1

Introduction

The feeblest of the four fundamental interactions governing the natural world is gravitation.[1]

The General Theory of Relativity[2] (GTR) is the formulation of gravitation set out by Albert Einstein in 1915 (Einstein, 1915c,d,a) and completed one year later (Einstein, 1916). It is the simplest possible gravitational theory compatible with his Special Theory of Relativity (STR) (Einstein, 1905). For contemporary comprehensive expositions of GTR, see, for example, Fok (1959), Synge (1960), Weinberg (1972), Hawking and Ellis (1973), Wald (1984), Stephani (1990), Cheng (2009), Padnanabhan (2010), Ohanian and Ruffini (2013), Zee (2013), Misner et al. (2017), Carroll (2019), Thorne and Blandford (2021), Schutz (2022), and Kenyon (2023). Some recent review articles, which appeared in the literature on the occasion of its last centenary, are, for example, Blandford (2015), Iorio (2015a), and Debono and Smoot (2016).

The time-honoured Law of Universal Gravitation proposed by Isaac Newton at the end of the seventeenth century in his immortal book *Philosophiæ Naturalis Principia Mathematica* (Newton, 1687; Chandrasekhar, 1995) describes it by means of a mysterious – remarkably, for Newton himself – force acting instantaneously between two or more material bodies, even if mutually separated in empty space by distances r much larger than their characteristic sizes D; as such, it benefits from the properties of the forces established by the three Newtonian laws of dynamics.

Instead, GTR adopts a completely different conceptual framework. According to it, gravitation is no longer best understood as a force, being, instead, a manifestation of the curvature, in a very specific sense, of a four-dimensional pseudo-Riemannian

[1] From the adjective *grăvis, e* ('heavy') and the noun *grăvĭtas, ātis*, ('weight, heaviness').
[2] From *Allgemeine Relativitätstheorie*.

Lorentzian manifold[3] known as spacetime (Oloff, 2023) with respect to the so-called 'flat' version of the spacetime employed by STR. Stated differently, the Einsteinian picture replaced the Newtonian concept of gravitational force with the notion of deformation of the chronogeometric[4] structure of spacetime (Damour, 2007) due to *all* forms of energy weighing it; as such, GTR can be defined as a chronogeometrodynamic theory of gravitation (Torretti, 1991). Indeed, the weight force on the Earth, which Newton unified with the agent determining the course of the heavens in the framework of his Universal Gravitation, is just an illusion due to the fact that we are born, live continuously, and die on the surface of our planet.[5] Actually, what we perceive as weight is not due to gravitation, but to the reaction force, of non-gravitational nature, exerted on our bodies by any physical surface we rest on; a chair, a floor, a bed. What kills us when we fall from a building is not gravity, but the non-gravitational reaction force by the ground. Indeed, if we are in free fall, that is, if we move subjected only to gravity and no[6] forces act on us, all the different parts of our body proceed with the same acceleration,[7] and we are not torn apart as would occur if gravity acted differently on bodies of diverse composition. Thus, as long as the regime of free fall continues, we are weightless, and the gravity seems to have been cancelled in our neighbourhood; for us, all things go as predicted by STR, we would obtain always the value of c in any experiment aimed at measuring the speed of light, and the worldlines of non-interacting, electrically neutral material objects appear as just straight in our freely falling experimental setup. It can be said that we are in a *local* (in both the spatial and temporal sense) inertial reference frame. It is one aspect of the so-called Equivalence Principle (EP).[8] In fact, such a removal of gravitation is not exact, being dictated by how

[3] According to differential geometry, a differentiable manifold is said to be pseudo-Riemannian (Benn and Tucker, 1987; Bishop and Goldberg, 1980) if it is endowed with a metric tensor that is everywhere nondegenerate, thus relaxing the requirement of positive-definiteness characterizing the Riemannian manifolds. A n_{d}-dimensional Lorentzian manifold is a special case of a pseudo-Riemannian manifold whose metric signature is $(1, n_{\mathrm{d}} - 1)$.

[4] From Χρόνος, 'Chronos', the personification of Time, not to be confused with Κρόνος, 'Kronos', the Titan father of Zeus, corresponding also to the Roman deity Saturn.

[5] From πλανήτης, -ου, ὁ, meaning 'wanderer', composed by the verb πλανάω ('I wander') and the masculine agent noun suffix -της.

[6] If gravity were a force, here one would have to prefix the adjective 'other' to the word 'forces'.

[7] The tale according to which Galilei experimentally proved it by dropping objects of different weights from the Leaning Tower of Pisa (Drake, 1978) is, in all likelihood, apocryphal (Adler and Coulter, 1978; Segre, 1989; Crease, 2006).

[8] So far, one has only talked about bodies whose self-gravity is negligible in holding their constituent parts together, and whose free fall is not affected by their reciprocal gravitational interaction. Such a weak version of the EP (Nobili and Anselmi, 2018) has been recently tested to a relative accuracy of $\simeq 10^{-15}$ (Touboul et al., 2022a) in the spaceborne experiment Micro-Satellite à traînée Compensée pour l'Observation du Principe d'Equivalence (MicroSCOPE) (Touboul et al., 2022b) with two objects made of platinum and titanium alloys, respectively, kept in free fall around the Earth inside a spacecraft which shielded them from any potentially disturbing non-gravitational influences. As shown by analyses of the motions of the Earth and the Moon in the field of the Sun with the Lunar Laser Ranging (LLR) technique (Williams et al., 2012; Müller et al., 2019; Biskupek et al., 2021) and, more recently, of the binary pulsar-white dwarf PSR

uniform the gravitational field is on the scale of our body and of the things that free fall in our vicinity along with us. The more uniform the field is, or the smaller our neighbourhood is, the more accurate the absence of gravity is. In any case, free falling non-interacting objects left to themselves will sooner or later move, more or less rapidly, towards or apart from each other because of the unavoidable non-uniformity of the gravitational field in which they all fall together. That is *not* an illusion, and there is *no* way of *wholly* removing such a state of affairs: it is the true essence of gravitation for Einstein (Taylor and Wheeler, 1992). In Newtonian language, one would explain the aforementioned pattern in terms of residual, or differential, gravitational forces, commonly dubbed tidal since they are the analogue of the lunar gravitational pulls which, varying from one end to the other over the entire extension of the terrestrial globe, raise the tides on it. Instead, in the language of spacetime, the worldlines of such objects 'tidally' driven towards or apart from each other no longer appear straight, being curved. Since, as remarked before, this is the key feature of gravity, in the Einsteinian framework it is said that gravity is a manifestation of the curvature of spacetime and GTR relies upon the EP. Thus, GTR is, at the same time, a theory of space and time, and of gravitation as well; furthermore, light and free massive particles move along geodesics of a curved spacetime, which are the generalization of straight lines taking place when gravity is absent. Their equation is

$$\frac{d^2x^\sigma}{d\lambda^2} = -\Gamma^{\sigma}_{\ \upsilon\iota}\frac{dx^\upsilon}{d\lambda}\frac{dx^\iota}{d\lambda},\ \sigma = 0, 1, 2, 3, \tag{1.1}$$

where λ is some affine[9] parameter which, in the case of a massive body, coincides with its proper time τ, while

$$\Gamma^{\sigma}_{\ \upsilon\iota} := \frac{1}{2}\mathrm{g}^{\sigma\kappa}\left(\frac{\partial \mathrm{g}_{\kappa\upsilon}}{\partial x^\iota} + \frac{\partial \mathrm{g}_{\kappa\iota}}{\partial x^\upsilon} - \frac{\partial \mathrm{g}_{\upsilon\iota}}{\partial x^\kappa}\right),\ \sigma, \upsilon, \iota = 0, 1, 2, 3 \tag{1.2}$$

are the Christoffel symbols of the second kind (Weinberg, 1972; Bishop and Goldberg, 1980; Misner et al., 2017); $\mathrm{g}^{\sigma\lambda}$ is the inverse of the spacetime metric tensor $\mathrm{g}_{\sigma\lambda}$. In terms of the temporal coordinate $x^0 := ct$, the equations of motion for a test particle retrievable from Equation (1.1) for $\lambda \rightarrow \tau$ and $\sigma = 1, 2, 3$, can be written as follows (Weinberg, 1972; Brumberg, 1991):

J0337+1715 (Ransom et al., 2014; Shao, 2016) in the field of another distant white dwarf, searching for violations of the EP in terms of the Nordtvedt effect (Nordtvedt, 1968b,a), the EP holds also in its stronger version, according to which the mutual gravitational attraction among bodies along with their own self-gravity is taken into account as well, to the $\simeq 10^{-4}$ (Hofmann and Müller, 2018) and $\simeq 10^{-6}$ (Archibald et al., 2018; Voisin et al., 2020) levels, respectively. The challenges of testing the EP in different regimes, including also the quantum realm in which it is not obvious that the former is valid, are reviewed in Tino et al. (2020).

[9] From *affīnis, e*, 'bordering on, adjacent, contiguous'.

$$\frac{d^2x^i}{dx^{0^2}} = -\Gamma^i{}_{\sigma\lambda}\frac{dx^\sigma}{dx^0}\frac{dx^\lambda}{dx^0} + \Gamma^0{}_{\sigma\lambda}\frac{dx^\sigma}{dx^0}\frac{dx^\lambda}{dx^0}\frac{dx^i}{dx^0},\ i = 1, 2, 3. \tag{1.3}$$

On the other hand, another crucial aspect of the EP consists of the fact that gravity can also be *emulated*, to a certain extent, by adopting an accelerated reference frame. Indeed, the motions of material objects referred to the latter are characterized by accelerations which depend neither on the mass nor on the composition of the former ones, which is just the distinctive trait of the gravitational interaction itself. Such a feature, together with STR, allows one to predict a number of peculiar phenomena pertaining to the propagation of electromagnetic waves and the motion of material objects which are unknown to the Newtonian gravitational picture. Suffice it to think about the Coriolis acceleration affecting a moving particle with respect to a rotating reference frame and the corresponding gravitomagnetic counterpart arising in GTR since the latter has to fulfil the Lorentz symmetry (Jantzen et al., 1992b; Schmid, 2023).

Since GTR is a *relativistic* theory of gravitation, and in STR *all* forms of energy are equivalent to mass, for Einstein, the source of gravitation, that is, of the spacetime curvature, is made by *several more* entities than for Newton and his scalar potential $\mathcal{U}$ *alone*. That is, a material body gravitates not only because it possesses its own rest energy, but also because it is compressed or dilated, or because it is distorted by internal stresses, and even if it moves. All that is encoded by the symmetric energy-momentum tensor $\mathsf{T}_{\sigma\lambda},\ \sigma, \lambda = 0,\ 1,\ 2,\ 3$ (Provost, 2017; d'Inverno and Vickers, 2022). Thus, there is *no* longer just a *single* gravitational potential sourced only by the matter density ρ, as in the Newtonian scheme, but now there are *ten* generally different quantities playing the role of gravitational potentials: the independent components of the symmetric spacetime metric tensor. The way the distribution of matter and energy actually deforms the spacetime ultimately determining the metric tensor is established by Einstein's field equations (Fok, 1959; Synge, 1960; Weinberg, 1972; Hawking and Ellis, 1973; Wald, 1984; Stephani, 1990; Cheng, 2009; Padnanabhan, 2010; Ohanian and Ruffini, 2013; Zee, 2013; Misner et al., 2017; Carroll, 2019; Thorne and Blandford, 2021; Schutz, 2022; Kenyon, 2023),

$$\mathsf{R}_{\sigma\lambda} - \frac{1}{2}\mathsf{g}_{\sigma\lambda}\mathsf{R} = \kappa_g\mathsf{T}_{\sigma\lambda},\ \sigma, \lambda = 0, 1, 2, 3, \tag{1.4}$$

which represent a set of complicated nonlinear partial differential equations. In Equation (1.4), $\mathsf{R}_{\sigma_\lambda}$ is the Ricci curvature tensor of the spacetime, defined by contracting two indices of the Riemann tensor (Weinberg, 1972; Bishop and Goldberg, 1980; Parker and Christensen, 1994b; Misner et al., 2017; Schutz, 2022),

$$\mathsf{R}^\epsilon{}_{\sigma\psi\lambda} := \frac{\partial\Gamma^\epsilon{}_{\lambda\sigma}}{\partial x^\psi} - \frac{\partial\Gamma^\epsilon{}_{\psi\sigma}}{\partial x^\lambda} + \Gamma^\epsilon{}_{\psi\chi}\Gamma^\chi{}_{\lambda\sigma} - \Gamma^\epsilon{}_{\lambda\chi}\Gamma^\chi{}_{\psi\sigma},\ \epsilon, \sigma, \psi, \lambda = 0, 1, 2, 3, \tag{1.5}$$

in the following way (Weinberg, 1972; Bishop and Goldberg, 1980; Parker and Christensen, 1994a; Misner et al., 2017):

$$\mathsf{R}_{\sigma\lambda} := \mathsf{R}^{\epsilon}{}_{\sigma\epsilon\lambda},\ \sigma, \lambda = 0, 1, 2, 3. \tag{1.6}$$

Furthermore,

$$\mathsf{R} := \mathsf{g}^{\mu\nu}\mathsf{R}_{\mu\nu} \tag{1.7}$$

is the trace of the Ricci tensor, and κ_g is Einstein's gravitational constant (Adler et al., 1975). Nonetheless, if the characteristic motions of the system at hand are quite slow, and the gravitational fields are weak and almost static, the general relativistic field equations reduce to just the Poisson equation

$$\nabla^2\mathcal{U} = 4\pi G\rho \tag{1.8}$$

for the potential $\mathcal{U}$ of the Newtonian theory. Such a correspondence fixes the value of Einstein's gravitational constant entering Equation (1.4) to[10]

$$\kappa_g := \frac{8\pi G}{c^4}, \tag{1.9}$$

where G is Newton's constant of gravitation. In view of its tensorial nature, if $\mathsf{R}^{\epsilon}{}_{\sigma\psi\lambda}$, $\epsilon, \sigma, \psi, \lambda = 0, 1, 2, 3$ vanishes in a given coordinate system, it is zero in *all* other coordinates as well; in this case, gravity is effectively absent even if the spacetime appears *formally* curved in some coordinates; they would refer to a merely accelerated reference frame. Indeed, the geodesic deviation equation, known also as Jacobi equation (Chicone and Mashhoon, 2002) in differential geometry, which expresses the tidal forces, that is, the true manifestation of gravity, within the GTR framework, is proportional just to the Riemann tensor (Wald, 1984; Ohanian and Ruffini, 2013; Carroll, 2019).

Of course, GTR is not limited only to providing a different theoretical scheme to frame and reproduce the same phenomena described by the Newtonian one. The Einsteinian theory is much richer than Newton's Universal Gravitation, predicting a whole set of new phenomena. Indeed, GTR is able to treat motions occurring in gravitational fields so intense – in the sense that their gravitational potentials are close to the speed of light squared c^2 – that they accelerate bodies to speeds close to c itself and bend the path of electromagnetic waves in unparalleled ways undergoing also exceptionally rapid variations in time and from a point in space to another nearby one. The most spectacular – and expensive, as well as long-lasting – tests of GTR, recently performed by large international teams after several decades, undoubtedly come from such strong regimes. Suffice it to think about the gravitational waves (Cervantes-Cota et al., 2016) emitted in the end-of-life stages of

[10] With such a choice, each component of $\mathsf{T}_{\sigma\lambda}$ has the dimensions of energy density, that is, energy per volume, or, equivalently, pressure.

binary black holes (BHs) (LIGO Scientific Collaboration and Virgo Collaboration, 2016) and neutron stars (LIGO Scientific Collaboration and Virgo Collaboration, 2017), detected so far by the Laser Interferometer Gravitational-wave Observatory (LIGO) and Virgo facilities, or the shadows of the supermassive black holes (SMBHs) at the centre of the supergiant elliptical galaxy Messier 87 (M87) (Event Horizon Telescope Collaboration, 2019) and in Sgr A* at the Galactic Centre (GC) (Event Horizon Telescope Collaboration, 2022) imaged by the Event Horizon Telescope (EHT) collaboration (Doeleman et al., 2009). In such domains, Newton fails miserably.

The first approximation of GTR to the next order to the purely Newtonian one, in which new terms in the equations of motion appear, is named post-Newtonian (pN); see, for example, Damour (1987), Asada and Futamase (1997), Blanchet (2003), Blanchet (2006), Futamase and Itoh (2007), Will (2018), and references therein. It is a computational scheme for solving the GTR field equations relying upon the assumptions that the characteristic speeds of the bodies under consideration are smaller than c and that the gravitational fields inside and around them are weak. Nonetheless, as pointed out by Will (2011b), such a framework turned out to be notably effective in describing also certain strong field and fast motion systems such as compact binaries made of at least one dense neutron star and inspiralling pairs of BHs emitting gravitational waves; the reasons for that are largely unknown (Will, 2011b). Thus, putting the pN approximation to the test in as many different scenarios and at the highest order of approximation as possible is of paramount importance to gain ever increasing confidence in it and extrapolating the validity of its effects to their counterparts in stronger regimes. In principle, such pN tests have the benefit that, if, on the one hand, the expected signals of interest have smaller magnitude with respect to the corresponding ones in the strong field regime, on the other hand, the knowledge of the competing features of motion of classical origin is relatively better, and the impact of their mismodelling can be more accurately assessed with respect to less accessible astrophysical scenarios whose environments are, generally, less reliably known. Furthermore, the measurement techniques routinely used, or under development, for tracking solar system's artificial or natural bodies like, for example, LLR, Satellite Laser Ranging (SLR) (Coulot et al., 2011), and Planetary Laser Ranging (PLR) (Dirkx et al., 2019) are becoming more and more accurate, allowing, in principle, one to detect increasingly smaller features of motion. As if that weren't enough, the technological efforts needed to measure such tiny effects could be useful one day in other, unsuspected fields. Last but not least, a somewhat opportunistic approach may be more easily followed by exploiting existing or planned missions directed to different goals, with a remarkable gain of time and money. In its technical realm of validity, the pN approximation has been, or is currently being, tested only to the first post-Newtonian (1pN) order,[11] since its 2pN effects are deemed too small to be currently measurable. Moreover, the tests done or currently underway largely refer to the mass monopole and, to a much lesser extent, the spin dipole moments of the source, namely its mass M and angular momentum $\boldsymbol{J}$. In particular, the perihelion[12] precessions of Mercury (Shapiro et al., 1972; Shapiro, 1990), of other inner planets of the solar system (Anderson et al., 1978, 1993), and of the asteroid Icarus (Shapiro et al., 1968, 1971) were measured long ago. More recently, Earth's geodetic satellites[13] (Pearlman et al., 2019), tracked with the SLR technique, were used (Lucchesi and Peron, 2010, 2014). Finally, the perinigricon[14] shift of the S star S2 in the field of the SMBH in Sgr A* was recently measured as well (GRAVITY Collaboration et al., 2020). Furthermore, the periastron[15] advance of a two-body system of comparable masses M_{A} and M_{B} was measured with different binary radiopulsars (Weisberg and Taylor, 1984; Stairs, 2003; Champion et al., 2004; Weisberg and Taylor, 2005; Kramer et al., 2006). As far as the 1pN orbital[16] effects induced by the angular momentum $\boldsymbol{J}$ of the primary, known collectively as the Lense–Thirring (LT) effect (Lense and Thirring, 1918; Mashhoon et al., 1984), are concerned, tests have been underway with SLR geodetic satellites since 1996 (Ciufolini et al., 1996). Some aspects of them, like their realistic accuracy, are currently being debated; see, for example, Renzetti (2013b) and references therein. So far, the only uncontroversial test of another 1pN feature due to the Earth's angular momentum is the one performed with the Gravity Probe B (Everitt, 1974) (GP-B) mission which measured the Pugh–Schiff precessions (Pugh, 1959; Schiff, 1960) of four spaceborne gyroscopes to a $\simeq$ 19% accuracy (Everitt et al., 2011, 2015), despite the fact that for many decades it was assumed that the final accuracy would be around 1% (Everitt, 1974; Everitt et al., 2001). Actually, to the 1pN level, other dynamical effects arise induced by mass and spin multipole moments of higher order (Soffel and Han, 2019).

In this book, extensive use is made of the Keplerian orbital elements (Brouwer and Clemence, 1961; Soffel, 1989; Brumberg, 1991; Klioner and Kopeikin, 1994; Bertotti et al., 2003; Roy, 2005; Kopeikin et al., 2011; Poisson and Will, 2014; Soffel and Han, 2019). They are the semimajor axis a, the eccentricity e, the inclination I, the longitude of the ascending node Ω, the argument of pericentre[17] ω, and

[11] It can be formulated to yield field equations for just two potentials (Soffel and Brumberg, 1991).

[12] From περί (+ accusative), meaning 'around, near, about, from', and Ἥλιος, -ου, ὁ, 'Hḗlios', the god of the Sun.

[13] From *sătellĕs*, *ĭtis*, meaning 'attendant upon a distinguished person', 'lifeguard'. For a discussion of the word satellite, its origin and its use in astronomy, see Sparavigna (2016).

[14] From περί (+ accusative), meaning 'around, near, about, from', and *nĭger, gra, grum* ('black').

[15] From περί (+ accusative), meaning 'around, near, about, from', and ἄστρον, -ου, τό ('celestial body, star').

[16] From *orbis, is*, 'a ring, circle, re-entering way, circular path, hoop, orbit'.

[17] From περί (+ accusative), meaning 'around, near, about, from', and κέντρον, -ου, τό, meaning, among other things, 'stationary point of a pair of compasses', 'centre (of a circle)'.

the mean anomaly at epoch[18] η. Such a choice, which, by no means, should be deemed obligatory since other orbital parameterizations also exist (Bond and Janin, 1981; Gurfil, 2004; Efroimsky, 2005; Kopeikin et al., 2011; Gurfil and Efroimsky, 2022; Pogossian, 2022), is motivated by their immediately intuitive meaning which greatly helps in visualizing the effects described with them. Furthermore, they are easy to use in order to suitably design space-based experiments and preliminarily assessing the impact of other competing dynamical effects of classical origin.

However, nowadays, *actual* tests of dynamical features of motion are usually performed *differently*. Large datasets are reduced in the following way. Highly detailed mathematical models of (a) the dynamics of the moving bodies, including pN effects X_{pN} to a certain degree of completeness (b) the propagation of the electromagnetic waves between the Earth's stations and the (re)transmitting/reflecting artificial or natural bodies of interest (c) the measurement devices, all containing several key parameters $\mathfrak{p}$ characterizing the physical and orbital features of the system's components at hand (masses, initial positions and velocities, bias of transponders, etc.), are fitted to huge amounts of data. The latter consist of measurements of the directly observable quantities[19] $\mathfrak{O}$. In such grand fits (Nordtvedt, 2000), $\mathfrak{p}$ are estimated in a least-square way[20] along with their errors and reciprocal correlations, all stored in the covariance matrix. Finally, time series of post-fit residuals[21] are produced by subtracting the *measured* values of the observables $\mathfrak{O}$ from their *analytical* counterparts calculated with the previously estimated values of $\mathfrak{p}$. In order to *realistically* assess the accuracy of the parameter(s) of interest, different data sets and background reference models can be used, and the resulting values $\mathfrak{p}$ are confronted with each other. In principle, such post-fit residuals should account for, among other things, all the mismodelled – or even unmodelled – dynamics. Thus, if they are statistically compatible with *zero*, there is the temptation to straightforwardly compare them to their analytically predicted counterparts in order to infer *upper bounds* on X_{pN} if the latter is not included in the dynamical models fit to the observations. Furthermore, should the post-fit residuals be considered *different from zero* at a statistically significant level, one would be likely tempted to claim a *measurement* of the unmodelled effect X_{pN} of interest. This is a widely adopted practice in the literature. Actually, *great care* is needed

[18] There is not a symbol commonly adopted for it in the literature. Suffice it to say that, for example, η is used by Milani et al. (1987), while in the notation by Brumberg (1991) the mean anomaly at epoch is l_0. Furthermore, Kopeikin et al. (2011) denote it as $\mathcal{M}_0$, while Bertotti et al. (2003) adopt $\epsilon^{'}$.

[19] The Keplerian orbital elements do not belong to them, being computed from observations through some intermediate steps.

[20] Recently, the Bayesian approach also has been gaining ground (Mariani et al., 2023).

[21] It is possible to produce time-dependent 'residuals' of the Keplerian orbital elements (Lucchesi and Balmino, 2006; Lucchesi, 2007) only when the spacecraft motion proceeds steady and seamlessly, without interruptive orbital manoeuvres needed for, for example, pointing an antenna towards the Earth.

in proceeding as just outlined, especially when the expected size of the pN signal one is interested in is *not much larger* than the measurement errors[22] (Fienga and Minazzoli, 2024). Indeed, if X_{pN} is *not* modelled, its possible signature may be more or less *absorbed* in some of – or all – the parameters $\mathfrak{p}$ estimated in the fit, like, for example, the initial conditions. Thus, it would be partially or totally *removed* from the post-fit residuals. In this case, one would infer artificially *too tight* constraints on (some of the parameters of) X_{pN}, when, instead, the real impact of the latter on the system's dynamics actually is *larger*. Furthermore, if the post-fit residuals produced *without modelling* X_{pN} are significantly *different from zero*, it may be that their resulting anomalous pattern is *not* due to X_{pN} at all, as one would hope, being, instead, caused by some fortunate *mutual partial cancellation* of completely different effects leaving a signature which, by chance, has just the characteristics of X_{pN} one is looking for. Then, the *correct* way to proceed consists of *explicitly* modelling the pN feature of motion X_{pN} one wants to test and *simultaneously estimating* the parameter(s) $\mathfrak{p}_{X_{\mathrm{pN}}}$ characterizing it[23] along with all the other ones. Then, one can compare the post-fit residuals produced in this way with, say, those generated without modelling X_{pN} at all to see if significant differences, larger than the measurement error level, can be spotted. Finally, the errors of $\mathfrak{p}_{X_{\mathrm{pN}}}$ along with their correlations with the other simultaneously estimated parameters in the covariance matrix obtained just by modelling X_{pN} are to be inspected. See Section K.3 for a discussion of a case in which this standard approach is, for some reasons, disregarded.

A clarification is in order when one talks about *tests* of pN gravity. Let $\mathfrak{B}$ be the theoretical prediction of a certain pN effect, namely an analytical formula usually containing, among other things, one or more parameter(s) characterizing the physical properties of the environment in which the former takes place; they could be, for example, the masses and some other relevant physical quantities (angular momenta, multipole moments) of, say, a two-body system. Let it be assumed that there is an agreement, within the experimental errors, between $\mathfrak{B}$ and a corresponding measured or observed quantity. Then, one can correctly speak of a *genuine test* of the effect under consideration only if the parameters entering $\mathfrak{B}$ are known *independently from that very same effect*; for example, they could have been previously determined by exploiting different, even non-dynamical, features. Conversely, if the theory at hand is widely accepted in the common knowledge at the time, $\mathfrak{B}$

[22] The scope of data reductions is to finally produce post-fit residuals as small as the measurement errors.

[23] A widely adopted set of parameters usually estimated in pN gravity tests are those belonging to the so-called parametrized post-Newtonian (PPN) formalism (Will, 2018), among which β_{PPN} and γ_{PPN}, both equal to 1 in GTR, are those that attract the greatest interest. The PPN scheme can be applied to all metric gravitational theories, namely, those relying upon the EP. The speed of light c remains constant in it, and the metric tensor $g_{\sigma\lambda}$ is always assumed symmetric.

and the corresponding measured value can be used just to measure or constrain the parameter(s) entering the former.

The same considerations hold also for the plethora of long-range, or infrared, modified models of gravity (Brax et al., 2004; Nojiri and Odintsov, 2007; De Felice and Tsujikawa, 2010; Maartens and Koyama, 2010; Capozziello and de Laurentis, 2011; Skordis, 2011; Clifton et al., 2012; Ferraro, 2012; de Rham, 2014; Capozziello et al., 2015; Ruggiero and Radicella, 2015; Cai et al., 2016; Joyce et al., 2016; Maggiore, 2017; Mashhoon, 2017; Kobayashi, 2019; Roshan and Mashhoon, 2022) that have been continually churned out mainly since the accelerated cosmic expansion was discovered in 1998 (Riess et al., 1998; Perlmutter et al., 1999; Riess, 2000; Astier and Pain, 2012; Schmidt, 2012) and, more recently, since the issue of the Hubble tension gained prominence (Cervantes-Cota et al., 2023; Hu and Wang, 2023; Vagnozzi, 2023; Capozziello et al., 2024). Also the puzzle of nonbaryonic dark matter at galactic scales (Merrifield, 2005; Garrett and Duda, 2011; Bullock and Boylan-Kolchin, 2017; Wechsler and Tinker, 2018; de Martino et al., 2020) prompted the birth of several alternative theoretical frameworks among which the most prominent one is, perhaps, the MOdified Newtonian Dynamics (MOND) paradigm (Milgrom, 1983a,b,c; Sanders and McGaugh, 2002; Bekenstein, 2009; Famaey and McGaugh, 2012; Milgrom, 2014; Bugg, 2015; McGaugh, 2015; Banik and Zhao, 2022). For epistemological discussions about the MOND/dark matter debate, see Duerr and Wolf (2023). Another model put forth to cope with, among other things, the dark matter issue is the Scalar Tensor Vector Gravity (STVG), or MOdified Gravity (MOG) (Brownstein and Moffat, 2006a,b; Moffat, 2006; Moffat and Toth, 2009; Harikumar, 2022). For a comparison between MOND and MOG and other less known theories trying the explain the same phenomenology, see Pascoli (2024), and references therein. Recently, also the Modified General Relativity (MGR) paradigm popped up (Nash, 2019; Das and Sur, 2022; Nash, 2023). A further theoretical scenario arising in the framework of the long-lasting attempts to find a consistent quantum theory of gravity is the effective field theory called[24] Standard Model Extension (SME) (Kostelecký, 2004; Kostelecký and Potting, 2005, 2009). Among other things, it encompasses local Lorentz violations in the gravity sector which may manifest themselves to a pN level with several phenomena including also orbital effects (Bailey and Kostelecký, 2006). For a recent review of modern tests of Lorentz invariance, see, for example, Mattingly (2005), and references therein. Another theoretical scheme encompassing violations of the Lorentz symmetry is the Einstein–Æther theory, a generally covariant theory of gravity coupled to a dynamical, unit timelike vector field that breaks the aforementioned symmetry (Jacobson and Mattingly, 2004;

[24] Here, the reference is to the Standard Model of elementary particles and fields (Gouttenoire, 2023).

Eling et al., 2006; Jacobson, 2008). Reliably testing such proposed modifications of the currently known laws of gravitation in local systems with, for example, orbital motions is of paramount importance in order to gain knowledge on them *independently* of the very same effects for which they were introduced which, otherwise, would remain their sole, ad hoc justification.

This book, in the wake of the meritoriously celebrated texts by[25] Soffel (1989), Brumberg (1991), and Soffel and Han (2019), treats the effect of pN and alternative gravity on different quantities (Keplerian orbital elements, astrometric angles RA and decl., radial velocity of spectroscopic binaries, variation of the times of arrival in binary pulsars, characteristic timescales and sky-projected spin-orbit angles in transiting exoplanets,[26] two-body range and range rate) within a unified and uniform calculational scheme for arbitrary orbital geometries and generic orientations of the spin axes of the sources of the gravitational field in space. It mainly adopts the language of celestial mechanics, being aimed at the widest possible audience of readers typically working on celestial mechanics, astronomy, and astrodynamics in astronomical observatories, laser-ranging stations, and data centres. Spatially isotropic or harmonic coordinates[27] are adopted (Soffel and Brumberg, 1991). Furthermore, the coordinate time t is used to calculate temporal intervals; they coincide with those obtained by an observer comoving with the orbiting particle in terms of its proper time[28] τ up to corrections of the order of $\mathcal{O}\left(1/c^4\right)$.

The book is organized as follows.

The general scheme needed to calculate the desired post-Keplerian[29] (pK) orbital effects is outlined in Chapter 2. In it, after an overview of the Keplerian picture for a restricted two-body system in Section 2.1, the pK variations of the osculating Keplerian orbital elements are treated in Section 2.2; the first-order shifts in the perturbing acceleration are worked out in Section 2.2.1, while the second-order ones are dealt with in Section 2.2.2. The mixed effects arising when two pK accelerations enter simultaneously the equations of motion are the subject of Section 2.2.3. The methods for calculating the pK corrections to various characteristic orbital

[25] To a different level, see also O’Leary (2021).

[26] From ἐκ- (ἐξ- before a vowel), meaning, among other things, ‘out of, forth from; outside of, beyond’, and the adjectival form ἔξω (‘outer, external’, or ‘foreign’). With reference to our solar system, an exoplanet is, then, a planet outside of it.

[27] As explained by Brumberg (2010), in order to effectively cope with the problem of the coordinate-dependent quantities in relativistic celestial mechanics and astrometry, in 1991 the International Astronomical Union (IAU) recommended to adopt one specific type of coordinates once and forever: the harmonic coordinates, determined by four specific non-tensorial differential relations to be added to the tensorial field equations of GTR (Fok, 1959; Weinberg, 1972; Brumberg and Kopeikin, 1989b; Damour et al., 1991).

[28] The coordinate and the proper times coincide, up to corrections of higher order in $1/c$, when the orbiter is quite distant from the source of the gravitational field.

[29] Here, by post-Keplerian (pK) I mean dynamical features arising from *any* acceleration, Newtonian or not, different from the simple Newtonian inverse-square one. Then, in *this* sense of the term pK, the classical acceleration due to, say, the primary’s oblateness is pK.

temporal intervals are presented in Section 2.3: they are the anomalistic (Section 2.3.1), draconitic (Section 2.3.2) and sidereal (Section 2.3.3) periods, which all coincide with each other in the Keplerian case. Section 2.4 illustrates how to calculate the pK shifts of a generic observable quantity for which an analytical model can be given; the cases treated are (a) the radial velocity of a spectroscopic binary in Section 2.4.1, (b) some characteristic timescales in transiting exoplanets in Section 2.4.2, (c) the rate of change of the sky-projected spin-orbit angle for such kinds of exoplanets, dealt with in Section 2.4.3, (d) the variation of the times of arrival (TOAs) of binary pulsars in Section 2.4.4, and (e) the astrometric angles RA and dec. in Section 2.4.5. Finally, the pK shifts of the two-body range and range-rate are calculated in Section 2.5.

Chapter 3 is devoted to the calculation of various 1pN gravitoelectric features of motion for a test particle (Section 3.1) and a binary system of bodies with comparable masses (Section 3.2): the Keplerian orbital elements in Section 3.1.1 (test particle) and Section 3.2.1 (binary system), the anomalistic (Section 3.1.2 for a test particle and Section 3.2.2 for a binary system), draconitic (Section 3.1.3 for a test particle and Section 3.2.3 for a binary system), and sidereal (Section 3.1.4 for a test particle and Section 3.2.4 for a binary system) orbital periods, RA and dec. (Section 3.1.5), the two-body range and range rate (Section 3.1.6), the radial velocity (Section 3.2.5), the characteristic timescales of transiting exoplanets (Section 3.2.6), and the TOAs of binary pulsars (Section 3.2.7).

The 2pN gravitoelectric orbital precessions of a binary system are calculated in Chapter 4.

The 1pN LT acceleration, sourced by the source's spin dipole moment(s) and dubbed also as 'gravitomagnetic', is treated in Chapter 5 along with several features of motion induced by it: the Keplerian orbital elements (Section 5.1), the anomalistic (Section 5.2), draconitic (Section 5.3), and sidereal (Section 5.4) orbital periods, the gravitomagnetic clock effect (Section 5.5), the radial velocity (Section 5.6), the characteristic timescales of transiting exoplanets (Section 5.7), the sky-projected spin-orbit angle (Section 5.8), the TOAs of binary pulsars (Section 5.9), RA and dec. (Section 5.10), and the two-body range and range rate (Section 5.11).

Other 1pN gravitomagnetic orbital precessions, due to the spin octupole moment of the central body, are dealt with in Chapter 6.

Several Newtonian features of motion due to the quadrupole mass moment(s) of the source are the subject of Chapter 7: the Keplerian orbital elements (Section 7.1), the anomalistic (Section 7.2), draconitic (Section 7.3), and sidereal (Section 7.4) orbital periods, the radial velocity (Section 7.5), the characteristic timescales of transiting exoplanets (Section 7.6), the sky-projected spin-orbit angle (Section 7.7), the TOAs of binary pulsars (Section 7.8), RA and dec. (Section 7.9), and the two-body range and range rate (Section 7.10).

The 1pN orbital precessions of the order of $\mathcal{O}\left(J_2/c^2\right)$ are calculated for a test particle in Chapter 8.

Newtonian and pN tidal orbital precessions of a test particle orbiting a primary induced by a distant third body are calculated in Chapter 9. In particular, in Section 9.1, the impact of the pN precessions of the axes of the reference frame comoving with the two-body system in geodesic motion in the spacetime of the third body is omitted, being, instead, treated in Section 9.2.

The orbital precessions induced by some categories of popular modified models of gravity are treated in Chapter 10: they are due to power-law (Section 10.1), Yukawa-like (Section 10.2), $1/r$ (Section 10.3), empirical once-per-revolution (Section 10.4), constant (Section 10.5), and tidal-like (Section 10.6) extra-accelerations. The effects of some dark matter distributions are the subject of Section 10.7. Models encompassing violations of the Lorentz symmetry in the gravitational sector are treated as well (Section 10.8).

Appendix A collects a list of acronyms and abbreviations.

Notations and definitions are listed in Appendix B.

In Appendix C, it is shown how to calculate pK Lagrangians, to be used as disturbing functions in the Lagrange equations for the variations of the Keplerian orbital elements, from the spacetime metric tensor.

Appendix D presents some useful coefficients accounting for the various spin-orbit configurations.

Appendix E contains the coefficients entering the LT instantaneous shifts of the orbital elements.

The coefficients of the instantaneous orbital shifts due to the Newtonian J_2 acceleration are listed in Appendix F.

Appendix G collects the coefficients of the total net mixed orbital shifts of the order of $\mathcal{O}\left(J_2/c^2\right)$.

Appendix H displays the explicit expressions of the coefficients of the orbital precessions of tidal origin.

The coefficients of the averaged disturbing functions of the power-law and exponential dark matter density profiles along with those of the resulting orbital precessions can be found in Appendix I.

In Appendix J, numerical values for the relevant physical parameters of some major bodies of the solar system (the Sun, the Earth, and Jupiter) are provided along with those of the double pulsar.

Appendix K contains the numerical values of the several pK orbital effects calculated for various natural and artificial bodies in the solar system and outside it: the Sun's planets (Section K.1), the spacecraft Juno[30] around Jupiter (Section K.2), the

[30] From *Iūnō, ōnis*, Roman deity, wife of Jupiter.

Earth's Laser GEOdynamics Satellite (LAGEOS) (Section K.3), the double pulsar PSR J0737–3039 (Section K.4), the triple pulsars (Section K.5), and the star S4716 in the GC (Section K.6).

Appendix L reviews some space-based missions aimed at testing several pN orbital effects recently proposed by the author: Highly Elliptical Relativity Orbiter (HERO) (Section L.1), In-Orbit Relativity Iuppiter Observatory, or IOvis Relativity In-Orbit Observatory (IORIO) (Section L.2), Elliptical Uranian Relativity Orbiter (EURO) (Section L.3), LEnse–Thirring Sun–Geo Orbiter (LETSGO) (Section L.4), and ELXIS (Section L.5). Further missions proposed by other authors are presented in Section L.6.