Canad. Math. Bull. Vol. **43** (2), 2000 pp. 174–182

Stable Parabolic Bundles over Elliptic Surfaces and over Riemann Surfaces

Christian Gantz and Brian Steer

Abstract. We show that the use of orbifold bundles enables some questions to be reduced to the case of flat bundles. The identification of moduli spaces of certain parabolic bundles over elliptic surfaces is achieved using this method.

1 Introduction

If $q\colon Y \to \Sigma$ is an elliptic fibration ([27] and Section 2) the procedure of pulling back an orbifold or V-bundle from Σ induces a correspondence between the moduli space of stable bundles $\mathcal{E}' \to Y$ with $c(\mathcal{E}') = 1 + c_1(\mathcal{E}') + c_2(\mathcal{E}') \in q^*H^*([\Sigma])$ and that of stable V-bundles over Σ (Theorem 3.4). This has been shown by S. Bauer using algebraic geometry [3] and the purpose of this paper is to point out how the use of orbifold structures enables one to twist by a line V-bundle and so reduce to the flat case where the matter is taken care of by the (hard) theorems of S. K. Donaldson and of M. S. Narasimhan and C. S. Seshadri.

The result (and proof) extends to parabolic bundles. This is shown in Section 4 and Sections 5 and 6 show, using work of S. Bauer and P. B. Kronheimer and T. S. Mrowka, that the moduli spaces are complex manifolds which admit Kähler forms and determinant line bundles which coincide under the correspondence; a result certainly expected and "well-known" if not, it seems, in print. The paper outlines parts of [14] and the authors thank S. Bauer, P. B. Kronheimer and T. Peternell for helpful remarks.

2 Elliptic surfaces

Throughout, let $q\colon Y \to \Sigma$ be an elliptic surface, *i.e.*, Y is a compact complex surface, Σ a compact Riemann surface and $q^{-1}(\sigma)$ an elliptic curve for generic, *i.e.*, all but finitely many, $\sigma \in \Sigma$ [13], [2]. We assume that any non-generic fibre is either a rational curve of multiplicity one with one self-intersection (called a singular fibre) or a multiple elliptic curve and furthermore that there is at least one singular fibre. (Moishezon shows that all elliptic surfaces are deformation equivalent to these [21].)

Theorem 2.1 **([27], [9])** *If $U_0 \subseteq \Sigma$ is a ball and $\pi^{-1}(U_0)$ contains all singular fibres but no multiple elliptic curves then $\pi^{-1}(U_0)$ is simply connected.*

Received by the editors February 4, 1998; revised January 10, 2000.

The first author was fully supported by the Rhodes Trust, Oxford, and the Alfried Krupp von Bohlen und Halbach – Stiftung, Essen.

AMS subject classification: 14J27, 32L07, 14H60, 14D20.

If $Y_\sigma = q^{-1}(\sigma)$ has multiplicity $m > 1$, let $\tilde{U}$ and B be discs in $\mathbb{C}$, $\phi\colon B \to U$ a chart with $U \subseteq \Sigma$, $\phi(0) = \sigma$ and construct a *uniformization* of U by

$$\tilde{U} \xrightarrow{z^m} B \xrightarrow{\phi} U$$

where $\langle \eta = e^{2\pi i/m} \rangle = \mathbb{Z}_m \subseteq \mathbb{C}$ acts on $\tilde{U}$ in the standard way. We do this for all multiple points and then consider Σ as an orbifold. Here it suffices to think of orbifolds as manifolds together with local coverings branched along some co-dimension one submanifold. An *orbifold bundle* or *V-bundle* over U is an equivariant bundle

$$(\tilde{U} \times \mathbb{C}^r, \mathbb{Z}_m) \to (\tilde{U}, \mathbb{Z}_m),$$
$$\text{where} \quad \eta(\tilde{u}, v_1, \dots, v_r) = (\eta\tilde{u}, \eta^{a_1} v_1, \dots, \eta^{a_r} v_r)$$

for *isotropies* $1 \le a_i \le m$. A V-bundle over Σ consists of such local bundles and ordinary ones over non-multiple points which patch together. Connections in V-bundles are locally invariant ones which patch together.

Theorem 2.2 **(Seifert, [12])** *Smooth V-bundles over Σ are classified by rank, degree (which is rational) and isotropies.*

For any $y \in Y_\sigma$ in a fibre of multiplicity m we can choose coordinates (z_1, z_2) on $U' \ni y$ such that $\phi^{-1} \circ q(z_1, z_2) = z_2^m$. Hence $\phi^{-1} \circ q$ lifts locally to a regular map

$$\sqrt[m]{\phi^{-1} \circ q} = z_2 \colon U' \to \tilde{U}$$

uniquely up to the action of $\mathbb{Z}_m$. Now q is a map of orbifolds. Because it is now regular away from finitely many points, Theorem 2.1 implies that $q_*\colon \pi_1(Y) \to \pi_1^V(\Sigma)$ is an isomorphism, where the orbifold fundamental group $\pi_1^V(\Sigma)$ is an extension of $\pi_1(\Sigma)$ by unipotent elements corresponding to the multiple points. The correspondence between representations of the fundamental group and flat bundles extends to orbifolds. So the flat bundles over Y correspond to the flat V-bundles over Σ. It also follows that the first Betti number of Y is even and, by [20], that Y *is Kähler*.

A divisor on Σ is a finite sum $D = \sum_{i \in I} \sigma_i n_i / m_i$ where $n_i \in \mathbb{Z}$ and m_i is the multiplicity of $\sigma_i \in \Sigma$. The vertical divisors on Y are precisely the pull-backs of divisors on Σ. Hence, [24], the line bundles $\mathcal{O}(D')$ over Y with vertical divisor correspond to the line V-bundles over Σ.

3 V-Bundles

Let $P_0 \in \Sigma$ be generic with respect to q, let $P_0' = q^{-1}(P_0)$ and let m_0 be a positive integer. We form the orbifolds $\Sigma_V = (\Sigma, P_0, m_0)$ and $Y_V = (Y, P_0', m_0)$ by adding local coverings, branched along P_0 and P_0':

$$\pi\colon \tilde{U} \to B$$
$$z \mapsto z^{m_0}$$

https://doi.org/10.4153/CMB-2000-025-9 Published online by Cambridge University Press

where $\phi\colon B \to U$ is a chart centered at P_0 and, for a chart $\phi'\colon V \times B \to U'$ of Y, where V is a disc and $\phi'(V \times \{0\}) = P'_0 \cap U'$, we take

$$\begin{aligned} V \times \tilde{U} &\to V \times B \\ (z_1, z_2) &\mapsto (z_1, z_2^{m_0}). \end{aligned}$$

The group $\mathbb{Z}_{m_0}$ acts both on $\tilde{U}$ and on $V \times \tilde{U}$. A V-bundle over Σ is also a V-bundle over Σ_V and similarly for Y_V. To speak about degrees and anti-self-duality we need an *orbifold Kähler form* on Y_V, *i.e.*, a positive, closed $(1,1)$-form on $Y \setminus P'_0$ which extend smoothly to a positive form on the local coverings.

Proposition 3.1 **([17, Appendix 2])** *For each Kähler form ω' on Y there exists an orbifold Kähler form ω'_V on Y_V such that $[\omega'_V] = [\omega'] \in \mathrm{H}^2(Y, \mathbb{R})$.*

Proposition 3.2 *If $\mathcal{E}' \to Y$ is stable then so is $\mathcal{E}' \to Y_V$, i.e., for all $\mathcal{F}' \to Y_V$ with $\operatorname{rank} \mathcal{F}' < \operatorname{rank} \mathcal{E}'$ and holomorphic $f\colon \mathcal{F}' \to \mathcal{E}'$, injective over some point, we have $\deg_V \mathcal{F}' / \operatorname{rank} \mathcal{F}' < \deg \mathcal{E}' / \operatorname{rank} \mathcal{E}'$; similarly for bundles over Σ.*

Proof There are trivialising sections $s_i\colon V \times \tilde{U} \to \mathcal{F}'|_{V \times \tilde{U}}$, $i = 1, \dots, r' = \operatorname{rank} \mathcal{F}'$, with $\eta \circ s_i = \eta^{a(i)} s_i(z_1, \eta z_2)$ for $a(i) \in \{1, \dots, m_0\}$. Similarly trivialising $\mathcal{E}'|_{V \times \tilde{U}}$, with trivial isotropies, write $f^j\colon \mathcal{F}'|_{V \times \tilde{U}} \to \mathbb{C}$ for the j-th component of f. Then

$$f^j\big(s_i(z_1, \eta z_2)\big) = \eta^{-a(i)} f^j\big(\eta \circ s_i(z_1, z_2)\big) = \eta^{-a(i)} f^j s_i(z_1, z_2)$$

since f is locally equivariant. Hence, $f^j \circ s_i\colon V \times \tilde{U} \to \mathbb{C}$ vanishes to order at least $m_0 - a(i)$ along $V \times \{0\}$. We define the sheaf $\mathcal{F}'_\infty|_{U'}$ to consist of sections

$$s = \sum_{i=1}^{r'} g_i z_2^{a(i) - m_0} s_i \in \Omega^0(\mathcal{F}'|_{V \times \tilde{U}})$$

where $g_i\colon V \times \tilde{U} \to \mathbb{C}$ are holomorphic and invariant, *i.e.*, holomorphic functions on U'. Hence $\mathcal{F}'_\infty|_{U'}$ is a genuine vector bundle over U'. It can be glued into $\mathcal{F}'|_{Y \setminus P'_0}$ to form a global genuine bundle $\mathcal{F}'_\infty \to Y$. Clearly, f induces a holomorphic map $f_\infty\colon \mathcal{F}'_\infty \to \mathcal{E}'$. Unless $\mathcal{F}'$ had trivial isotropies, f is injective over some point in $Y \setminus P'_0$ and hence f_∞ is. Also,

$$\deg \mathcal{F}'_\infty = \deg_V \mathcal{F}' + \langle \omega'_V, P'_0 \rangle \sum_{i=1}^{m_0} \big(m_0 - a(i)\big) \geq \deg_V \mathcal{F}'. \qquad \blacksquare$$

Lemma 3.3 *If $E' \to Y_V$ has $\mathrm{c}_2(E') - \frac{1}{2}\mathrm{c}_1^2(E') = 0$ then any ASD connection is flat.*

Proof $0 = \langle 8\pi^2\big(\mathrm{c}_2(E') - \frac{1}{2}\mathrm{c}_1^2(E')\big), Y_V \rangle = \int_{Y_V} \operatorname{Tr}(F^2) = \|F^-\|^2 - \|F^+\|^2.$ $\blacksquare$

Theorem 3.4

(i) *For arbitrary fixed Kähler form ω' on Y, pulling back gives a correspondence between stable V-bundles $\mathcal{E} \to \Sigma$ and those stable bundles $\mathcal{E}' \to Y$ with $\mathrm{c}(\mathcal{E}') \in q^*\mathrm{H}^*(\Sigma, \mathbb{Q})$.*

(ii) *If $P = (P_1, \ldots, P_n) \subset \Sigma$ are generic, $P' = q^{-1}(P)$ and $(m_1, \ldots, m_n) \in \mathbb{Z}^n_{>0}$ and if Σ_V and Y_V denote the corresponding orbifolds then the stable V-bundles $\mathcal{E} \to \Sigma_V$ corresponds to the stable V-bundles $\mathcal{E}' \to Y_V$ with $\mathrm{c}(\mathcal{E}') \in q^*\mathrm{H}^*(\Sigma, \mathbb{Q})$.*

Hence, stability of $\mathcal{E}'$ is independent of the choice of ω' if $\mathrm{c}(\mathcal{E}') \in q^*\mathrm{H}^*(\Sigma, \mathbb{Q})$. Part (i) of this theorem has been shown, under some conditions on ω' and with the additional condition $\det \mathcal{E}' \in q^*\mathrm{Pic}_V(\Sigma)$, by Bauer [3].

Proof (i) Let $\mathcal{E}' \to Y$ be stable with respect to ω' and suppose that $\mathrm{c}_2(\mathcal{E}') = 0$ and $\mathrm{c}_1(\mathcal{E}') = q^*\theta$ for some $\theta \in \mathrm{H}^2(\Sigma, \mathbb{Q})$. Choose $l \in \mathbb{Z}$ and $m_0 \in \mathbb{Z}_{>0}$ such that $\frac{rl}{m_0} = \langle \theta, \Sigma \rangle$ and construct $\Sigma_V = (\Sigma, P_0, m_0)$ and $Y_V = (Y, P_0', m_0)$ as before. Consider the holomorphic line V-bundles $\mathcal{L} = [-\frac{l}{m_0}P_0] \to \Sigma_V$ and $\mathcal{L}' = q^*\mathcal{L} = [-\frac{l}{m_0}P_0'] \to Y_V$.

By Proposition 3.2, $\mathcal{E}' \otimes \mathcal{L}'$ is stable. Now $m(\sigma)\langle Y_\sigma, \omega' \rangle = \kappa$ is constant, where $m(\sigma)$ is the multiplicity of $Y_\sigma \subseteq Y$. So Poincaré duality implies that

$$\deg_V(\mathcal{E}' \otimes \mathcal{L}') = \langle \big(\mathrm{c}_1(\mathcal{E}') + r\mathrm{c}_1(\mathcal{L}')\big) \cup \omega', Y \rangle = \frac{rl}{m_0}\kappa - r\frac{l}{m_0}\kappa = 0.$$

After Donaldson [10], and generalisations [17], [26], $\mathcal{E}' \otimes \mathcal{L}'$ admits a unique irreducible, hermitian ASD connection. Now $\mathrm{c}(\mathcal{E}' \otimes \mathcal{L}') \in q^*\mathrm{H}^*(\Sigma)$ and by Lemma 3.3 this ASD connection is flat.

The map $q_*\colon \pi_1^V(Y_V) \to \pi_1^V(\Sigma)$ is again an isomorphism by Theorem 2.1. Hence $\mathcal{E}' \otimes \mathcal{L}' = q^*\mathcal{F}$ for unique flat, irreducible $\mathcal{F} \to \Sigma_V$ which is stable, by the theorem of Narasimhan & Seshadri [23], and generalisations [12], [8]. Since $\mathcal{E}'$ has trivial isotropies along P_0', so has $\mathcal{E}$ over P_0 and is therefore a V-bundle over Σ.

(ii) The proof is entirely similar. ■

4 Parabolic Bundles

Let $P = \{P_1, \ldots, P_n\} \subset \Sigma$ be generic and $P' = \{P_j' = q^{-1}(P_j)\}_{j=1,\ldots,n} \subseteq Y$. A *weighted* bundle $E' \to (Y, P')$ is a bundle $E' \to Y$ with proper *filtrations* and *weights*

$$E'|_{P_j'} = E'_{j,1} \supset E'_{j,2} \supset \cdots \supset E'_{j,l_j} \neq 0,$$
$$0 \leq \alpha'_{j,1} < \alpha'_{j,2} < \cdots < \alpha'_{j,l_j} < 1$$

for $j = 1, \ldots, n$. We call $\mu'_{j,k} = \mathrm{rank}(E'_{j,k}/E'_{j,k+1})$ the *multiplicity* of $\alpha'_{j,k}$. Let $\alpha_j' = \mathrm{diag}(\alpha'_{j,k})$ be the diagonal matrix of rank r with entries $\alpha'_{j,k}$ with multiplicities. If bundle and filtrations are holomorphic then we speak of a *parabolic bundle*. For $E \to (\Sigma, P)$ the $\{E_{j,k}\}$ are filtrations of $E|_{P_j}$ with weights $\{\alpha_{j,k}\}$ and multiplicities $\{\mu_{j,k}\}$. Thinking of vectors in $E|_P$ as having the obvious weights ($+\infty$ the weight of zero vectors), a *morphism* is a bundle map never decreasing weights.

https://doi.org/10.4153/CMB-2000-025-9 Published online by Cambridge University Press

When we use bundle operators (like c_i, det or deg) on a weighted bundle E we mean them to be applied to the underlying bundle $|E|$. By definition,

$$(*)\qquad \operatorname{par} c_1(E') = c_1(E') + \sum_{j=1}^{n} \operatorname{Tr}(\alpha'_j)\operatorname{PD}(P'_j) \in H^2(Y,\mathbb{R}).$$

Similarly $\operatorname{par} c_1(E) \in H^2(\Sigma,\mathbb{R})$. Set $\operatorname{par}\deg E = \langle \operatorname{par} c_1(E), \Sigma\rangle \in \mathbb{R}$ and for given ω' let $\operatorname{par}\deg E' = \langle \operatorname{par} c_1(E') \cup \omega', Y\rangle \in \mathbb{R}$. Since $P \cdot P = 0$ in the group $H^4(Y,\mathbb{R})$ we have

$$(**)\qquad \begin{aligned} \operatorname{par} c_2(E') &= \frac{1}{2}\operatorname{par} c_1^2(E') + \left(c_2(E') - \frac{1}{2}c_1^2(E')\right) \\ &\quad - \sum_{j=1}^{n}\sum_{k=1}^{l_j} \alpha'_{j,k}\operatorname{PD}\big(\deg(E'_{j,k}/E'_{j,k+1})\big). \end{aligned}$$

Definition 4.1 A parabolic bundle $\mathcal{E}' \to (Y,P')$ is called *stable* if, for all non-zero parabolic maps $\mathcal{F}' \to \mathcal{E}'$ injective over some point and with $\operatorname{rank}\mathcal{F}' < \operatorname{rank}\mathcal{E}'$, we have $\operatorname{par}\deg\mathcal{F}'/\operatorname{rank}\mathcal{F}' < \operatorname{par}\deg\mathcal{E}'/\operatorname{rank}\mathcal{E}'$. Similarly for $\mathcal{E} \to (\Sigma, P')$.

Theorem 4.2 *Pulling back induces a correspondence between stable parabolic V-bundles* $\mathcal{E} \to (\Sigma, P)$ *and those stable parabolic bundles* $\mathcal{E}' \to (Y,P')$ *with* $\operatorname{par} c(\mathcal{E}') = 1 + \operatorname{par} c_1(\mathcal{E}') + \operatorname{par} c_2(\mathcal{E}') \in q^* H^*(\Sigma,\mathbb{R})$.

We recover (i) of Theorem 3.4 if there is no parabolic structure.

Definition 4.3 A *weighted* hermitian metric h' on $E' \to (Y,P')$ is a metric on $E'|_{Y-P'}$ such that for any $y \in P'_j$ and coordinate z which is normal to P'_j there exist smooth trivialising sections near y, respecting the parabolic filtration of $E'|_{P'_j}$, such that $h = |z|^{2\alpha'_j}$ as hermitian matrix.

The weighted metric h' induces a Chern connection on $\mathcal{E}'|_{Y\setminus P'}$, with ∂-part extending smoothly over P' only if α vanishes. A weighted connection is *reducible* if the weighted bundle decomposes invariantly under the connection.

Theorem 4.4 (**[6]**, **[10]**, **[22]**, **[26]**) *A degree zero parabolic bundle over a Kähler surface is stable if and only if it admits a unique irreducible weighted ASD metric.*

Theorem 4.5 (**[19]**, **[5]**, **[23]**, **[24]**) *A parabolic V-bundle* $\mathcal{E}$ *over a compact orbifold curve* Σ*, parabolic only on a set* $P = \{p_1, \ldots, p_n\}$ *of generic points, is stable if and only if it admits an irreducible weighted Yang-Mills metric; that is, an irreducible projectively flat weighted metric.*

To define what is meant by Yang-Mills a Kähler (orbifold) metric has to be taken on Σ, but the assertion holds for any choice. It is straightforward to see that if $\operatorname{par}\deg\mathcal{E} = 0$ such a metric determines an irreducible representation of $\pi_1^V(\Sigma - P)$ into $SU(n)$- and conversely an irreducible representation determines a stable bundle of parabolic degree 0.

https://doi.org/10.4153/CMB-2000-025-9 Published online by Cambridge University Press

Lemma 4.6

(i) *[14], [22] If* $\operatorname{par} c_2(\mathcal{E}') - \frac{1}{2} \operatorname{par} c_1^2(\mathcal{E}') = 0$ *for the parabolic bundle* $\mathcal{E}' \to (Y, P')$ *then any ASD connection is flat.*

(ii) *[19], [14] A flat hermitian bundle over* $Y \setminus P'$ *(respectively* $\Sigma \setminus P$*) extends uniquely to a parabolic hermitian bundle over* Y *(respectively* Σ*).*

(iii) *[14] Let* $P_0 \in \Sigma \setminus P$ *be generic and* $P_0' = q^{-1}(P_0)$*. If* $\mathcal{E}' \to (Y, P')$ *is stable parabolic then so is* $\mathcal{E}' \to (Y, P' \cup P_0')$*. Similarly in the case of bundles over* Σ*.*

(iv) $q^*\colon \pi_1(Y \setminus P') \to \pi_1^V(\Sigma \setminus P)$ *is an isomorphism.*

Parts (i), (iii) and (iv) are proved as for V-bundles, and (ii) is not very difficult. Theorem 4.2 is now proved along the same lines as Theorem 3.4.

5 Moduli Spaces

Let $\mathcal{E}$ be a stable parabolic V-bundle over (Σ, P) or a stable V-bundle over $\Sigma_V = (\Sigma, P, m_1, \ldots, m_n)$. Let $\mathcal{T}$ be the V-bundle underlying $\operatorname{Par}\operatorname{End}_0 \mathcal{E}$ if $\mathcal{E}$ is parabolic (*i.e.*, forget the weights). If $\mathcal{E}$ is a V-bundle then transform its V-structure at the points P into a parabolic structure first [12]. Let $\mathcal{E}' = q^*\mathcal{E}$ and $\mathcal{T}' = q^*\mathcal{T}$. Now $\mathcal{T}$ and $\mathcal{T}'$ are holomorphic (V-)bundles. We have the deformation complexes of smooth sections

$$\begin{array}{ccccc}
\Omega^0_\Sigma(\mathcal{T}) & \xrightarrow{\bar\partial_{\mathcal{T}}} & \Omega^{0,1}_\Sigma(\mathcal{T}) & \xrightarrow{\bar\partial_{\mathcal{T}}} & 0 \\
\downarrow & & \downarrow{\scriptstyle q^*} & & \downarrow \\
\Omega^0_Y(\mathcal{T}') & \xrightarrow{\bar\partial_{\mathcal{T}'}} & \Omega^{0,1}_Y(\mathcal{T}') & \xrightarrow{\bar\partial_{\mathcal{T}'}} & \Omega^{0,2}_Y(\mathcal{T}').
\end{array}$$

Let D be a divisor on Σ with $\mathcal{O}(D) = \det \mathcal{E}$, E the underlying smooth (weighted) V-bundle of $\mathcal{E}$ and $\mathcal{M}(E, D)$ the space of stable structures on E with determinant $\mathcal{O}(D)$. Similarly, define $\mathcal{M}(E', D')$ for $E' = q^*E$ and $D' = q^*D$. Applying the extension of standard deformation theory to parabolic bundles makes $\mathcal{M}(E', D')$ into a Hausdorff complex space [17, Proposition 8.23], [22], with a description near $\mathcal{E}'$ given by the zero set of a holomorphic map

$$H^{0,1}_{\bar\partial}(Y, \mathcal{T}') \to H^{0,2}_{\bar\partial}(Y, \mathcal{T}').$$

Because $H^{0,2}(\Sigma, \mathcal{T}) = 0$, $\mathcal{M}(E, D)$ is a complex manifold with a chart near $\mathcal{E}$ given by $H^{0,1}_{\bar\partial}(\Sigma, \mathcal{T})$. Also, $\mathcal{M}(E, D)$ is connected and we let $\mathcal{M}_{\mathcal{E}'}(E', D')$ be the connected component of $\mathcal{E}'$.

Theorem 5.1 $q^*\colon \mathcal{M}(E, D) \to \mathcal{M}_{\mathcal{E}'}(E', D')$ *is an isomorphism of complex manifolds.*

Proof Bauer's theorem [3, Cor. 1.6] generalises and $q^*\colon H^{0,1}_{\bar\partial}(\Sigma, \mathcal{T}) \to H^{0,1}_{\bar\partial}(Y, \mathcal{T}')$ is an isomorphism [14]. Theorem 4.2 or Theorem 3.4 finishes the proof. ■

Proposition 5.2 *A smooth line V-bundle* $\pi\colon L \to \Sigma$ *is trivial if* $\pi'\colon L' = q^*L \to Y$ *is trivial.*

https://doi.org/10.4153/CMB-2000-025-9 Published online by Cambridge University Press

Proof Write SL for the circle bundle associated to the complex line bundle L and let $q'\colon \mathrm{SL}' \to \mathrm{SL}$ for the induced map of circle bundles. We have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K' & \longrightarrow & \pi_1(\mathrm{SL}') & \xrightarrow{\pi'_*} & \pi_1(Y) & \longrightarrow & 0 \\
\downarrow & & \downarrow \hat{q} & & \downarrow q'_* & & \downarrow q_* & & \downarrow \\
0 & \longrightarrow & K & \longrightarrow & \pi_1^V(\mathrm{SL}) & \xrightarrow{\pi_*} & \pi_1^V(\Sigma) & \longrightarrow & 0,
\end{array}$$

where $K = \langle k \rangle$, $K' = \langle k' \rangle$ for generic fibres $k = \hat{q}(k')$ of SL and k' of SL'. Since q_* is an isomorphism and the rows exact, q'_* is an isomorphism if $\hat{q}$ is one.

Assume there is a V-homotopy $H\colon [0,1] \times [0,1] \to \mathrm{SL}$ with boundary k^n; that is, a continuous family $h_t \in \Omega_V(\mathrm{SL})$, $0 \le t \le 1$, where $\Omega_V(\mathrm{SL})$ is the orbifold loop space [12, p. 50] and $h_t(s) = H(t,s)$. Since q is regular away from finitely many points, there exists $H'\colon [0,1] \times [0,1] \to Y$ lifting $\pi \circ H$. Hence there exists $\tilde{H}\colon [0,1] \times [0,1] \to \mathrm{SL}'$ lifting H and H'. If $k \in S(L_x)$ then $\partial H' = \pi' \circ \partial \tilde{H} \subseteq Y_x$ and $\partial H = q' \circ \partial \tilde{H} = k^n$. We may suppose that $x \in U_0$, where U_0 is as in Theorem 2.1, and we are working on $\mathrm{SL}'|_{q^{-1}(U_0)} = S^1 \times q^{-1}(U_0)$. We can lift some homotopy in $q^{-1}(U_0)$ with boundary $\partial H'$ to one contracting $\partial \tilde{H}$ into one fibre of SL'. This must be $(k')^n$. Hence q'_* is an isomorphism.

Seifert proved that

$$\pi_1^V(\mathrm{SL}) = \langle a_j, b_j, g_i, k : [a_j, k] = [b_j, k] = [g_i, k] = 1 = g_i^{m_i} k^{\beta_i} = k^{-b} \prod_1^g [a_j, b_j] \prod_1^n g_i \rangle,$$

for genus g and m_i the multiplicity of σ_i. The isotropy $\beta_i \mod m_i$ at σ_i and $\deg L = b + \sum_1^n \beta_i / m_i$ are independent of the choices of lifts g_i, a_j and b_j of the generators of $\pi_1^V(\Sigma)$. Since SL' is trivial, there are lifts such that all β_i and b are zero. By Theorem 2.2, L is trivial. ■

Recall the definitions (*) and (**) of par $c_i(E^*)$ in Section 4.

Theorem 5.3 *If $E' \to (Y, P')$ is weighted, if* $\operatorname{par} \mathrm{c}(E') \in q^* \mathrm{H}^*(\Sigma, \mathbb{R})$ *and if the set $\mathcal{M}(E')$ of stable structures is non-empty then there exists a unique line V-bundle $L \to \Sigma$ with $q^* L = \det E' = \det |E'|$. Conversely, E' is determined by its parabolic weights and L. Hence, for weighted $E \to \Sigma$ we have $q^* E = E'$ if and only if the weights coincide and $L = \det E$. So $\mathcal{M}(E') \simeq \bigsqcup_E \mathcal{M}(E)$ where the union is over such E.*

Proof By Theorem 4.2, there exists a weighted E with $q^*(E) = E'$. Hence $L = \det E$, which is unique by Proposition 5.2. Now, $|E'|$ is determined by L since $\mathrm{c}_2(|E'|) = 0$. The parabolic filtrations of $E' = q^*(E)$ are by trivial bundles. Two different such filtrations of $|E'|_{P'_j}$ are related by a map $P'_j \to \mathrm{Sl}(r, \mathbb{C})$ which can be extended to an isomorphism of $|E'|$ being the identity outside a tubular neighbourhood of P'_j since $\mathrm{Sl}(r, \mathbb{C})$ is simply connected. ■

https://doi.org/10.4153/CMB-2000-025-9 Published online by Cambridge University Press

6 Determinant Line Bundles

Now let $\mathcal{E} \to \Sigma_V = (\Sigma, P, m_1, \dots, m_n)$ be a stable V-bundle of rank r. As in the genuine case, we have a Kähler form Ω_V on $\mathcal{M}(E, D)$ induced by

$$\tilde{\Omega}_V(a, b) = \frac{1}{4\pi^2} \int_{\Sigma_V} \mathrm{Tr}\{a^* b - b^* a\}$$

for $a, b \in \Omega^{0,1}_{\Sigma_V}(\mathrm{End}_0 E)$. Here, the adjoint is with respect to a metric h on E and conjugation on forms. Let m denote the least common multiple of the orders of *all* marked points in Σ_V. Choose ω'_V on $Y_V = (Y, P', m_1, \dots, m_n)$ such that $\int_{Y_\sigma} \omega'_V = m$ for generic $\sigma \in \Sigma$. We have the Kähler form Ω'_V on $\mathcal{M}_{\mathcal{E}'}(E', D')$ induced by

$$\tilde{\Omega}'_V(a', b') = \frac{1}{4\pi^2} \int_{Y_V} \mathrm{Tr}\{(a')^{*'} b' - (b')^{*'} a'\} \wedge \omega'_V$$

for $a', b' \in \Omega^{0,1}_{Y_V}(\mathrm{End}_0 E')$ where adjoints are here with respect to $h' = q^* h$. If Y has no orbifold structure and if ω' is Poincaré dual to an imbedded Riemann surface then Donaldson & Kronheimer construct a line bundle $\mathcal{L}'_{DK} \to \mathcal{M}_{\mathcal{E}'}(E', D')$ with Chern form $r\Omega'_V$ [11, p. 252].

Theorem 6.1

(i) $\Omega'_V = mq^* \Omega_V$.
(ii) *There exists a hermitian holomorphic line bundle $\mathcal{L}' \to \mathcal{M}_{\mathcal{E}'}(E', D')$ with Chern form $r\Omega'_V$.*
(iii) *The bundle $\mathcal{L}' = \mathcal{L}'_{DK}$ if $\mathcal{M}(E, D)$ is simply connected and if both T and ω' are as above.*

If no proper smooth subbundle of E is of equal slope then $\mathcal{M}(E, D) = \mathcal{M}_{\mathcal{E}'}(E', D')$ is simply connected [12]. This generalises Atiyah & Bott's case of coprime rank and degree. In case (iii), ω' is integral and hence $\int_{Y_\sigma} \omega'_V = m$ for generic σ. (Note that in [11], where Y can be any projective surface, the line bundle over $\mathcal{A}_Y$ needs to be tensored r times to descend.)

Proof (i)

$$\begin{aligned}\tilde{\Omega}'_V(q^* a, q^* b) &= \frac{1}{4\pi^2} \int_{Y_V} \mathrm{Tr}\{(q^* a)^{*'} (q^* b) - (q^* b)^{*'} (q^* a)\} \wedge \omega'_V \\ &= \frac{1}{4\pi^2} \int_{Y_V} q^* \mathrm{Tr}\{a^* b - b^* a\} \wedge \omega'_V = \tilde{\Omega}_V(a, b) \int_{Y_\sigma} \omega'_V = m \tilde{\Omega}_V(a, b).\end{aligned}$$

(ii) Biswas & Raghavendra and, for rank two, Konno construct a hermitian holomorphic line bundle $\mathcal{L} \to \mathcal{M}(E, D)$ with Chern form $mr\Omega_V$.

(iii) If $\mathcal{M}(E, D)$ is simply-connected then the Chern form uniquely determines the hermitian holomorphic line bundle. ∎

https://doi.org/10.4153/CMB-2000-025-9 Published online by Cambridge University Press

References

[1] M. F. Atiyah and R. Bott, *The Yang-Mills equation over Riemann surfaces.* Philos. Trans. Roy. Soc. London A **308**(1982), 523-0615.
[2] W. Barth, C. Peters and A. Van de Ven, *Compact complex surfaces.* Springer, Berlin-Heidelberg-New York, 1984.
[3] S. Bauer, *Parabolic bundles, elliptic surfaces and* SU(2)*-representation spaces of genus zero Fuchsian groups.* Math. Ann. **290**(1991), 509–526.
[4] S. Bauer and C. Okonek, *The algebraic geometry of representation spaces associated to Seifert fibred homology 3-spheres.* Math. Ann. **286**(1990), 45–76.
[5] O. Biquard, *Fibrés paraboliques stables et connexions singulières plates.* Bull. Soc. Math. France **119**(1991), 231–257.
[6] ———, *On parabolic bundles over a complex surface.* J. London Math. Soc. **53**(1996), 302–316.
[7] I. Biswas and N. Raghavendra, *Determinants of parabolic bundles on Riemann surfaces.* Proc. Indian Acad. Sci. Math. (1) **103**(1993), 41–71.
[8] H. U. Boden, *Representations of orbifold groups and parabolic bundles.* Comment. Math. Helv. **66**(1991), 389–447.
[9] I. Dolgachev, *Algebraic surfaces with* $p_g = q = 0$. In: Algebraic surfaces CIME 1977, Liguori, Napoli, 1981, 97–215.
[10] S. K. Donaldson, *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles.* Proc. London Math. Soc. **50**(1985), 1–26.
[11] S. K. Donaldson and P. B. Kronheimer, *The Geometry of Four-Manifolds.* Clarendon Press, Oxford, 1990.
[12] M. Furuta and B. Steer, *Seifert fibred homology 3-spheres and the Yang-Mills equations on Riemann surfaces with marked points.* Adv. Math. **96**(1992), 38–102.
[13] P. Griffith and J. Harris, *Principles of Algebraic Geometry.* John Wiley & Sons, New York, 1978.
[14] C. Gantz, *On parabolic bundles.* D.Phil. thesis, Oxford University, 1996.
[15] H. Konno, *On the natural line bundle on the moduli space of stable parabolic bundles.* Commun. Math. Phys. **155**(1993), 311–324.
[16] P. B. Kronheimer and T. S. Mrowka, *Gauge theory for embedded surfaces, I.* Topology **32**(1993), 773–826.
[17] ———, *Gauge theory for embedded surfaces, II.* Topology **34**(1995), 37–98.
[18] M. Lübke, *Stability of Einstein-Hermitian vector bundles.* Manuscripta Math. **42**(1983), 245–257.
[19] V. B. Mehta and C. S. Seshadri, *Moduli of vector bundles on curves with parabolic structures.* Math. Ann. **248**(1980), 205–239.
[20] Y. Miyaoka, *Kähler metrics on elliptic surfaces.* Proc. Japan Acad. **50**(1974), 533–536.
[21] B. Moishezon, *Complex surfaces and connected sums of complex projective planes.* Lecture Notes in Math. **603**, Springer, Berlin, 1977.
[22] A. Munari, *Singular instantons and parabolic bundles over complex surfaces.* D.Phil. thesis, Oxford University, 1993.
[23] M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on compact Riemann surfaces.* Ann. Math. **82**(1963), 540–567.
[24] E. B. Nasatyr and B. Steer, *The Narasimhan-Seshadri theorem for parabolic bundles: an orbifold approach.* Philos. Trans. Roy. Soc. London A **353**(1995), 137–171.
[25] J. A. Poritz, *Parabolic vector bundles and Hermitian-Yang-Mills connections over a Riemann surface.* Internat. J. Math. **4**(1993), 467–501.
[26] B. Steer and A. Wren, *The Donaldson-Hitchin-Kobayashi correspondence for parabolic bundles over orbifold surfaces.* Preprint, Oxford, 1992.
[27] M. Ue, *On the diffeomorphism types of elliptic surfaces with multiple fibres.* Invent. Math. **84**(1986), 633–643.

Mathematical Institute
Oxford OX1 3LB
UK