J. Austral. Math. Soc. (*Series A*) **30** (1980), 252–255

C-COMMUTATIVITY

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(Received 29 September 1979)

Communicated by D. E. Taylor

Abstract

An associative ring R with identity is said to be c-commutative for $c \in R$ if $a, b \in R$ and $ab = c$ implies $ba = c$. Taft has shown that if R is c-commutative where c is a central, nonzero divisor of R then $R[[x]]$ is c-commutative. We give examples to show that neither condition on c (that is, central or nonzero divisor) can be omitted. We show that if $R[x]$ is $h(x)$-commutative for any $h(x) \in R[x]$ then so is R with any finite number of (commuting) indeterminates adjoined. Examples are given to show that $R[[x]]$ need not be c-commutative even if $R[x]$ is. Finally, examples are given to answer Taft's question for the special case of a zero-commutative ring.

1980 Mathematics subject classification (*Amer. Math. Soc.*): primary 16 A 70; secondary 16 A 05.

An associative ring R with identity is said to be c-commutative for $c \in R$ if $a, b \in R$ and $ab = c$ implies $ba = c$. Taft has shown (see the foot-note in Hemr (1970)) that if R is c-commutative where c is a central, nonzero divisor of R then $R[[x]]$ is c-commutative. He raises the question of whether either condition on c (that is, central or nonzero divisor) can be omitted. We give examples to show that neither condition can be omitted. However, the following question remains open: If c is a noncentral, nonzero divisor and R is c-commutative is $R[x]$ c-commutative? We show that if $R[x]$ is $h(x)$-commutative for any $h(x) \in R[x]$ then so is R with any finite number of (commuting) indeterminates adjoined. Examples are given to show that $R[[x]]$ need not be c-commutative even if $R[x]$ is. Of course by Taft's result c is either a noncentral element in R or is a zero-divisor in R. Finally, examples are given to answer Taft's question for the special case of zero-commutative ring.

EXAMPLE 1. A ring R with a central zero divisor c such that R is c-commutative but $R[x]$ is not. Let Z_2 denote the ring of integers modulo two. Let R be the ring $Z_2\{a_0, a_1, b_0, b_1\}$ (noncommuting indeterminates) subject to the relations:

(1) $a_0 b_0 = b_0 a_0$.

(2) $a_0 b_0 + a_1 b_0 = 0$.

(3) $a_1 b_1 = 0$.
(4) $b_0 a_1 + b_1 a_0 + b_1 a_1 = 0$.
(5) All monomials of order greater than two are zero.

Then $c = a_0 b_0$ gives the desired example.

We include the proof for this example as a sample of the techniques. If f is an element of R then we can write $f = f_0 + f_1 + f_2$ where f_i is a form of degree i and in fact $f_0 \in Z_2$. Let $f = f_0 + f_1 + f_2$ and $g = g_0 + g_1 + g_2$ be two elements of R such that $fg = c$. If g is a unit, $g = 1 + g_1 + g_2$ so clearly $cg = c$ and hence $f = c$. Thus, $gf = c$. If f is a unit we argue similarly. If neither f nor g is a unit then f and g can be taken of degree one, $f = \alpha_0 a_0 + \alpha_1 a_1$ and $g = \beta_0 b_0 + \beta_1 b_1$ (where $\alpha_i, \beta_i \in Z_2$) or vice versa. In either case by using the relations one can show $gf = c$. Therefore, R is c-commutative. To see that $R[x]$ is not c-commutative note that $c = (a_0 + a_1 x)(b_0 + b_1 x)$ by the relations but $c \neq (b_0 + b_x)(a_0 + a_1 x)$.

EXAMPLE 2. A ring T with a noncentral element c such that T is c-commutative but $T[x]$ is not. Begin with $R = Z_2\{a_0, a_1, b_0, b_1, a'_0, a'_1, b'_0, b'_1\}$ subject to the relations (1) to (5) of Example 1 and relations (1)′ to (4)′ (for the elements with primes) analogous to (1) to (4) above. Let $c = a_0 b_0$ and $c' = a'_0 b'_0$.

It can be shown, using a proof similar to the proof of Example 1, that R is c-commutative. However, c is in the centre of R. We shall extend R to a ring T which is c-commutative and in which c is not central. Let $\sigma(d) = d'$ and $\sigma(d') = d$ for $d \in \{a_0, a_1, b_0, b_1\}$. σ is an automorphism of R. Form the twisted polynomial ring $T = R[t, \sigma]$ over R. That is, the additive group of T is the additive group of $R[t]$, and multiplication in T is defined by the rule $tf = \sigma(f)t$ for $f \in R$, and its consequences. Since $tc = c't \neq ct$, c is not in the centre of T. Furthermore, the polynomial ring $T[x]$ in one (commuting) indeterminate is not c-commutative. To see this note that by the relations on R: $c = (a_0 + a_1 x)(b_0 + b_1 x)$ but $c \neq (b_0 + b_1 x)(a_0 + a_1 x)$.

It remains to show that T is c-commutative. Let $f(t) = \sum_{i=0}^{m} f_i t^i$ and $g(t) = \sum_{j=0}^{n} g_j t^j$ be two elements of T such that $f(t)g(t) = c$. We show that $g(t)f(t) = c$ by considering various cases resulting from the equation $f_0 g_0 = c$ in R. We illustrate with one such case: $f_0 = c$ and $g_0 = 1 + g_{01} + g_{02}$ where g_{0i} is a form of degree i in R for $i = 1$ and 2. First we argue that each coefficient f_i of f has a zero constant term. Let E_k denote the equation resulting from equating the coefficient of t^k in $f(t)g(t)$ and the coefficient of t^k in c. Equation E_1 is: $0 = cg_1 + f_1(1 + \sigma(g_{01}) + \sigma(g_{02}))$. It follows that f_1 has zero constant term. By using equation E_i and induction on the subscript i we easily show that each f_i has a zero constant term. Using this fact and the equations, another induction will show that for $0 \leqslant i \leqslant m$, $f_i = \gamma c$ where $\gamma \in Z_2$. If $f_m \neq 0$ for $m > 0$ we can show, by considering equations $E_m, E_{m+1}, \ldots, E_{m+n}$ in reverse order, that each g_j has a zero constant term. In particular g_0 has a zero constant term. But since $g_0 = 1 + g_{01} + g_{02}$ this would be a contradiction. Thus $f(t) = c$, $c \in R$. Then for $1 \leqslant k \leqslant n$ equation E_k becomes

$cg_k = 0$. It follows that for $1 \leqslant k \leqslant n, g_k$ has a zero constant term and $g_k \sigma^l(c) = 0$, $l = 1, 2, \ldots, m$. So $g(t) f(t) = c$. This completes the proof for the first case. The other cases are similar.

Next we note that if R is 0-commutative and if $f(x)$ and $g(x)$ are linear polynomials in $R[x]$ then $f(x) g(x) = 0$ if and only if $g(x) f(x) = 0$. However, the following is an example of a 0-commutative ring R such that $R[x]$ is not 0-commutative. This answers a question raised by Chowdhury (1971).

EXAMPLE 3. A 0-commutative ring R such that $R[x]$ is not 0-commutative. Let $R = Z_2\{a_0, a_1, b_0, b_1, b_2\}$ subject to the relations:

(1) $a_0 b_0 = 0$ and $b_0 a_0 = 0$.
(2) $a_1 b_0 + a_0 b_1 = 0$.
(3) $a_1 b_1 + a_0 b_2 = 0$.
(4) $a_1 b_2 = 0$ and $b_2 a_1 = 0$.
(5) $(b_0 + b_1 + b_2)(a_0 + a_1) = 0$.
(6) All monomials of order greater than two are zero.

The proof is similar to the proof of Example 1 and is omitted. We note in passing that $Z_n\{a, b\}[x]$ is 0-commutative for all integers $n \geqslant 1$. We have one affirmative result. But first we state an easy lemma which is essentially contained in the well-known Noether normalization lemma.

LEMMA. *If $p(x, y)$ and $q(x, y)$ are elements of $R[x, y]$ then $p(x, y) = q(x, y)$ if and only if $p(x, x^k) = q(x, x^k)$ for sufficiently large k.*

THEOREM. *If $R[x]$ is $h(x)$-commutative for $h(x) \in R[x]$ then so is $R[x, y]$.*

PROOF. If $f(x, y) \cdot g(x, y) = h(x)$ then for all $k \geqslant 0$, $f(x, x^k) g(x, x^k) = h(x)$. So $g(x, x^k) f(x, x^k) = h(x)$ for all $k \geqslant 0$ so $g(x, y) f(x, y) = h(x)$ by the lemma.

COROLLARY. *If $R[x_1]$ is $h(x_1)$-commutative then so is $R[x_1, x_2, \ldots, x_n]$ for all integers $n \geqslant 1$.*

We now give examples to show that $R[[x]]$ may not be c-commutative even if $R[x]$ is. First note that if c is a noncentral element of R then, in general, nothing can be said about the c-commutativity of $R[x]$. However, we can say that $R[[x]]$ is not c-commutative. To see this choose $b \in R$, $b \neq 0$ such that $bc \neq cb$. Then

$$c = (1 + bx)(c - bcx + b^2 cx^2 - b^3 cx^3 + \ldots)$$

but

$$c \neq (c - bcx + b^2 cx^2 - b^3 cx^3 + \ldots)(1 + bx).$$

Note also that if R is c-commutative then c commutes with all units and all elements of the Jacobson radical of R (so with all nilpotents in R). Thus if R is local and c-commutative then c is in the centre of R.

EXAMPLE 4. A ring R with a noncentral element a such that $R[x]$ is a-commutative but $R[x]$ is not. Let $R = Z_2\{a,b\}$. No relations this time!

EXAMPLE 5. A ring R and a central zero divisor c in R such that $R[x]$ is c-commutative but $R[[x]]$ is not. Let $R = Z_4\{a_0, a_1, b_0, b_1, b_2, \ldots\}$ subject to the relations:

(0) $a_0 b_0 = 2$ and $b_0 a_0 = 2$,
(n) $a_0 b_n + a_1 b_{n-1} = 0$ for $n = 1, 2, 3, \ldots$, and
(∞) all monomials of order greater than two are zero.

$R[x]$ is 2-commutative but $R[[x]]$ is not since the relations imply

$$2 = (a_0 + a_1 x)(b_0 + b_1 x + b_2 x^2 + \ldots)$$

but $2 \neq (b_0 + b_1 x + b_2 x^2 + \ldots)(a_0 + a_1 x)$.

EXAMPLE 6. A ring R such that $R[x]$ is 0-commutative but $R[[x]]$ is not. Let $R = Z_2\{a_0, a_1, b_0, b_1, b_2, \ldots\}$ subject to the relations:

(0) $a_0 b_0 = 0$ and $b_0 a_0 = 0$,
(n) $a_0 b_n + a_1 b_{n-1} = 0$ for all $n \geqslant 1$, and
(∞) All monomials of order greater than two are zero.

The first named author received support from Samford University through its sabbatical year programme and through Samford University Research Grant No. 65.

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https://doi.org/10.1017/S1446788700016542 Published online by Cambridge University Press