{\mu n}$, defined locally by $(P, t) \mapsto (P, t^\mu)$ for $P \in \mathbb{P}^2$ and $t \in \mathbb{C}$, we can regard the equation $t_n^\mu - h = 0$ as a defining equation of $p_{\mathrm{pow}}^{-1}(\mathrm{Im}(h))$ in $\mathcal{T}_n$ for a continuous map $h: \mathcal{C} \to \mathcal{T}_{\mu n}$ from a reduced divisor $\mathcal{C}$ on $\mathbb{P}^2$ to $\mathcal{T}_{\mu n}$ satisfying $p_{\mu n} \circ h = \mathrm{id}_{\mathcal{C}}$.

Theorem 2.4 *Let $B = \sum_{i=1}^{m-1} B_i$ and let $\mathcal{C}$ be two plane curves on $\mathbb{P}^2$ with no common components. Assume that the degree of $\mathcal{B} = \sum_{i=1}^{m-1} i \cdot B_i$ is divisible by m, that all irreducible components of $\mathcal{C}$ are smooth, and that $\mathcal{C}$ is smooth at all intersection points of $\mathcal{C}$ and B. Let $\phi_{\mathcal{B}}: X_{\mathcal{B}} \to \mathbb{P}^2$ be the cyclic cover of degree m branched at $\mathcal{B}$, and put $n := \deg \mathcal{B}/m$. Let $F_{\mathcal{B}} \in \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(mn))$ be a global section defining $\mathcal{B}$. Then, the connected number $c_{\phi_{\mathcal{B}}}(\mathcal{C})$ is the maximal divisor λ of m such that there exists a continuous map $h: \mathcal{C} \to \mathcal{T}_{\mu n}$, where $\mu := m/\lambda$, satisfying the following conditions:*

(i) $p_{\mu n} \circ h = \mathrm{id}_{\mathcal{C}}$;
(ii) $h^\lambda = h^{\otimes \lambda}: \mathcal{C} \to \mathcal{T}_{\mu n}^{\otimes \lambda} = \mathcal{T}_{mn}$ *coincides with the restriction $F_{\mathcal{B}}|_{\mathcal{C}}$ of $F_{\mathcal{B}}$ to $\mathcal{C}$;*
(iii) *for each irreducible component C of $\mathcal{C}$, there exists a global section g_C in* $\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(\mu n))$ *such that $h|_C = g_C|_C$.*

Remark 2.5 Assume the same hypotheses as Theorem 2.4, and let $L \subset \mathbb{P}^2$ be a general line; *i.e.*, L intersects transversally with the curve $B + \mathcal{C}$. Since $p_{\mu n}: \mathcal{T}_{\mu n} \to \mathbb{P}^2$ coincides over $U := \mathbb{P}^2 \smallsetminus L$ with the projection $U \times \mathbb{C} \to U$, the existence of a continuous map $h: \mathcal{C} \to \mathcal{T}_{\mu n}$ satisfying (i), (ii), and (iii) in Theorem 2.4 is equivalent to the existence of a polynomial $g'_C \in \Gamma(U, \mathcal{O}_{\mathbb{P}^2}) \cong \mathbb{C}[x, y]$ of degree μn for each irreducible component C of $\mathcal{C}$ satisfying the following conditions:

(i) $(g'_C)^\lambda - F_{\mathcal{B}}|_U$ is an element of the ideal defining C in $\Gamma(U, \mathcal{O}_{\mathbb{P}^2})$;
(ii) $g'_C(P) = g'_{C'}(P)$ at each intersection $P \in C \cap C'$ for any irreducible components C and C'.

Corollary 2.6 *Assume the same hypotheses as Theorem 2.4, and that $\mathcal{C}$ is a nodal curve. Then, the connected number $c_{\phi_{\mathcal{B}}}(\mathcal{C})$ is the maximal divisor λ of m such that there exists a divisor D on $\mathbb{P}^2$ with $\lambda D|_{\mathcal{C}} = \mathcal{B}|_{\mathcal{C}}$ as Cartier divisors on $\mathcal{C}$:*

$$c_{\phi_{\mathcal{B}}}(\mathcal{C}) = \max\left\{ \lambda \;\middle|\; \lambda \text{ is a divisor of } m \text{ and } \lambda D|_{\mathcal{C}} = \mathcal{B}|_{\mathcal{C}} \text{ for some divisor } D \right\}.$$

3 Proofs

In this section, we prove Theorem 2.4 and Corollary 2.6.

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Proof of Theorem 2.4 Let $B = \sum_{i=1}^{m-1} B_i$ and $\mathcal{C}$ be two curves on $\mathbb{P}^2$ with no common components. Assume that all irreducible components of $\mathcal{C}$ are smooth, that $\mathcal{C}$ is smooth at each intersection of B and $\mathcal{C}$, and the degree of $\mathcal{B} = \sum_{i=1}^{m-1} i \cdot B_i$ is divisible by m. Let $\phi_{\mathcal{B}} : X_{\mathcal{B}} \to \mathbb{P}^2$ be the cyclic cover of degree m branched at $\mathcal{B}$. Let $p_n : \mathcal{T}_n \to \mathbb{P}^2$, $F_{\mathcal{B}}$ and let $\phi'_{\mathcal{B}} : X'_{\mathcal{B}} \to \mathbb{P}^2$ be the notation used in the construction of $\phi_{\mathcal{B}} : X_{\mathcal{B}} \to \mathbb{P}^2$ in Section 2. Let $\kappa_{\mathcal{B}} : X_{\mathcal{B}} \to X'_{\mathcal{B}}$ be the morphism of the normalization. Since $X'_{\mathcal{B}}$ is smooth over $\mathbb{P}^2 \smallsetminus B$, the normalization $\kappa_{\mathcal{B}}$ gives an isomorphism $\kappa'_{\mathcal{B}}$ over $\mathbb{P}^2 \smallsetminus B$:

$$\kappa'_{\mathcal{B}} := \kappa|_{X_{\mathcal{B}} \smallsetminus \phi^{-1}(B)} : X_{\mathcal{B}} \smallsetminus \phi_{\mathcal{B}}^{-1}(B) \xrightarrow{\sim} X'_{\mathcal{B}} \smallsetminus (\phi'_{\mathcal{B}})^{-1}(B).$$

Hence, the connected number $c_{\phi_{\mathcal{B}}}(\mathcal{C})$ is equal to the number of connected components of $(\phi'_{\mathcal{B}})^{-1}(\mathcal{C} \smallsetminus B)$. Let $f_{\mathcal{C}} = 0$ be a defining equation of $\mathcal{C}$.

If there exists a continuous map $h : \mathcal{C} \to \mathcal{T}_{\mu n}$ satisfying conditions (i), (ii), and (iii), then the equation

$$t_n^{\mu} - \zeta_{\lambda}^{j-1} h = f_{\mathcal{C}} = 0 \tag{3.1}$$

defines a subset $\widetilde{\mathcal{C}}_j$ of $(\phi'_{\mathcal{B}})^{-1}(\mathcal{C})$ in $\mathcal{T}_n$ for each $j = 1, \ldots, \lambda$ such that $\bigcup_{j=1}^{\lambda} \widetilde{\mathcal{C}}_j = (\phi'_{\mathcal{B}})^{-1}(\mathcal{C})$, and $\widetilde{\mathcal{C}}_j \smallsetminus (\phi'_{\mathcal{B}})^{-1}(B)$ $(j = 1, \ldots, \lambda)$ are disconnected, where ζ_λ is a primitive λ-th root of the unity. Hence, we have $c_{\phi_{\mathcal{B}}}(\mathcal{C}) \geq \lambda$. Thus, it is sufficient to prove that there exists a continuous map $h : \mathcal{C} \to \mathcal{T}_{\mu n}$ satisfying conditions (i), (ii), and (iii) such that $\widetilde{\mathcal{C}}_j \smallsetminus (\phi'_{\mathcal{B}})^{-1}(B)$ is connected, where $\widetilde{\mathcal{C}}_j$ is the subset of $(\phi'_{\mathcal{B}})^{-1}(\mathcal{C})$ defined by equation (3.1).

Let $\mathcal{C} = \sum_{i=1}^{k} C_i$ be the irreducible decomposition of $\mathcal{C}$. We prove the existence of such $h : \mathcal{C} \to \mathcal{T}_{\mu n}$ by induction on the number k of irreducible components of $\mathcal{C}$. If $\mathcal{C}$ is irreducible, hence $\mathcal{C}$ is smooth, then Theorem 2.4 is equivalent to [10, Theorem 2.1] (*cf.* [14, Theorem 2.7]). Moreover, the closure of each connected component of $\phi_{\mathcal{B}}^{-1}(\mathcal{C} \smallsetminus B)$ is defined by equation (3.1) in $\mathcal{T}_n$ for $j = 1, \ldots, \lambda$.

Suppose that $k > 1$, and let $\mathcal{C}_1 := \sum_{i=1}^{k-1} C_i$ and $\mathcal{C}_2 := C_k$ with $c_{\phi_{\mathcal{B}}}(\mathcal{C}_i) = \nu_i$ for $i = 1, 2$. Moreover, we suppose that the closure $\widetilde{\mathcal{C}}_{ij}$ of each connected component $\widetilde{\mathcal{C}}'_{ij}$ of $(\phi'_{\mathcal{B}})^{-1}(\mathcal{C}_i \smallsetminus B)$ is defined by $t_n^{\mu_i} - \zeta_{\nu_i}^{j-1} h_i = f_i = 0$ in $\mathcal{T}_n$ for $i = 1, 2$ and $j = 1, \ldots, \nu_i$, where $\mu_i := m/\nu_i$, $f_i = 0$ is a defining equation of $\mathcal{C}_i$, and $h_i : \mathcal{C}_i \to \mathcal{T}_{\mu_i n}$ is a continuous map satisfying conditions (i), (ii), and (iii) for $\mathcal{C}_i$. Let ν be the greatest common divisor of ν_1 and ν_2, and let γ_i and δ be the integers ν_i/ν and $m/(\gamma_1\gamma_2\nu)$, respectively. Then we have $\mu_1 = \delta\gamma_2$, $\mu_2 = \delta\gamma_1$, and $m = \delta\gamma_1\gamma_2\nu$. We fix a primitive m-th root ζ_m of unity, and we can assume that $\zeta_{\nu_i} = \zeta_m^{\mu_i}$. For each intersection $P \in \mathcal{C}_1 \cap \mathcal{C}_2$, we also fix an open neighborhood U_P of P in $\mathbb{P}^2$ such that $p_n^{-1}(U_P) \cong U_P \times \mathbb{C}$. For the continuous maps $h_i : \mathcal{C}_i \to \mathcal{T}_{\mu_i n}$ $(i = 1, 2)$ (resp. $F_{\mathcal{B}} : \mathcal{C} \to \mathcal{T}_n$), let $h_{i,P} : U_P \to \mathbb{C}$ (resp. $F_{\mathcal{B},P} : U_P \to \mathbb{C}$) be the function such that $h_i(Q) = (Q, h_{i,P}(Q)) \in \mathcal{T}_{\mu_i n}$ (resp. $F_{\mathcal{B}}(Q) = (Q, F_{\mathcal{B},P}(Q))$) for any $Q \in U_P$ under a fixed isomorphism $p_n^{-1}(U_P) \cong U_P \times \mathbb{C}$. For $P \in \mathcal{C}_1 \cap \mathcal{C}_2$, let d_P be a μ_1-th root of $h_{1,P}(P)$. Then we have

$$F_{\mathcal{B},P}(P) = h_{1,P}^{\nu_1}(P) = h_{2,P}^{\nu_2}(P) = d_P^m \quad \text{and} \quad h_{2,P}(P) = \zeta_{\nu_2}^{a_P} d_P^{\mu_2}$$

for some integer $0 \leq a_P < \nu_2$. Then we have the following claim.

Claim 1 *The intersection $\widetilde{\mathcal{C}}_{1j_1} \cap \widetilde{\mathcal{C}}_{2j_2} \cap (\phi'_{\mathcal{B}})^{-1}(P)$ is non-empty if and only if $j_1 - j_2 \equiv a_P \pmod{\nu}$.*

Proof The intersection $\widetilde{\mathcal{C}}_{1j_1} \cap \widetilde{\mathcal{C}}_{2j_2} \cap (\phi'_{\mathcal{B}})^{-1}(P)$ is non-empty if and only if there exists a complex number $t_P \in \mathbb{C}$ such that $t_P^{\mu_1} = \zeta_{\nu_1}^{j_1-1} d_P^{\mu_1}$ and $t_P^{\mu_2} = \zeta_{\nu_2}^{a_P+j_2-1} d_P^{\mu_2}$. The latter condition is equivalent to

$$j_1 - j_2 + (\beta_1\gamma_1 - \beta_2\gamma_2)\nu \equiv a_P \pmod{m}$$

for some integers β_1 and β_2. Hence, if $\widetilde{\mathcal{C}}_{1j_1} \cap \widetilde{\mathcal{C}}_{2j_2} \cap (\phi'_{\mathcal{B}})^{-1}(P) \neq \varnothing$, then $j_1 - j_2 \equiv a_P \pmod{\nu}$.

Conversely, if $j_1 - j_2 \equiv a_P \pmod{\nu}$, then $a_P = j_1 - j_2 + b\nu$ for some integer b. Since γ_1 and γ_2 are coprime, there exist two integers α_1 and α_2 such that $\alpha_1\gamma_1 - \alpha_2\gamma_2 = 1$; hence, by putting $\beta_i = b\alpha_i$ for $i = 1, 2$, we have $\beta_1\gamma_1 - \beta_2\gamma_2 = b$. This implies that $\widetilde{\mathcal{C}}_{1j_1} \cap \widetilde{\mathcal{C}}_{2j_2} \cap (\phi'_{\mathcal{B}})^{-1}(P) \neq \varnothing$.

This proves Claim 1.

Claim 2 *Let $\widetilde{\mathcal{C}}_{1j}$ and $\widetilde{\mathcal{C}}_{1j'}$ be closures of connected components $\widetilde{\mathcal{C}}'_{1j}$ and $\widetilde{\mathcal{C}}'_{1j'}$ of $(\phi'_{\mathcal{B}})^{-1}(\mathcal{C}_1 \smallsetminus B)$, respectively. Then $\widetilde{\mathcal{C}}_{1j}$ and $\widetilde{\mathcal{C}}_{1j'}$ are contained in the closure $\widetilde{\mathcal{C}}$ of a connected component of $(\phi'_{\mathcal{B}})^{-1}(\mathcal{C} \smallsetminus B)$ if and only if*

$$[j - j']_\nu \in \big\langle [a_P - a_Q]_\nu \mid P, Q \in \mathcal{C}_1 \cap \mathcal{C}_2 \big\rangle \subset \mathbb{Z}/\nu\mathbb{Z},$$

where $[r]_\nu$ is the image of the integer r in $\mathbb{Z}/\nu\mathbb{Z}$.

Proof Two connected components $\widetilde{\mathcal{C}}'_{1j}, \widetilde{\mathcal{C}}'_{1j'}$ of $(\phi'_{\mathcal{B}})^{-1}(\mathcal{C}_1 \smallsetminus B)$ are contained in a connected component $\widetilde{\mathcal{C}}'$ of $(\phi'_{\mathcal{B}})^{-1}(\mathcal{C} \smallsetminus B)$ if and only if there exists a path $p\colon [0,1] \to (\phi'_{\mathcal{B}})^{-1}(\mathcal{C} \smallsetminus B)$ such that $p(0) \in \widetilde{\mathcal{C}}'_{1j}$ and $p(1) \in \widetilde{\mathcal{C}}'_{1j'}$. We can assume that $\phi'_{\mathcal{B}} \circ p(0)$ and $\phi'_{\mathcal{B}} \circ p(1)$ are not intersection points of $\mathcal{C}_1$ and $\mathcal{C}_2$ and that $\phi'_{\mathcal{B}} \circ p(s) \in \mathcal{C}_1 \cap \mathcal{C}_2$ for $0 < s < 1$ if and only if, for any $0 < \epsilon \ll 1$, either $\phi'_{\mathcal{B}} \circ p(s+\epsilon) \in \mathcal{C}_1$ and $\phi'_{\mathcal{B}} \circ p(s-\epsilon) \in \mathcal{C}_2$ or $\phi'_{\mathcal{B}} \circ p(s+\epsilon) \in \mathcal{C}_2$ and $\phi'_{\mathcal{B}} \circ p(s-\epsilon) \in \mathcal{C}_1$. Since $\phi'_{\mathcal{B}} \circ p(0)$ and $\phi'_{\mathcal{B}} \circ p(1)$ are two points of $\mathcal{C}_1$, the number of points $s \in (0,1)$ with $\phi'_{\mathcal{B}} \circ p(s) \in \mathcal{C}_1 \cap \mathcal{C}_2$ is even. Let $\{s_1, \dots, s_{2k'}\}$ be the set of points $s_i \in (0,1)$ with $\phi'_{\mathcal{B}} \circ p(s_i) \in \mathcal{C}_1 \cap \mathcal{C}_2$ and $0 < s_1 < \cdots < s_{2k'} < 1$. Let $\widetilde{\mathcal{C}}'_{1j_i}$ and $\widetilde{\mathcal{C}}'_{2l_i}$ for $i = 1, \dots, k'$ be the connected components of $(\phi'_{\mathcal{B}})^{-1}(\mathcal{C}_1 \smallsetminus B)$ and $(\phi'_{\mathcal{B}})^{-1}(\mathcal{C}_2 \smallsetminus B)$ such that $p(s) \in \widetilde{\mathcal{C}}'_{1j_i}$ for $s_{2i} < s < s_{2i+1}$ and $p(s) \in \widetilde{\mathcal{C}}'_{2l_i}$ for $s_{2i-1} < s < s_{2i}$, respectively, where $s_{2k'+1} = 1$, hence $\widetilde{\mathcal{C}}'_{1j_{k'}} = \widetilde{\mathcal{C}}'_{1j'}$. Putting $P_i := \phi'_{\mathcal{B}} \circ p(s_i)$ for $i = 1, \dots, 2k'$, by Claim 1, we obtain

$$j_{i-1} - l_i \equiv a_{P_{2i-1}} \pmod{\nu}, \tag{3.2}$$

$$j_i - l_i \equiv a_{P_{2i}} \pmod{\nu} \tag{3.3}$$

for $i = 1, \dots, k'$, where $j_0 = j$. Hence, if $\widetilde{\mathcal{C}}'_{1j}$ and $\widetilde{\mathcal{C}}'_{1j'}$ are contained in a connected component $\widetilde{\mathcal{C}}'$ of $(\phi'_{\mathcal{B}})^{-1}(\mathcal{C} \smallsetminus B)$, then we have

$$j - j' \equiv \sum_{i=1}^{k'} (a_{P_{2i-1}} - a_{P_{2i}}) \pmod{\nu}.$$

Conversely, suppose that $j - j' \equiv \sum_{i=1}^{k'} (a_{P_{2i-1}} - a_{P_{2i}}) \pmod{\nu}$. We can find integers $0 < j_i < \nu_1$ and $0 < l_i < \nu_2$ for $i = 1, \dots, k'$ satisfying equations (3.2) and (3.3). By Claim 1, there exists a path $p\colon [0,1] \to (\phi'_{\mathcal{B}})^{-1}(\mathcal{C} \smallsetminus B)$ with $p(0) \in \widetilde{\mathcal{C}}'_{1j}$ and $p(1) \in \widetilde{\mathcal{C}}'_{1j'}$.

Therefore, $\widetilde{\mathcal{C}}'_{1j}$ and $\widetilde{\mathcal{C}}'_{1j'}$ are contained in a connected component $\widetilde{\mathcal{C}}'$ of $(\phi'_{\mathcal{B}})^{-1}(\mathcal{C} \setminus B)$. This proves Claim 2.

Fix an intersection point P_0 of $\mathcal{C}_1$ and $\mathcal{C}_2$; we can assume that $a_{P_0} = 0$ after multiplying $\zeta_{\nu_2}^{-a_{P_0}}$ to h_2. Then we have

$$\left\langle [a_P - a_Q]_\nu \mid P, Q \in \mathcal{C}_1 \cap \mathcal{C}_2 \right\rangle = \left\langle [a_P]_\nu \mid P \in \mathcal{C}_1 \cap \mathcal{C}_2 \right\rangle \cong \mathbb{Z}/\lambda\mathbb{Z},$$

where λ is the greatest common divisor of ν and a_P for $P \in \mathcal{C}_1 \cap \mathcal{C}_2$. We put $\lambda' := \nu/\lambda$. Then we obtain

$$h_{1,P}^{\gamma_1\lambda'}(P) = d_P^{\mu_1\gamma_1\lambda'} = d_P^{\delta\gamma_1\gamma_2\lambda'},$$
$$h_{2,P}^{\gamma_2\lambda'}(P) = (\zeta_{\nu_2}^{a_P} d_P^{\mu_2})^{\gamma_2\lambda'} = \zeta_{\nu_2}^{\gamma_2\nu(a_P/\lambda)} d_P^{\mu_2\gamma_2\lambda'} = d_P^{\delta\gamma_1\gamma_2\lambda'}$$

for any $P \in \mathcal{C}_1 \cap \mathcal{C}_2$. We define the continuous map $h\colon \mathcal{C} \to \mathcal{T}_{\mu n}$ by $h(P) = h_i^{\gamma_i\lambda'}(P)$ if $P \in \mathcal{C}_i$ for $i = 1, 2$, where $\mu := \gamma_1\mu_1\lambda' = \gamma_2\mu_2\lambda' = m/\lambda$. The map h satisfies conditions (i), (ii), and (iii). Moreover, by Claim 2, equation (3.1) defines a subset $\widetilde{\mathcal{C}}_j$ of $(\phi')^{-1}(\mathcal{C})$ such that $\widetilde{\mathcal{C}}_j \setminus (\phi')^{-1}(B)$ is connected for $j = 1, \dots, \lambda$. ■

Proof of Corollary 2.6 The existence of a divisor D on $\mathbb{P}^2$ with $\lambda D|_{\mathcal{C}} = \mathcal{B}|_{\mathcal{C}}$ implies the existence of a global section $H \in \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(\mu n))$ with $H^\lambda|_{\mathcal{C}} = F_{\mathcal{B}}|_{\mathcal{C}}$. Hence it is sufficient to prove that for any continuous map $h\colon \mathcal{C} \to \mathcal{T}_{\mu n}$ satisfying conditions (i), (ii) and (iii) in Theorem 2.4, there exists a global section $H \in \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(\mu n))$ such that $H|_{\mathcal{C}} = h$.

Let $\mathcal{C} = \sum_{i=1}^k C_i$ be the irreducible decomposition of $\mathcal{C}$. We prove the above statement by induction on the number k of irreducible components of $\mathcal{C}$. In the case $k = 1$, it is obvious by condition (iii). Suppose $k > 1$, and put $\mathcal{C}_1 := \sum_{i=1}^{k-1} C_i$ and $\mathcal{C}_2 := C_k$. Let $h\colon \mathcal{C} \to \mathcal{T}_{\mu n}$ be a continuous map satisfying (i), (ii), and (iii). By the assumption of induction, there exist two global sections $H_1, H_2 \in \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(\mu n))$ such that $H_i|_{\mathcal{C}_i} = h|_{\mathcal{C}_i}$ for $i = 1, 2$. We have the exact sequence

$$\mathcal{O}_{\mathbb{P}^2}(\mu n - c_1) \otimes \mathcal{O}_{\mathbb{P}^2}(\mu n - c_2) \xrightarrow{(f_1, -f_2)} \mathcal{O}_{\mathbb{P}^2}(\mu n) \longrightarrow \mathcal{O}_Z(\mu n) \longrightarrow 0,$$

where c_i is the degree of $\mathcal{C}_i$, f_i is a global section of $\mathcal{O}_{\mathbb{P}^2}(c_i)$ defining $\mathcal{C}_i$ for $i = 1, 2$ and Z is the scheme defined by the ideal sheaf generated by f_1 and f_2. Since $\mathcal{C}$ is a nodal curve, $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect transversally. Hence, $\mathcal{O}_Z(\mu n)$ is the skyscraper sheaf with support $\mathcal{C}_1 \cap \mathcal{C}_2$ such that its stalk at each intersection $P \in \mathcal{C}_1 \cap \mathcal{C}_2$ is isomorphic to $\mathbb{C}$. Since $H_i|_{\mathcal{C}_i} = h|_{\mathcal{C}_i}$ for $i = 1, 2$, the image of $H_1 - H_2$ on $\mathcal{O}_Z(\mu n)$ is zero. Hence, there exist two global sections $g_i \in \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(\mu n - c_i))$, $i = 1, 2$, such that $f_1 g_1 - f_2 g_2 = H_1 - H_2$. Then $H := H_1 - f_1 g_1 = H_2 - f_2 g_2$ satisfies $H|_{\mathcal{C}} = h$. ■

4 Artal Arrangements of Degree b

In [1], Artal gave a Zariski pair $(\mathcal{A}_1, \mathcal{A}_2)$, where $\mathcal{A}_i$ is an arrangement of a smooth cubic C_i and three of its inflectional tangents $L_{i,j}$ $(j = 1, 2, 3)$ for each $i = 1, 2$; *i.e.*, $C_i \cap L_{i,j}$ is just one point. Putting $P_{i,j}$ as the tangent point of C_i and $L_{i,j}$, the three points $P_{1,j}$ $(j = 1, 2, 3)$ are collinear, but the three points $P_{2,j}$ are not collinear. He distinguished the embedded topology of $\mathcal{A}_1$ and $\mathcal{A}_2$ by their Alexander polynomials. It is possible to distinguish their embedded topology by other invariants; the existence

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of dihedral coverings, the splitting number, and the linking set (see [5, 15]). In this section, we generalize Artal's Zariski pair by using the connected number. First we define k-Artal arrangements of degree b.

Definition 4.1 Let B be a smooth curve on $\mathbb{P}^2$ of degree $b \geq 3$ having k total inflection points $P_1, \dots, P_k$, and let L_i $(i = 1, \dots, k)$ be the tangent line of B at P_i. We call $B + \sum_{i=1}^{k} L_i$ a *k-Artal arrangement of degree b* if no three lines of L_i $(i = 1, \dots, k)$ meet at just one point. In the case of $k = 3$, we call a 3-Artal arrangement of degree b an *Artal arrangement of degree b*.

Remark 4.2 In [5], a k-Artal arrangement is defined as an arrangement of one smooth cubic and k of its inflectional tangents. A k-Artal arrangement in [5] is our k-Artal arrangement of degree 3.

We construct Zariski k-plets of Artal arrangements of degree $b \geq 4$ by using Fermat curves. For an integer $\mu \geq 2$, let F_μ be the curve defined by $x^\mu + y^\mu + z^\mu = 0$. For each $i = 1, 2, 3$ and $j = 1, \dots, \mu$, let $P^\mu_{i,j}$ be the point

$$P^\mu_{1,j} = (1 : \zeta_{2\mu}^{2j-1} : 0), \quad P^\mu_{2,j} = (0 : 1 : \zeta_{2\mu}^{2j-1}), \quad P^\mu_{3,j} = (\zeta_{2\mu}^{2j-1} : 0 : 1),$$

where $\zeta_{2\mu}$ is a primitive 2μ-th root of unity. For $\mu \geq 3$, F_μ is the Fermat curve of degree μ, and $P^\mu_{i,j}$ are the 3μ total inflection points of F_μ. Let $L^\mu_{i,j}$ be the tangent line of F_μ at $P^\mu_{i,j}$. Note that $L^\mu_{1,j}$, $L^\mu_{2,j}$, and $L^\mu_{3,j}$ are defined by

$$\zeta_{2\mu}^{2j-1} x - y = 0, \quad \zeta_{2\mu}^{2j-1} y - z = 0, \quad \text{and} \quad \zeta_{2\mu}^{2j-1} z - x = 0,$$

respectively, and that $L^\mu_{1,j_1} \cap L^\mu_{2,j_2} \cap L^\mu_{3,j_3} = \varnothing$ for any j_1, j_2, j_3. From Carnot's theorem (*cf.* [7, Lemma 2.2]), we have the following lemma.

Lemma 4.3 *Fix integers μ, j_1, j_2, j_3. There exists a divisor D on $\mathbb{P}^2$ of degree d such that $D|_{L^\mu_{i,j_i}} = dP^\mu_{i,j_i}$ for all $i = 1, 2, 3$ if and only if $(\zeta_{2\mu}^{2j_1+2j_2+2j_3-3})^{2d} = 1$.*

Then we obtain the following proposition by Corollary 2.6 and Lemma 4.3.

Proposition 4.4 *Let $h_\mu, f_{\mu,i}$ $(i = 1, 2, 3)$ be the following polynomials:*

$$h_\mu := x^\mu + y^\mu + z^\mu, \qquad f_{\mu,1} := \zeta_{2\mu} x - y,$$
$$f_{\mu,2} := \zeta_{2\mu} y - z, \qquad f_{\mu,3} := \zeta_{2\mu}^{2\mu-1} z - x.$$

Let ν be a positive integer, and put $b := \mu\nu$. Let $B_{b,\mu}$ be the curve of degree b defined by

$$f_{\mu,1} f_{\mu,2} f_{\mu,3}\, g + h_\mu^\nu = 0,$$

where g is a general homogeneous polynomial of degree $b - 3$, i.e., *$B_{b,\mu}$ is smooth. Let $\phi_{B_{b,\mu}} : X_{b,\mu} \to \mathbb{P}^2$ be the cyclic cover of degree b branched at $B_{b,\mu}$, and put $L_\mu := L^\mu_{1,1} + L^\mu_{2,1} + L^\mu_{3,\mu}$. Then $c_{\phi_{B_{b,\mu}}}(L_\mu) = \nu$.*

Since the connected number $c_{\phi_B}(L)$ is divisible by b for an Artal arrangement of degree b, $B + L$, where $L = L_1 + L_2 + L_3$, Proposition 4.4 implies that for any number ν

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possible as the connected number of Artal arrangements of degree b, *i.e.*, any divisor ν of b, there exists an Artal arrangement of degree b, $B + L$, with $c_{\phi_B}(L) = \nu$.

Theorem 4.5 *Let b be a positive integer, and let $\mu_1, \dots, \mu_k$ be distinct divisors of b. Assume that all B_{b,μ_i} are smooth. Let $\mathcal{B}_{b,i}$ be the Artal arrangement $B_{b,\mu_i} + L_{\mu_i}$ of degree b constructed as above. Then $(\mathcal{B}_{b,1}, \dots, \mathcal{B}_{b,k})$ is a Zariski k-plet.*

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