AN ELEMENTARY PROOF OF A THEOREM BY MATSUMOTO

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Abstract. Matsumoto proved in, [*Prime end rotation numbers of invariant separating continua of annular homeomorphisms*, Proc. Amer. Math. Soc. **140**(3) (2012), 839–845.] that the prime end rotation numbers associated to an invariant annular continuum are contained in its rotation set. An alternative proof of this fact using only simple planar topology is presented.

§1. Introduction

The rotation number was introduced by Poincaré to study the dynamics of circle homeomorphisms $f\colon S^1 \to S^1$. Given a lift $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ of f, the rotation number of $\tilde{f}$ is defined as $\rho(\tilde{f}) = \lim_{n\to\infty}(\tilde{f}^n(x) - x)/n \in \mathbb{R}$, for any $x \in \mathbb{R}$. The limit is independent of x and only depends on the lift $\tilde{f}$ up to an integer constant. The rotation number $\rho(f) = \rho(\tilde{f}) \bmod \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$ measures the speed at which points rotate under the iteration of f and essentially classifies the dynamics.

The definition of rotation number does not extend smoothly to homeomorphisms of the annulus $f\colon S^1 \times [-1, 1] = \mathbb{A} \to \mathbb{A}$. Consider the universal cover of $\mathbb{A}$ identified to $\mathbb{R} \times [-1, 1]$ and let $\tilde{f}\colon \mathbb{R} \times [-1, 1] \to \mathbb{R} \times [-1, 1]$ be a lift of f. Denote $(x)_1$ the first coordinate of a point $x \in \mathbb{R} \times [-1, 1]$. Then $\lim_{n\to\infty}\big((\tilde{f}^n(x))_1 - (x)_1\big)/n$ now depends on x and, even worse, may not exist. Instead of looking at orbits it is useful to consider f-invariant probabilities μ in $\mathbb{A}$ and define

$$\rho(\tilde{f}, \mu) = \int_{\mathbb{A}} \big(\tilde{f}\big(s(y)\big)\big)_1 - \big(s(y)\big)_1 d\mu(y),$$

where $s\colon S^1 \times [-1, 1] \to \mathbb{R} \times [-1, 1]$ is a section of the universal cover $\pi\colon \mathbb{R} \times [-1, 1] \to S^1 \times [-1, 1]$, i.e. $s\pi = \mathrm{id}$. More generally, denote

$$\rho_{\mathrm{mes}}(\tilde{f}, X) = \{\rho(\tilde{f}, \mu) : \mu \text{ is an } f\text{-inv. Borel prob. measure, } \mathrm{supp}(\mu) \subset X\},$$

for any f-invariant set $X \subset \mathbb{A}$. Since the space of invariant Borel probabilities endowed with the weak topology is compact and convex it follows that $\rho_{\mathrm{mes}}(\tilde{f}, X)$ is a compact interval.

A continuum $X \subset \mathrm{int}(\mathbb{A})$ is *essential* if the two boundary components $S^1 \times \{-1\}$ and $S^1 \times \{1\}$ of $\mathbb{A}$ belong to different connected components, denoted, respectively, U_- and U_+, of $\mathbb{A} \setminus X$. It is called an (essential) *annular* continuum if, additionally, $\mathbb{A} \setminus X = U_- \cup U_+$. The previous notions of rotation can be applied to study the dynamics of invariant annular continua. In contrast with the one-dimensional case, the coexistence of different rotation numbers is typical. An example is the Birkhoff attractor Λ [L88], which is the global attractor of a dissipative diffeomorphism of the open annulus. Even though Λ has empty interior, it contains infinitely many periodic orbits with different rotation numbers.

There is yet another way of measuring the rotation of an invariant annular continuum. After identifying $S^1 \times \{1\}$ to a point, U_+ is transformed into an invariant open topological disk. Carathéodory's prime end theory (see [M82]) permits to compactify this new domain with a boundary circle, the set of prime ends of U_+, producing a closed topological disk $\hat{U}_+$. The construction being topological allows the homeomorphism f to be extended to a homeomorphism $\hat{f}\colon \hat{U}_+ \to \hat{U}_+$. Furthermore, a lift $\tilde{f}$ of f uniquely determines a lift $\hat{F}\colon \mathbb{R} \to \mathbb{R}$ of the restriction of $\hat{f}$ to the circle of prime ends of U_+, boundary of $\hat{U}_+$, and vice versa. The *upper prime end rotation number* of the lift $\tilde{f}$ in X is defined as the rotation number of $\hat{F}$ and denoted $\rho_+(\tilde{f}, X)$. The *lower prime end rotation number* $\rho_-(\tilde{f}, X)$ is defined analogously. One can think of these rotation numbers as measures of the rotation of the boundary of X as seen from the exterior.

An alternative intuitive approach to the prime end rotation numbers in terms of accessible points is discussed in [BG91]. A point p is called *accessible* from a domain U, $p \notin U$, provided there is an arc $\gamma\colon [0, 1] \to U \cup \{p\}$ such that $\gamma([0, 1)) \subset U$ and $\gamma(1) = p$. Denote $\tilde{U}_+, \tilde{X}$ the lifts of U_+, X to the universal cover $\mathbb{R} \times [-1, 1]$ of $\mathbb{A}$. Let $x \neq x' \in \tilde{X}$ be accessible from U_+ and $\gamma, \gamma'\colon [0, 1] \to \mathbb{R} \times [-1, 1]$ be two disjoint arcs such that $\gamma(0), \gamma'(0) \in \mathbb{R} \times \{1\}$, $\gamma([0, 1)), \gamma'([0, 1)) \subset \tilde{U}_+$ and $\gamma(1) = x$ and $\gamma'(1) = x'$. Denote $\gamma(0) = (r, 0), \gamma'(0) = (r', 0)$ and define $x \prec x'$ if and only if $r < r'$. Then, $\prec$ defines

a linear order in the set of points of X accessible from U_+. For any x, y in that set and $n \in \mathbb{Z}$ there is a unique $k = k(x, y)$ such that $T^k(y) \preceq x \prec T^{k+1}(y)$, where T denotes the deck transformation of the universal cover. One can prove that $\lim_{n \to \infty} \big(k\big(\tilde{f}^n(x), y\big)\big)/n$ is independent of x and y and is equal to $\rho_+(\tilde{f}, X)$.

The goal of this article is to give an elementary proof of the following theorem due to Matsumoto [M12].

THEOREM 1. *Let $f \colon \mathbb{A} \to \mathbb{A}$ be a homeomorphism isotopic to the identity and $X \subset \mathrm{int}(\mathbb{A})$ an invariant annular continuum. For any lift $\tilde{f}$ of f*

$$\rho_+(\tilde{f}, X), \rho_-(\tilde{f}, X) \in \rho_{\mathrm{mes}}(\tilde{f}, X).$$

Recall that a classical result due to Epstein shows that $f \colon \mathbb{A} \to \mathbb{A}$ is isotopic to the identity if and only if preserves orientation and each of the boundary circles.

Matsumoto's proof of Theorem 1 uses Le Calvez's deep theorem on the existence of a foliation by Brouwer lines for any orientation preserving homeomorphism of $\mathbb{R}^2$ in its equivariant form [L05] for the torus $\mathbb{T}^2$. The proof then goes on concluding the result in each of several cases, depending on the topological type of the aforementioned foliation. In this paper an alternative proof of Theorem 1 is presented. The arguments involve only basic facts from planar topology and prime end theory making our approach elementary in nature.

Theorem 1 allows to estimate the size of the rotation set of X, $\rho_{\mathrm{mes}}(\tilde{f}, X)$, without precise information of the dynamics within X. It can be subsequently applied to conclude the existence of periodic orbits in X of any rotation number $p/q \in [\rho_\pm(\tilde{f}, X), \rho_\mp(\tilde{f}, X)] \subset \rho_{\mathrm{mes}}(\tilde{f}, X)$ provided some extra hypothesis is satisfied: either $f_{|X}$ is chain-recurrent (polishing an argument due to Franks [F88], see [K15, M12]) or f is area-preserving [FL03] or X is a cofrontier [BG91] or, more generally, a circloid [K15].

In order to ease the notation, for any integer k the action $T^k(S)$ of the deck transformation T on a set $S \subset \mathbb{R} \times [-1, 1]$ will be denoted by $S + k$. Additionally, the projection $p(S)$ of S under the first coordinate map $p \colon \mathbb{R} \times [-1, 1] \to \mathbb{R}$ will be denoted by $(S)_1$.

§2. Proof of Theorem 1

Next lemma follows directly from the definitions.

LEMMA 2. *For any integer k,*

$$\rho_{\mathrm{mes}}(T^k \tilde{f}^n, X) = n\rho_{\mathrm{mes}}(\tilde{f}, X) + k, \qquad \rho_{\pm}(T^k \tilde{f}^n, X) = n\rho_{\pm}(\tilde{f}, X) + k.$$

The proof of Theorem 1 presented here only deals with the upper prime rotation number and shows that $\rho_+(\tilde{f}) \geqslant \inf \rho(\tilde{f}, X)$, the other cases being completely analogous.

Argue by contradiction: suppose there are integers p, q such that $\rho_+(\tilde{f}) < p/q < \inf \rho(\tilde{f}, X)$. As a consequence of Lemma 2, $\rho_+(T^{-p} \tilde{f}^q) < 0 < \inf \rho(T^{-p} \tilde{f}^q, X)$. Thus, after renaming, it is possible to assume

$$\rho_+(\tilde{f}) < 0 < \inf \rho(\tilde{f}, X). \tag{1}$$

Some notation to describe the shape of $\tilde{U}_+$ is now introduced. Let $\eta = \max\{y \in \mathbb{R} : (0, y) \in \tilde{X}\}$ and $\beta \colon [0, 1] \to \mathbb{R} \times [-1, 1]$ be a vertical arc with endpoints $\beta(0) = (0, 1) \in \mathbb{R} \times \{1\}$ and $\beta(1) = (0, \eta) \in \tilde{X}$. Denote $x_0 = \beta(1)$ and use β also to denote the image of the arc β. This abuse of notation is present throughout the text. The arcs $\beta' = \beta \setminus \{x_0\}$ and $\beta' + 1$ bound a region in $\tilde{U}_+$ which contains the segment $(0, 1) \times \{1\}$. The closure of this region, as a subset of $\tilde{U}_+$, will be denoted by V and thought of as a fundamental region. Clearly,

$$\tilde{U}_+ = \bigcup_{k \in \mathbb{Z}} (V + k)$$

and $(V - 1) \cap V = \beta'$. Define

$$(V + k)^+ = \bigcup_{j \geqslant k} (V + j), \qquad (V + k)^- = \bigcup_{j \leqslant k} (V + j).$$

Note that $x_0 \in \tilde{X}$ is accessible both from $\tilde{U}_+$ and from V. The following lemma is based on the interpretation of the prime end rotation number in terms of accessible points and their induced order $\prec$ as was discussed in the introduction.

LEMMA 3.

(1) *For every point x accessible from $\tilde{U}_+$ there exists $k \in \mathbb{Z}$ such that x is accessible from $V + k$.*

(2) *If x is accessible from $V + k$ then $\tilde{f}^n(x)$ is accessible from $(V + k)^+$, if $n \leqslant 0$, or from $(V + k)^-$, if $n \geqslant 0$.*
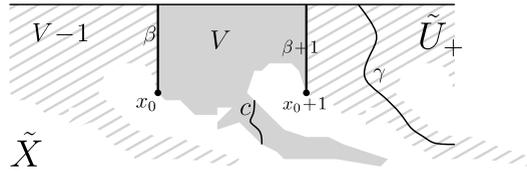
Figure 1.
c is a crosscut of $\tilde{U}_+$ and γ is a hair of $V + 1$.

(3) Suppose $x \in \tilde{X}$ is accessible from $V + k_1$ and $\tilde{f}^{-1}(x)$ is accessible from $V + k_2$. Then, for any point $z \in \tilde{X}$ accessible from $\tilde{U}_+$ there exists an integer n so that $\tilde{f}^n(z)$ is accessible from $V + j$, for some $k_1 \leqslant j \leqslant k_2$.

A *crosscut* of $\tilde{U}_+$ is an arc c whose endpoints lie in $\tilde{X}$ and whose interior is contained in $\tilde{U}_+$. By definition, the endpoints of c are accessible from $\tilde{U}_+$. Recall a standard fact from prime end theory: c separates $\tilde{U}_+$ in exactly two connected components.

DEFINITION 4. An arc $\gamma\colon [0, 1] \to V$ is said to be a *hair* of V if $\gamma(0) \in \mathbb{R} \times \{1\}$. More generally, an arc $\gamma\colon [0, 1] \to \tilde{U}_+$ is a *hair* if $\gamma - k$ is a hair of V for some $k \in \mathbb{Z}$. In that case γ is called a *hair* of $V + k$ (see Figure 1).

LEMMA 5. *There exist* $m \geqslant 1$ *satisfying* $\left(\tilde{f}^m(x)\right)_1 \geqslant (x)_1 + 1$ *for every* $x \in \tilde{X}$. *Furthermore, there exists* $M > 1$ *such that if* x *is a point in* V *for which every hair* γ *of* V *ending at* x *satisfies* $\mathrm{diam}\left((\gamma)_1\right) > M$ *then* $\left(\tilde{f}^m(x)\right)_1 \geqslant (x)_1 + 1/2$.

Proof. For the first part, suppose on the contrary that there are integers $\{n_i\}_{i \geqslant 1} \to +\infty$ and points $\{x_i\}_{i \geqslant 1}$ in $\tilde{X}$ such that $\left(\tilde{f}^{n_i}(x_i)\right)_1 < (x_i)_1 + 1$. The probability measures defined on X by

$$\mu_i = \frac{1}{n_i} \sum_{j=0}^{n_i-1} \delta_{f^j(\pi(x_i))}$$

satisfy $\rho(\tilde{f}, \mu_i) = \left(\tilde{f}^{n_i}(x_i)\right)_1 / n_i - (x_i)_1 / n_i < 1/n_i$. The space of Borel probability measures on X endowed with the weak topology being compact and metric, there is a subsequence $\{\mu_{i_j}\}_{j \geqslant 1}$, $i_j \to +\infty$, of $\{\mu_i\}$ whose limit is a Borel probability measure μ. By continuity of the pushforward operator

$$f_*(\mu) - \mu = \lim_j f_*(\mu_{i_j}) - \mu_{i_j} = \lim_j \frac{1}{n_{i_j}} \left(\delta_{f^{n_{i_j}}(\pi(x_i))} - \delta_{\pi(x_i)} \right) = 0,$$

μ is f-invariant, and by the weak convergence $\mu_{i_j} \to \mu$

$$\rho(\tilde{f}, \mu) = \lim_j \rho(\tilde{f}, \mu_{i_j}) \leqslant \lim_j 1/n_{i_j} = 0,$$

which contradicts (1).

For the second, let $\tilde{Y} = \{z \in \mathbb{R} \times [-1,1] : (\tilde{f}^m(z))_1 \geqslant (z)_1 + 1/2\}$ be a neighborhood of $\tilde{X}$. Clearly, the projection $Y = \pi(\tilde{Y})$ of $\tilde{Y}$ onto $\mathbb{A}$ is a compact neighborhood of X and $\mathrm{dist}(X, \partial Y) = \delta > 0$. If the statement does not hold one can find points $z_n \in V \setminus \tilde{Y}$ such that any hair γ of V ending at z_n satisfies $\mathrm{diam}((\gamma)_1) > n$, for every $n \geqslant 1$. It is possible to choose an infinite subsequence $\{z_{n_j}\}_j$ of $\{z_n\}_n$ so that the balls centered at z_{n_j} of radius δ are pairwise disjoint and contained in V. This is impossible because V has finite area.   □

For simplicity, for the rest of the proof replace f by f^m, where m is as in the previous lemma. Then, $(\tilde{f}(x))_1 \geqslant (x)_1 + 1$ for any $x \in \tilde{X}$ and (1) still holds.

The following object gives a way to roughly describe the shape of V. Construct hairs γ_n, $n \geqslant 1$, in V such that $l_{n+1} < l_n$, $r_{n+1} > r_n$, where $l_n = \min(\gamma_n)_1$ and $r_n = \max(\gamma_n)_1$.

There are three mutually exclusive cases depending on V:

(i)   It is not possible to have $\lim_n l_n = -\infty$.
(ii) There is an infinite sequence of hairs $\{\gamma_n\}_n$ such that $\lim_n l_n = -\infty$ and $\{r_n\}_n$ is bounded.
(iii) For any infinite sequence of hairs $\{\gamma_n\}_n$ such that $\lim_n l_n = -\infty$, always $\lim_n r_n = +\infty$.

They correspond to: (i) $(V)_1$ is bounded from below, (ii) $(V)_1$ unbounded from below but bounded from above and (iii) $(V)_1$ is unbounded both from below and above.

The proof of Theorem 1 deals separately with these three cases. Lemmas 3 and 5 are extensively used to derive a contradiction with Inequality (1) in each of them.

Case (i): $(V)_1$ *is bounded from below.*

In this case there exists $L \in \mathbb{R}$ such that every point $x \in \tilde{X}$ accessible from V satisfies $L < (x)_1$. Consider k so that $\tilde{f}(x_0)$ is accessible from $V + k$. Lemma 3 ensures $k \leqslant 0$. It follows that for any point $z \in \tilde{X}$ accessible from V^- there exists $n \geqslant 0$ such that $\tilde{f}^{-n}(z)$ is accessible from $V + j$,

for some $k \leqslant j \leqslant 0$. The lower bound on $(V)_1$ implies $L + k < \left(\tilde{f}^{-n}(z)\right)_1$ so, by Lemma 5, $L + k < (z)_1$, which is absurd. □

Case (ii): $(V)_1$ *is unbounded from below but bounded from above.*

The union of γ_n, $n \geqslant 1$, is contained in V and divides $\tilde{U}_+$ in many connected components. Denote B the component which contains $(V - 1)^-$. Clearly, B is unbounded from the left but satisfies $\sup(B)_1 \leqslant M$, where M is an upper bound for $\{r_n\}_n$.

Let $x \in \tilde{X}$ be accessible from $V - 1 \subset B$. For every $n \geqslant 1$, by Lemma 3 the point $\tilde{f}^n(x)$ is accessible from $(V - 1)^-$ and thus from B as well. However, for large $n \geqslant 0$, Lemma 5 implies $\left(\tilde{f}^n(x)\right)_1 > M$ and, in particular, $\tilde{f}^n(x)$ cannot belong to the adherence of B. □

Case (iii): $(V)_1$ *is unbounded both from below and above.*

This case is more involved and some preliminary results are needed. First, the shape of the region V is shown to be snake–like. This idea is made precise in the following proposition.

PROPOSITION 6. *There exist sequences* $\{L_n\}_n, \{R_n\}_n$ *of real numbers such that*

(1) $R_1 > 1$, $L_1 < 0$.
(2) $\{L_n\}_n$ *is decreasing and tends to* $-\infty$.
(3) $\{R_n\}_n$ *is increasing and tends to* $+\infty$.

such that

(i) *If* γ *is a hair of* V *and* $\left(\gamma(1)\right)_1 < L_n$ *then* $R_n \in (\gamma)_1$.
(ii) *If* γ *is a hair of* V *and* $\left(\gamma(1)\right)_1 > R_n$ *then* $L_{n-1} \in (\gamma)_1$.

Proof. Let α, γ be hairs of V. Notice the following simple fact: γ does not intersect any translated $\alpha + k$ unless $k = 0$. As a consequence:

($\star$) If $(\gamma(1))_1 > \max(\alpha)_1 + 1$ then $\min(\gamma)_1 < \min(\alpha)_1 + 1$.
($\star\star$) If $(\gamma(1))_1 < \min(\alpha)_1 - 1$ then $\max(\gamma)_1 > \max(\alpha)_1 - 1$.

The proof goes on following a mechanical routine using the sequence of hairs $\{\gamma_n\}_n$ and the associated scalar sequences $\{l_n\}_n, \{r_n\}_n$. First, choose m_1 so that $r_{m_1} > 2$ and $l_{m_1} < 1$. Take $R_1 = r_{m_1} - 1$ and $L_1 = l_{m_1} - 1$ and note that by ($\star\star$) with $\alpha = \gamma_{m_1}$ the statement (i) holds for $n = 1$. Then, take m_2' so that $l_{m_2'} < L_1 - 1$ and set $R_2 = \max(\gamma_{m_2'})_1 + 1$. Clearly ($\star$) forces (i) to hold for $n = 2$. Let m_2 such that $r_{m_2} > R_2 + 1$ and define $L_2 = \min(\gamma_{m_2})_1 - 1$. Using again ($\star\star$) it follows that (i) holds for $n = 2$. This
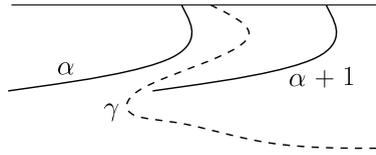
Figure 2.
Figure for statement $(\star)$.

procedure can be continued indefinitely and yields the sequences $\{L_n\}_n$ and $\{R_n\}_n$. ∎

The next two lemmas are obtained as corollaries of the previous proposition.

LEMMA 7. *For every $L < 0$ there exists $R > 1$ such that if γ is a hair contained in V^- and $(\gamma(1))_1 \geqslant R$ then $L \in (\gamma)_1$.*

Proof. Fix n so that $L_{n-1} \leqslant L$ and define $R = R_n$. Then γ is a hair in V^- whose endpoint is not on the left of R_n, that is, $(\gamma(1))_1 \geqslant R = R_n$. By Proposition 6, $L_{n-1} \in (\gamma)_1$ and, consequently, $L \in (\gamma)_1$. ∎

LEMMA 8. *For every $R > 1$ there exists $L' < 0$ such that if γ is a hair contained in V^+ and $(\gamma(1))_1 \leqslant L'$ then $R \in (\gamma)_1$.*

In the setting of Lemma 8, there is a crosscut c in $\tilde{U}_+$ which separates $\gamma(1)$ from $\mathbb{R} \times \{1\}$ and such that $(c)_1 = \{R\}$.

Let finish the proof of case (iii). Apply Lemma 7 to $L = -M$ (where M comes from Lemma 5) to obtain $R > 1$. Consider the family

$$\mathcal{A} = \{c \text{ is a crosscut of } \tilde{U}_+, c \subset (V + k)^- \text{ such that } \min(c)_1 \geqslant R + k\}.$$

By definition, $\mathcal{A}$ is invariant by integer translations. It is not empty because $(V)_1$ is not bounded from above. In addition, $\mathcal{A}$ is $\tilde{f}$-invariant. Indeed, if $c \in \mathcal{A}$ then by Lemma 5 it is automatically contained in the region $\{x \in \mathbb{R} \times [-1, 1] : (\tilde{f}(x))_1 > (x)_1\}$. Consequently, $\min(\tilde{f}(c))_1 > \min(c)_1 \geqslant R + k$. Since $c \subset (V + k)^-$, it then follows from Lemma 3 that $\tilde{f}(c)$ is contained in $(V + k)^-$. Thus, $\tilde{f}(c) \in \mathcal{A}$.

Apply now Lemma 8 to R to obtain L'. Denote $x_n = \tilde{f}^n(x_0)$ the orbit of x_0. Since $(x_n)_1 \to -\infty$ as n tends to $-\infty$, there is $m > 0$ such that $(x_{-m})_1 \leqslant L'$. By Lemma 3, the points x_n are accessible from V^+ for $n \leqslant 0$, so $x_{-m} + k$ is accessible from V for some $k \leqslant 0$. Since $(x_{-m} + k)_1 \leqslant L' + k \leqslant L'$, remark

after Lemma 8 provides a crosscut c of $\tilde{U}_+$ in V which separates $x_{-m} + k$ from $\mathbb{R} \times \{1\}$ and $\min(c)_1 = R$. Thus, $c \in \mathcal{A}$ and $c_{-m} = c - k \in \mathcal{A}$.

The crosscut $c_{-m} \in \mathcal{A}$ separates x_{-m} from $\mathbb{R} \times \{1\}$ in $\tilde{U}_+$. Thus, $\tilde{f}^m(c_{-m}) \in \mathcal{A}$ separates $\tilde{f}^m(x_{-m}) = x_0$ from $\mathbb{R} \times \{1\}$. However, the arc β joins x_0 and $\mathbb{R} \times \{1\}$ and does not meet any element of $\mathcal{A}$ because $(\beta)_1 = \{0\}$ is disjoint to $[R, +\infty)$. □

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