Arabic Sciences and Philosophy, 35 (2025): 1-37 doi:10.1017/S0957423924000092

READING BOOK II WITHIN ARITHMETICAL-ALGEBRAIC PRACTICES

THE CASE OF AL-KARAǦĪ, WITH A CONTINUATION IN AL-ZANǦĀNĪ

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Abstract. In Arabic treatises on algebra, Book II of Euclid's *Elements* quickly became a traditional work of reference, especially for justifying quadratic equations. However, in many of these treatises we find a representation of Euclid's notions that deviates from the "original Euclid." In this article, I focus on the way in which propositions of Book II were understood and reported by al-Karaǧī (11th c.) in two of his algebraic writings. Inspired by the variety of arithmetical practices of his time, al-Karaǧī transposed these Euclidean propositions from geometrical objects to numbers and applied them to an algebraic context. This allowed him to combine various argumentative strategies deriving from different fields. Building upon al-Karaǧī's work, al-Zanǧānī (13th c.) no longer needed to mention Euclid and instead conceived of a justification of quadratic equations (the "cause" of the equation) which is completely internal to algebra. These case studies provide evidence for the use of the *Elements* as a toolbox for the development of algebra. More importantly, they shed further light upon a typical feature of medieval mathematics, namely the existence of a plurality intrinsic in the name "Euclid."

Résumé. Dans les traités d'algèbre arabes, le Livre II des *Éléments* d'Euclide devient rapidement une référence traditionnelle, notamment dans la justification du procédé de résolution des équations quadratiques. Cette référence s'écarte toutefois significativement de l'Euclide original. Dans cet article, j'examine les relectures des propositions du livre II effectuées par al-Karaǧī (xɪ^e siècle) dans deux de ses écrits algébriques. Inspiré par la variété des pratiques arithmétiques de son époque, al-Karaǧī applique à des nombres les propositions euclidiennes originairement conçues pour des objets géométriques, pour ensuite les utiliser dans un cadre algébrique. Il parvient ainsi à combiner diverses stratégies argumentatives issues de différents domaines. La démarche d'al-Karaǧī sera au fondement du travail d'al-Zanǧānī (xɪɪɪ^e siècle). Ce dernier ne mentionne plus le nom d'Euclide et il conçoit une justification des équations quadratiques (la «cause» de l'équation) totalement interne à l'algèbre. Ces exemples témoignent de l'usage des *Éléments* comme une boîte à outils pour le développement de l'algèbre et ils mettent davantage en lumière une caractéristique typique des mathématiques médiévales, à savoir l'existence d'une pluralité intrinsèque au nom «Euclide».

As numerous studies have shown, the medieval reception of Euclid's *Elements* is an uncertain and intricate field of investigation.[1] Not only did the translations available to medieval mathematicians often diverge considerably from the original text, they also diverged from one another. In other words, medieval scholars read a text called "The Elements" which could vary significantly according to the available copies and the mathematical milieu in which those copies were read and translated. From a historical point of view, the direct consequence of this state of affairs is that, before examining the philological, scientific and philosophical aspects of the Arabic versions of the *Elements*, the starting question should always be: which "Euclid" are we talking about? The question becomes even more crucial when one considers that, in addition to producing translations and commentaries, mathematicians of different time periods and socio-cultural contexts also employed parts of Euclid's text while composing their own original treatises on geometry, algebra, or arithmetic. In doing so, they rephrased certain propositions, removed or modified the demonstrations, and sometimes even applied the propositions to different contexts or to problems that do not belong to the Euclidean corpus.[2] Indeed, scholarship has shown that certain authors tended to consider the *Elements* as a toolbox from which they could select materials in order to formulate their own new mathematical insights.[3]

[1] An emblematic article which clearly shows how complicated it is to trace the history of the Euclidean transmission in both Arabic and Latin sources is Bernard Vitrac, Ahmed Djebbar and Sabine Rommevaux, "Remarques sur l'histoire du texte des Éléments d'Euclide," *Archive for History of Exact Sciences*, 55 (2001), p. 221–295. With regard to the genesis of the Arabic versions of Euclid's Books, see in particular the publications of De Young and Brentjes, such as Gregg De Young, "The Latin Translation of Euclid's 'Elements' attributed to Gerard of Cremona in relation to the Arabic Translation," *Suhayl*, 4 (2004), p. 311–384 and Sonja Brentjes, "An Exciting New Arabic Version of Euclid's Elements: MS Mumbai, Mullā Fīrūz R.I.6," *Revue d'histoire des mathématiques*, 12 (2006), p. 163–197.

[2] See for instance Roshdi Rashed and Bijan Vahabzadeh, *Al-Khayyām mathématicien* (Paris: Albert Blanchard, 1999). Outside the Arabic context, see Sabine Rommevaux, *Clavius: Une clef pour Euclide au xvi^e siècle* (Paris: Vrin, 2005) and Leo Corry, *Distributivity-like Results in the Medieval Traditions of Euclid's Elements* (Springer, 2021).

[3] The idea of the use of the *Elements* (and of Greek classics more in general) as a toolbox was introduced by Saito in the 90's. See for instance Ken Saito, "Mathematical reconstructions out, textual studies in: 30 years in the historiography of Greek mathematics," *Revue d'histoire des mathématiques*, 4 (1998), p. 131–142. It was further developed in Reviel Netz, *The Shaping of Deduction in Greek Mathematics* (Cambridge University Press, 1999). More recently, it was examined during

Among extant texts, a representative example of this attitude towards the *Elements* can be identified in certain Arabic treatises dealing with algebra and, more broadly, arithmetic.[4] In such treatises one frequently finds the Euclidean definition of number (Book VII); certain notions related to the theory of proportions (Book V); as well as, in advanced texts, the study of irrational quantities (Book X). In algebraic contexts, the part of the *Elements* most often employed by Arabic mathematicians is, without doubt, Book II. This book deals with plane geometry, and its propositions involve straight lines and surfaces which are compared according to the congruence of lines and surfaces (i. e. according to the typical geometrical criteria adopted by Euclid).[5] Since Antiquity, Book II has raised numerous questions among mathematicians. Certain scholars soon realized that its propositions could easily be applied not only to geometrical magnitudes but also to numbers, and this led to the realization that geometrical magnitudes can be treated (and computed) just like numbers. Thanks to this unifying dimension, scholars also realized that this part of the Euclidean corpus could offer important insights for the investigation of algebra as well. The special status of Book II has also been at the origin of lively historiographical debates, such as the dispute over "geometric algebra," which developed during the 20th century.[6]

the workshop "Mathematics and its Ancient Classics Worldwide: Translations, Appropriations, Reconstructions, Roles." The present article is inspired by the results of that workshop. In Karine Chemla, Vincenzo De Risi, and Antoni Malet, "Mathematics and its Ancient Classics Worldwide: Translations, Appropriations, Reconstructions, Roles," *Oberwolfach Reports*, 18 (2021), p. 1347–1406, I have reported all the contributions to the workshop.

[4] We identify "arithmetic" as a heterogeneous set of theoretical approaches, accountancy practices, and computational methods of the time. This set includes several studies of the object "number" (such as Euclidean number theory, or logistics), as well as manuals related to specific computational practices (digital, sexagesimal, "Indian arithmetic," "aerian arithmetic," etc.). The practices transmitted by the *ḥussāb* (masters of computations) constitute the most widespread part of the discipline. Al-Karağī's work is a typical example of the plurality intrinsic to the definition of arithmetic. Indeed, as we will see, our author not only summarized the Euclidean arithmetical books, but also wrote a book on aerial arithmetic and one (currently lost) on the so-called "Indian arithmetic."

[5] See on this topic Euclid, *Les Éléments*, ed. and trans. by Bernard Vitrac, vol. 1 (Paris: Presses universitaires de France, 1990), p. 501–512.

[6] The expression "geometric algebra" was introduced in 1896 by Hieronymus Georg Zeuthen in his *Geschichte der Mathematik im Altertum und Mittelalter* to refer to the algebraic theory he considered to be implicit in Book II. According to Zeuthen's thesis, Greek scholars already knew a form of algebra which was, however, disguised as geometry. In formulating this theory, Zeuthen was strongly influenced by the

In this article, I will focus on the way in which certain propositions of Book II were rephrased and applied to the arithmetic of unknown algebraic quantities by the mathematician Abū Bakr al-Karağī (who lived between the end of the 10th and the beginning of the 11th century), as well as by his successor al-Zanğānī (who died in the mid-13th century). [7] Among al-Karağī's preserved texts, one can identify the algebraic treatise *Al-faḫrī fī ṣināᶜat al-ğabr wa al-muqābala*,[8] the arithmetical treatise *Al-kāfī fī al-ḥisāb*,[9] and a second algebraic treatise, entitled *Al-badīᶜ fī al-ḥisāb*.[10] These three texts were written around the year 1000 in Baghdad. Later on, they were largely commented upon and/or extended by al-Karağī's pupils and successors[11]. In this way, they contributed to

work of Tannery, who interpreted Book II on the basis of a "geometric algorithm" which could be seen at work in Euclidean proofs, and which – in his view – provides evidence of Euclid's numerical treatment of geometry. During the '70s, the idea of geometric algebra was debated between supporters and opponents of an algebraic interpretation of the Euclidean book. See on this topic Euclid, *Les Éléments*, ed. Vitrac, vol. 1, p. 367; Jean Christianidis (ed.), *Classics in the History of Greek Mathematics* (Dordrecht: Kluwer, 2004), p. 383–461; Michalis Sialaros and Jean Christianidis, "Situating the Debate on 'Geometrical Algebra' within the Framework of Premodern Algebra," *Science in Context*, 29 (2016), p. 129–150; and Jens Høyrup, "What Is Geometric Algebra, and What Has It Been in Historiography?" *AIMS Mathematics*, 2 (2016), p. 128–160. The expression "geometric algebra" has been reused by Rashed to designate the specific features of the algebraic approach grounded on geometry and developed in the second half of the 9th century by Ṯābit b. Qurra and his contemporaries. See Roshdi Rashed, *Thābit ibn Qurra: Science and Philosophy in Ninth-Century Baghdad* (De Gruyter, 2009), p. 153–158. Although inspired by this literature, my study will not deal with the topic of geometric algebra, but will rather focus on other aspects related to the reception of Euclid's Book II.

[7] With regard to the life and work of al-Karağī, see Roshdi Rashed, *Entre arithmétique et algèbre: Recherches sur l'histoire des mathématiques arabes* (Paris: Les Belles Lettres, 1984), p. 31–91; Jeffrey A. Oaks, "Diophantus, al-Karajī, and Quadratic Equations," in M. Sialaros (ed.), *Revolutions and Continuity in Greek Mathematics* (De Gruyter, 2018), p. 271–294; and Jean Christianidis and Jeffrey A. Oaks, *The Arithmetica of Diophantus* (Routledge, 2023), p. 146–158.

[8] See Franz Woepcke (ed.), *Extrait du Fakhrî: Traité d'algèbre par Aboû Bekr Mohammed ben Alhaçan Alkarkhî* (Paris: Imprimerie impériale, 1853). For an edition of *Al-faḫrī*, see Aḥmad Salīm Saᶜīdān, *Tārīḫ ᶜilm al-ğabr fī al-ᶜālam al-ᶜarabī*, vol. 1, "Algebra in Eastern Islam: Study built upon 'Al-fakhrī' of al-Karajī" (Kuwait, 1986).

[9] The text was edited by Sami Chalhoub, *Al-kāfī fī l-ḥisāb (Genügendes über Arithmetik) von Abū Bakr Muhamad ben al-Hasan al-Karağī (4–5. Jhd. / 10–11. Jhd. u.)* (Aleppo, 1986). An extensive summary of the book can be found in Adolf Hochheim, *Kâfî fîl hisâb (Genügendes über Arithmetik) des Abu Bekr Muhammed Ben Alhusein Alkarkhî* (Halle: Louis Nebert, 1878).

[10] The text was edited by Adel Anbouba, *L'algèbre Al-badīᶜ d'al-Karagī* (Beirut, Publications de l'Université libanaise, 1964).

[11] The mathematicians al-Šaqqāq (contemporary to al-Karağī), al-Šahrazūrī (end of

the development of an oriental arithmetical-algebraic tradition, which was still in vogue in the middle of the 13th century, when al-Zanǧānī wrote his treatise "Balance of the equation in the science of algebra and al-muqābala."[12]

In al-Karaǧī's writings one encounters several Euclidean subjects. The mathematician reformulates a group of propositions of Book II (in both *Al-faḫrī* and *Al-badīᶜ*); uses the theory of proportions to solve algebraic problems and to explain certain rules of aerial arithmetic (*Al-faḫrī* and *Al-kāfī*); writes a compendium of Euclid's arithmetical books – namely Books VII to IX – which includes the statement of the propositions but leaves out all the proofs (*Al-badīᶜ*); and transposes several propositions of Book X from magnitudes to numbers (*Al-badīᶜ*). In particular, al-Karaǧī's work on Book X allowed him to identify "new" numbers, namely irrational numbers, for which he formulated rules of computation.[13] Although all of these topics deserve attention, I will focus here on the first point mentioned above, namely al-Karaǧī's reading of Book II. My aim is to show that:

1. Al-Karaǧī read Book II and employed its propositions as a tool for his arithmetical-algebraic investigation. This attitude was not new at the time; rather, his adoption of it shows how there was a continuity between our author and several other traditions engaged in the investigation of numbers.

2. One can identify two stages of al-Karaǧī's arithmetical reading of Book II: a first stage in *Al-faḫrī* followed by a second one in *Al-badīᶜ*. Both readings are framed within algebra because algebra provides a framework that is less rigid and more open to new methods. Indeed, it allows one to combine the objects, methods and problems of geometry with arithmetical ones.

3. Al-Karaǧī's project would be continued a few centuries later by

the 11th century), and, most importantly, al-Samawʾal (12th century) were also representative scholars of al-Karaǧī tradition. The latter left us an important treatise on algebra, which has been edited and translated in Roshdi Rashed, *L'algèbre arithmétique au xiiᵉ siècle: "Al-bāhir d'al-Samawʾal"* (De Gruyter, 2021).

[12] For an edition, translation and commentary of the text, see ᶜIzz al-Dīn al-Zanǧānī, *Balance de l'équation dans la science d'algèbre et al-muqābala*, ed. and trans. by Eleonora Sammarchi (Paris, Classiques Garnier, 2022). My article stems from certain considerations developed in that book.

[13] See on this topic Marouane Ben Miled, *Opérer sur le continu: Traditions arabes du livre X des* Éléments *d'Euclide, avec l'édition et la traduction du commentaire d'Abū ᶜAbdi Allāh Muḥammad b. ᶜĪsā al-Māhānī* (Carthage, 2005) and Galina Matvievskaya, "The Theory of Quadratic Irrationals in Medieval Oriental Mathematics," *Annals of the New York Academy of Sciences*, 500 (2006), p. 253–277.

al-Zanǧānī, who no longer needed to mention the name of Euclid, and rather conceived of an independent form of reasoning for, and within, algebra itself.

My analysis will be developed as follows. I will begin by identifying three works that I believe inspired al-Karaǧī's reading of the Euclidean propositions. I will then describe the account developed in *Al-faḫrī* and that developed in *Al-badīᶜ*. In the latter book, al-Karaǧī shifts from numbers (*aᶜdād*) to algebraic entities, and justifies his propositions through a deductive sequence of equalities between non-instantiated expressions. Finally, I will compare al-Karaǧī's approach to that of al-Zanǧānī. In Chapter VII of his "Balance of the equation," al-Zanǧānī collects a group of propositions (*muʾāmarāt*). Some of them correspond to the propositions of Book II previously presented by al-Karaǧī. While the formulation chosen by al-Zanǧānī accurately corresponds to that of *Al-faḫrī*, the application of the propositions to the problems shows that they were meant for a different purpose: al-Zanǧānī uses them in order to develop proofs that rely entirely upon algebraic objects, tools and methods.

1. THREE ANTECEDENTS TO AL-KARAǦĪ'S ARITHMETICAL-ALGEBRAIC MENTION OF BOOK II

As I have already mentioned, starting in the second half of the 9th century the Euclidean work circulated widely in the Arabic world. I have been able to identify at least three antecedents to al-Karaǧī's approach. These sources belonged to traditions that inspired the milieu of arithmeticians-algebraists to which al-Karaǧī himself belonged. They can be considered representative of the heterogeneity of arithmetical approaches of the time, as well as of the plurality of traditions involved in the reception of Book II. These antecedents are:

1. Abū Kāmil's algebraic work;
2. Al-Nayrīzī's commentary on Books II, III and IV of the *Elements*;
3. The epistle "On the number" written by the Brethren of Purity.

It is important to recall that al-Karaǧī never refers explicitly to his sources. However, analysis of the content of his works and of the teaching context to which he belonged allows us to be quite confident that the notions contained in these earlier writings were known to al-Karaǧī and indeed were circulating among the arithmeticians-algebraists. It is therefore possible to consider these antecedents at least as indirect sources, and probably even as direct sources, for our author. Broadly speaking, by looking at these three sources it is interesting to note how similar projects can arise within different traditions and mathematical milieux. Before we examine each of them in turn, I want to emphasize that my analysis has no claim of exhaustiveness, and should rather be considered as preparing the ground for future research.

1.1. Abū Kāmil and the mention of propositions 5 and 6 of Book II in his Algebra

Among the three antecedents I have identified, the writings of Abū Kāmil are those which had the clearest influence upon the work of al-Karağī.[14] Despite the fact that Abū Kāmil's name is never mentioned by al-Karağī, we find striking similarities between their texts in the choice of the numerical examples that follow the rules of algebraic operations, as well as in the corpus of algebraic problems. Indeed, one can identify in al-Karağī's corpus several arithmetical problems solved through the science of algebra or via other arithmetical procedures that were already present in Abū Kāmil's text.[15]

In his algebraic writings, Abū Kāmil refers to Euclid on several occasions. The most famous of these references is the one that involves propositions 5 and 6 of Book II in the justification of the solution – i. e. in order to identify the "cause" (*ᶜillā*) – of the three forms of composite quadratic equations. Written in the second half of the 9th century, Abū Kāmil's algebraic work presents itself as an extension of al-Ḫwārizmī's algebra. Similarly focused on quadratic equations (and on the problems that these forms can solve), it also includes rules, proofs and problems, such as indeterminate (*sayyāla*) problems, that al-Ḫwārizmī did not consider.[16]

In Abū Kāmil's text, the reference to Euclid serves as a strategy for justifying the development of genuine arithmetic operations on sides and surfaces. Indeed, when he mentions Euclid's name and propositions 5 and 6 within the framework of the justification ("the cause") of quadratic equations, he combines the algebraic procedure with the

[14] On this subject, see Roshdi Rashed, *Classical Mathematics from al-Khwārizmī to Descartes* (Routledge, 2014) and Marc Moyon, *La géométrie de la mesure dans les traductions arabo-latines médiévales* (Brepols, 2017).

[15] See on this topic Eleonora Sammarchi, "Les collections de problèmes algébriques dans le 'Qisṭās al-muᶜādala fī ᶜilm al-jabr wa-l-muqābala' d'al-Zanjānī," *Médiévales*, 77 (2019), p. 25–40.

[16] Roshdi Rashed, *Abū Kāmil: Algèbre et analyse diophantienne* (De Gruyter, 2012), p. 578–679 and 732–762. It is important to recall that the indeterminate problems solved by Abū Kāmil derive not from the Diophantine tradition, but rather from other sources.

rigour that the axiomatic-deductive geometrical approach of the *Elements* guarantees. In his account, Abū Kāmil also formulates several justifications for the same form of equation, he shows how to obtain the numerical result of the square (without considering the value of the root), and points out the correspondence between the two propositions of Book II and the method called "application of areas" typical of propositions 28 and 29 of Book VI.[17] The procedural steps we have just mentioned help to further systematize the theory of quadratic equations. This theory becomes a traditional chapter in treatises on algebra and arithmetic, in both Arabic and Latin sources. However, if we look at the way in which the Euclidean text is reported by Abū Kāmil, we realize that these propositions are not applied in the classic Euclidean way. Indeed, the type of geometry used in this algebraic context is significantly different from the congruence of lines and surfaces that characterises Euclidean scholarly geometry.

Before looking at all these aspects in Abū Kāmil's text, it may be useful to recall how propositions are stated in one of the earliest translations of the *Elements*, namely one of al-Ḥağğāğ's versions (I consider here the MS Bibliothèque nationale [Paris] Persian 169). Let us for instance read the proposition corresponding to *Elements* II, 6:

> As for any line, if it is divided into halves, and there is thereupon added in its length a distance, the addition, then the tile of the whole of this by the addition, and the tile of half the first line by its like, altogether, are like the tile of half the first line with the addition by its like.
>
> For example: as for line AB, it has been divided into halves at G, and there has been added in its length a quantity, the addition, and that is BD. So I say that the tile of AD by DB, and of GB by its like, altogether, is like the tile of GD by its like. And that is what we wanted to demonstrate.[18]

Lo Bello also translates the same proposition according to MS Escorial Ar. 907, in which one can already find traces of interpretation of the proposition in terms of computations:

> And in the copy of al-Hajjaj, he says: as for any line, if it is divided into halves, and thereupon another line is added to its length, then what is from the *product* (*ḍarb*) of the totality of that, all of it, times the added line, and from the product of the first line times itself, altogether, is like what is from the product of half the first line, if the added line is joined to it, and thereupon all of it is multiplied times itself.[19]

[17] Rashed, *Abū Kāmil*, p. 304. On the area application method see Euclid, *Les Éléments*, ed. Vitrac, vol. 1, p. 377–385.

[18] Anthony Lo Bello, *The Commentary of al-Nayrīzī on Books II-IV of Euclid's Elements of Geometry* (Brill, 2009), p. 63.

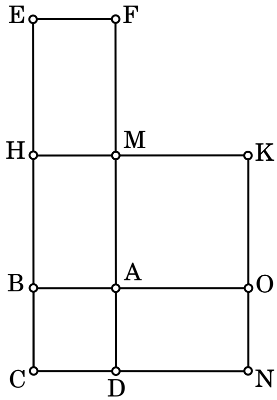

Fig. 1: Abū Kāmil, justification by the "cause" (in Rashed, *Abū Kāmil*, p. 254)

Taken together, these two versions highlight the contrasting reformulations later provided by Abū Kāmil, al-Karağī, and al-Zanğānī. Furthermore, they demonstrate that, since its initial translations, the transmission of the *Elements* has never been a neutral act. However, as I have already mentioned, there is no evidence that our authors had access to precisely these Arabic versions of the Euclidean text.

Let us now turn our attention to Abū Kāmil's text and examine the case of the first composite equation, i. e. "squares and roots are equal to a number." After the description of the procedure for solving the equation, Abū Kāmil seeks, like al-Ḫwārizmī, the "cause" of this procedure. This justification involves a geometric figure whose construction corresponds, step by step, to the procedure for solving the equation. Abū Kāmil writes:

> As for the cause of "the square plus ten roots equals thirty-nine," the procedure that leads you to the root (*jiḏr*) is the following. Let the square (*māl*) be a square (*murabbaᶜ*) ABCD to which we add the roots that are with it, i. e. ten roots, and this is the surface ABFE. It is clear that the straight line BE is ten in number, since the side of the surface ABCD, i. e. the straight line AB, multiplied by one, is the root of the surface ABCD. And when it is multiplied by ten, we obtain ten roots of the surface ABCD. The straight line BE is therefore ten; and the surface FECD is thirty-nine, since it is a square plus ten roots. This is obtained by multiplying the straight line EC by the straight line CD. But the straight line CD is equal to the straight line CB. Therefore, the product of the straight line EC by the straight line CB is thirty-nine. And the straight line EB is ten.
>
> We split the straight line EB into two halves at point H, so the straight line EB is split into two halves at point H and the straight line BC is added

[19] Lo Bello, *The Commentary of al-Nayrīzī*, p. 68.

https://doi.org/10.1017/S0957423924000092 Published online by Cambridge University Press

> to its length. *Therefore, the product of the straight line EC by the straight line CB, and the product of HB by itself, are equal to the product of the straight line HC by itself, according to what Euclid said in the second book of his work*.[20]

Although the geometrical construction corresponds to that already presented by al-Ḫwārizmī in his *Algebra*, the explicit reference to Euclid is an element of novelty. This mention provides evidence of the widespread circulation of early Arabic versions of the *Elements* during the second half of the 9th century.[21] More importantly, we should note the discrepancy between Abū Kāmil's formulation of proposition II, 6 (transcribed in italics in the quotation) and the version of MS Bibliothèque nationale Persian 169 attributed to al-Haǧǧāǧ, which remains closer to the traditional Euclidean version. Abū Kāmil multiplies geometric objects, such as straight lines, in the same way he would multiply numbers. Therefore, he replaces the congruence of surfaces with the computation of geometrical magnitudes. This is not the only instance of such a passage. For example, in the sixth form of quadratic equation (also justified by referring to proposition II, 6) he replaces the Euclidean formulation: "The rectangle contained by AD, BD taken with the square on CB is equal to the square on CD"[22] with a sentence such as: "The product of AG by GM plus MH by itself is equal to HG by itself." In that same passage, he also speaks of "subtracting" surfaces and "keeping the remainder."[23]

This different approach can be explained by the fact that Abū Kāmil aimed to find the numerical values of algebraic unknowns. By adopting the terminology of arithmetical operations, he developed a reasoning which no longer corresponds to the Euclidean one, and yet seems fully justified. According to his approach, the name of Euclid served as a guarantee of rigour, even when the general framework was crucially modified.

1.2. Al-Nayrīzī's commentary on Book II

With regard to the question of "which Euclid" al-Karaǧī and his school could have read, as we have already mentioned, we do not know which

[20] Trans. from Rashed, *Abū Kāmil*, p. 254.

[21] A similar mention of Euclid and of this kind of justification by the "cause" can be found in the short text "Restoring the problems of algebra by geometric demonstrations," written by Ṯābit b. Qurra in the same period. On this subject see Rashed, *Thābit ibn Qurra*, p. 160–169.

[22] Euclid, *Les Éléments*, ed. Vitrac, vol. 1, p. 335.

[23] Rashed, *Abū Kāmil*, p. 334.

https://doi.org/10.1017/S0957423924000092 Published online by Cambridge University Press

exact version(s) of the *Elements* (especially Book II) al-Karağī was able to consult. The analysis of the content and of the notions adopted by our author seems rather to indicate that he made use of several streams of the indirect tradition (translations, commentaries, etc.). Let us first recall that, in the case of Arabic sources, the direct transmission of the *Elements* comprises two families of versions: the "al-Ḥağğāğ family" (to which the proposition we quoted earlier belongs) and the – very heterogeneous – "Isḥaq-Ṯābit family."[24] On the other hand, the indirect transmission is composed of Arabic recensions and commentaries, as well as of several translations of later Greek commentaries on the *Elements*, such as those written by Heron of Alexandria, Pappus and Simplicius, to name only the most well-known. To this group, one should add individual readings or interpretations of specific parts of the *Elements* in the light of algebra, such as those found in al-Māhānī (9th century), al-Ahwāsī (10th century), Ibn al-Hayṯam (mid 10th–11th century), al-Ḫayyām, and many others.[25]

Within this general framework, the Commentary on Books II, III, and IV of the *Elements* written by al-Nayrīzī stands out.[26] It belongs to the indirect tradition, since al-Nayrīzī himself presents his text as a commentary on Heron of Alexandria's commentary on the *Elements*. Written around the year 900, the text focuses on Euclidean plane geometry. In it, several voices can be distinguished: that of Euclid, that of the commentator (whom Lo Bello identifies with al-Nayrīzī himself), as well as those of other scholars such as Heron and Simplicius.[27]

[24] See on this subject Gregg De Young, "The Arabic Textual Tradition of Euclid's 'Elements'" *Historia mathematica*, 11 (1984), p. 147–160. Brentjes's thesis is also worth mentioning. Because of the considerable number of variants that characterize "Isḥaq-Ṯābit" texts, Brentjes holds that this tradition lacks a single representative text. She thus rejects the division into two families and instead hypothesizes a single (but remarkably heterogeneous) tradition, which gathers together all these versions. Within her analysis, she found that the two books with the greatest internal homogeneity are precisely Books I and II. Indeed, these two books are strikingly similar to the edited Greek texts, as well as to Gerard of Cremona's translation. See on this topic Sonja Brentjes, "Textzeugen und Hypothesen zum arabischen Euklid in der Überlieferung von al-Ḥağğağ b. Yūsuf b. Maṭar (zwischen 786 und 833)," *Archive for History of Exact Sciences*, 47 (1994), p. 53–92 and Sonja Brentjes, "Two comments on Euclid's *Elements*? On the relation between the Arabic text attributed to al-Nayrīzī and the Latin text ascribed to Anaritius," *Centaurus*, 43 (2001), p. 17–55.

[25] With regard to the commentaries on Book X, see in particular Ben Miled, *Opérer sur le continu*.

[26] Lo Bello, *The Commentary of al-Nayrīzī*. See also Leo Corry, "Geometry and Arithmetic in the Medieval Traditions of Euclid's Elements: a View from Book II," *Archive for the History of Exact Sciences*, 67 (2013), p. 637–705, in particular p. 661–663.

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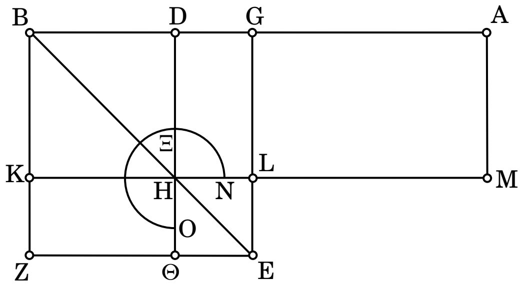

FIG. 2: Al-Nayrīzī, "The fifth figure of the Second Book" (adapted from Lo Bello, *The Commentary of al-Nayrīzī*, p. 30)

In Book II, propositions are introduced with the incipit "The first / second / third figure." Each proposition is first presented according to a geometrical account. Its structure includes the *protasis*, the *ecthesis*, the analysis and the synthesis, and a geometrical representation is given as well. Interestingly, al-Nayrīzī adds a numerical example to the first five propositions of the book. Let us have a look at the proposition corresponding to *Elements* II, 5. Al-Nayrīzī formulates the statement in the following way:

> If any straight line be divided into two segments equal to each other, and be divided again into two different segments, then the surface which the different segments enclose, with the square of the line that is between the two points of the two divisions, is equal to the square of half the line.[28]

After the statement, al-Nayrīzī proceeds to the construction of the geometric figure (Fig. 2). He then formulates the *ecthesis* (which begins with the expression "For example"), and develops the demonstration according to the criteria of analysis (the "proof").

It is at this point that he formulates a numerical example linking the geometric reading of the proposition to the arithmetic one:

> An example with numbers. Let us fix AB to be the number ten, and the two segments AG, GB, each one of them, five, and the segment AD seven. Then DB is left to be three. So, GD ends up as two. So it is clear that the totality of the multiplication of segment GB by its like is twenty-five, and that is equal to what is the total of the multiplication of AD times DB, and that is twenty-one, and of the multiplication of GD by its like, and that is

[27] Lo Bello points out that this commentary is not an abridged version of the *Elements*, but rather an excellent translation of the Euclidean text. See Lo Bello, *The Commentary of al-Nayrīzī*, p. XX. On Heron's metrological tradition, see Fabio Acerbi and Bernard Vitrac, *Héron d'Alexandrie: Metrica* (Pisa-Roma, Fabrizio Serra, 2014).

[28] Lo Bello, *The Commentary of al-Nayrīzī*, p. 29.

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four, and the sum of the two of them is twenty-five. And that is what we wanted to demonstrate.[29]

This example is followed by "the procedure of Heron in the proof of this figure" first by analysis, then by synthesis. As Lo Bello explains in the notes of his translation, this technique corresponds not so much to an analytical proof as rather to the association of numerical values with geometrical magnitudes. The demonstration is then derived from the previous propositions in the book. Moreover, one can notice that al-Nayrīzī adopts as numerical example the division of the number ten into two parts: seven and three. This example is typical of the *ḥussāb* and is found too in the algebraic books of al-Ḫwārizmī, Abū Kāmil, etc. It will also be frequently used by al-Karağī and his students.

The case of al-Nayrīzī reader of Heron, reader in turn of Euclid, is emblematic of the approach which consists in commenting on the classics while also developing original research. The outcome is a mixed text, in which ancient and medieval mathematical notions are combined. I consider al-Nayrīzī's commentary a relevant antecedent to al-Karağī's approach mainly for two reasons.

First, as I have just described, this reading of Book II transposes to numbers properties that are valid for magnitudes and thus places geometrical figures and numerical examples on the same level. Using numbers and geometrical magnitudes side-by-side was a typical feature of the Heronian approach and we will see that it corresponds precisely to the strategy adopted by al-Karağī.

Second, the milieu of the *ḥussāb*, to which al-Karağī belonged, was characterised by a deep interaction between metrological and arithmetical-algebraic practices, which still needs to be fully investigated.[30] In our specific case, for instance, it would be interesting to determine to what extent this interaction influenced al-Karağī's later writings on concrete uses of mathematics, such as hydraulics, the latter constituting the main subject of his treatise "On the extraction of un-

[29] Lo Bello, *The Commentary of al-Nayrīzī*, p. 31. In the footnote to this passage, Lo Bello emphasizes that, according to Klamroth, these numerical examples were taken from al-Ḥağğāğ's version of the *Elements*. Contrary to this thesis, Engroff, De Young, and Lo Bello himself consider that this example, and those that are included in the other propositions of the text, are an original addition of al-Nayrīzī.

[30] One can currently rely on the works of Moyon. See for instance Marc Moyon, "Algèbre et Practica geometriæ en Occident médiéval latin: Abū Bakr, Fibonacci et Jean de Murs," in S. Rommevaux, M. Spiesser and M. R. Massa Esteve (ed.), *Pluralité de l'algèbre à la Renaissance* (Paris: Honoré Champion, 2017), p. 33–65 and Moyon *La géométrie de la mesure*, p. 21–52.

derground water."[31] The comparative study of this commentary and of al-Karağī's work, which should prove to be a promising line of research, will therefore allow us to better understand not only the transmission of Euclid's geometry, but also the reception of non-Euclidean geometrical traditions, the circulation of the Heronian metrological corpus in the Islamicate world, and the intersection of scholarly and "practical" geometries.

1.3. The propositions of Book II in the Epistles of the Brethren in Purity

The third antecedent I want to highlight is the corpus of epistles of the Brethren in Purity (*Iḫwān al-Ṣafāʾ*). This corpus is a collection of fifty-two epistles dealing with various subjects which were written during the 10th century in the form of an encyclopedia. The authors are a group of scholars located between Basra and Baghdad.[32] Mathematics, especially arithmetic, are one of the main subjects addressed in the letters.

The first epistle, entitled "On the number," is one of the six philosophical-mathematical epistles edited and examined by Vaulx d'Arcy and is particularly significant for our present investigation.[33] In his study, Vaulx d'Arcy shows that the system of scientific disciplines described in the epistles is deeply inspired by al-Kindī's philosophy and builds upon the arithmetical theory developed by Nicomachus of Gerasa (flourished ca. 100 CE). Indeed, Nicomachus is mentioned several times in the epistles. According to the Neo-Pythagorean approach that characterizes the text, arithmetic is viewed as the primary form of science, upon which all other scientific disciplines are built. Consequently, the first definitions presented in the letter are those of unit and of number. After these definitions, the first epistle continues with the introduction of certain properties that characterize numbers and numerical sequences. Several sections of the epistle then deal with the nature of numbers (these can be perfect, deficient, abundant, amicable,

[31] See Giuseppina Ferriello, *L'estrazione delle acque nascoste: Trattato tecnico-scientifico di Karaji matematico-ingegnere persiano vissuto nel Mille* (Torino, Kim Williams Books, 2007).

[32] A detailed study of the Brethren in Purity can be found in Nader El-Bizri (ed.), *The Ikhwān al-Ṣafāʾ and their* Rasāʾil*: An introduction* (Oxford, 2008) and in Godefroid de Callataÿ, *Ikhwan al-Safaʾ: A Brotherhood of Idealists on the Fringe of Orthodox Islam* (Oxford: Oneworld, 2006).

[33] , Guillaume de Vaulx d'Arcy, *Les Épîtres des Frères en pureté (*Rasāʾil Ikhwān al-Ṣafā*): Mathématique et philosophie. Présentation et traduction de six épîtres par Guillaume de Vaulx d'Arcy* (Paris: Les Belles Lettres, 2019).

https://doi.org/10.1017/S0957423924000092 Published online by Cambridge University Press

etc.), whereas others are focused instead on arithmetical operations. The final part of the epistle presents certain philosophical and theological considerations regarding the primacy of arithmetic over the other disciplines of science.

The Neo-Pythagorean approach evident in "On the number" is shared by the tradition of the *ḥussāb* as well. For, although the texts belonging to the Arabic science of calculations (*ᶜilm al-ḥisāb*) are mainly focused on accounting and computational practices and problems, an analysis of their content reveals that they make use of several notions deriving from the Neo-Pythagorean tradition. This can be easily explained when looking at the heterogeneous nature of the content of *ḥisāb*. An emblematic example is represented by the notion of distance (*buᶜd*), which characterizes the multiplication and division of ranks (arithmetic and algebraic)[34] and is expressed according to the idea of counting forward and backward, namely according to an accounting strategy which was typical of logistics. This notion appears in the definition of algebraic powers that al-Karağī formulates in the first chapter of *Al-faḫrī*, as well as in the definition of arithmetic powers included in *Al-kāfī*. We can also find it in the works of both al-Samawᵓal and al-Zanğānī.[35]

With regard to Book II of Euclid, one section of the epistle is particularly noteworthy. Its title is: "Questions taken from [Book] II of Euclid's work on the *Elements*," and it is composed of the first ten propositions

[34] In his arithmetical treatise *Al-kāfī*, al-Karağī provided the reader with a definition of the rank (*martaba*). He identifies the first rank with the place of the units; the second with that of the tenths, the third with that of the hundreds, and so forth. Similarly, in his algebraic treatise *Al-faḫrī*, he identifies the first rank with that of the *units* (in the sense of numbers), the second is that of the roots (i. e. the things, or x), the third is that of the *squares* (i. e. x^2), the fourth is that of the *cubes* (i. e. x^3), and so forth. In recent scholarship, a definition of the term "rank" is given in Roshdi Rashed, *Lexique historique de la langue scientifique arabe* (Hildesheim: Georg Olms, 2017), p. 283–286.

[35] I quote the passage in which al-Zanğānī aims to explain how to name algebraic ranks: "If we want to name a known distance (*buᶜd*) from the units, then we will repeatedly subtract one unit from the known distance, and divide the remainder by three. We take for each unit of the result of the division a cube (*kaᶜb*), and apply them to each other. If the remainder of the dividend is less than three – and if it is two –, we will take for it a square (*māl*) and apply it to the cubes. And if it is a unit, we will take away from the terms "cube" a unit, replace it with a square-square (*māl-māl*) and apply it to the rest of the cubes. As if we want to name the rank thirteen from the units. We remove one unit from thirteen, we divide the remainder by three, and we obtain four. We take one cube each and apply one part to the other. The result is: a cube-cube-cube-cube." Trans. from al-Zanğānī, *Balance de l'équation*, ed. Sammarchi, p. 88.

of Book II. These are stated for numbers and justified by means of a numerical example.[36]

As an example, let's examine the proposition corresponding to *Elements* II, 4. The author of the epistle writes the proposition as follows:

> Let a number be divided into two parts, we say: the product of this number multiplied by itself is equal to the product of each part multiplied by itself, and of one multiplied by the other twice. For example, ten is divisible into two parts, seven and three, so we say: the product of ten multiplied by itself is equal to the product of seven multiplied by itself and of three multiplied by itself, and [to which we add] twice seven times three.[37]

One can notice that, in this account of the proposition, the author eliminates all geometrical references and directly transposes to numbers the property that was, in the Euclidean account, conceived for straight lines and surfaces. Consequently, the validity of the proposition relies exclusively on the numerical example. The latter consists in dividing ten units into two parts, namely seven and three, to which the specific conditions of the proposition are posited. We have already mentioned that dividing 10 into 7 and 3 would become a typical example of algebraic texts as well. It also corresponds to a type of problem solved since late antiquity by both Diophantus and the *ḥussāb*.[38]

Scholarship has shown that the Brethren in Purity aimed to turn Book II into a tool for a science of numbers grounded on the odd-even relationship and on the notion of figurative number, both of which were features of the Neo-Pythagorean study of number as opposed to the Euclidean one (which relied upon the notion of primality). Similarly, in the science of algebra and *al-muqābala*, Book II becomes a tool used to justify procedures and new "algebraic" numbers. In both cases, these readings are examples of the way the Euclidean text was reappropriated for the purpose of working with numbers (arithmetic or algebraic). Although the investigation of the Brethren in Purity does not concern algebraic procedures, it is important to note how the arithmetical reading of Book II they promote corresponds to the formulation of the propositions 5 and 6 made by Abū Kāmil, and – as we will see – to that of al-Karağī too.

[36] In a note, Vaulx d'Arcy clarifies that: "The present reformulation can be taken from Tābit b. Qurra, who wrote in the same period a *tahdīb* of Euclid's *Elements*. This *tahdīb* could constitute a clearing of all geometrical elements in order to obtain the arithmetical form that can be found here." Trans. from Vaulx d'Arcy, *Les Épîtres des Frères en pureté*, p. 115.

[37] Trans. from Vaulx d'Arcy, *Les Épîtres des Frères en pureté*, p. 116.

[38] See Rashed, *Abū Kāmil* and Christianidis and Oaks, *The Arithmetica of Diophantus*.

As in the case of al-Nayrīzī's commentary, I am convinced that a deeper investigation of the mutual influences between Neo-Pythagorean sources and the arithmetical-algebraic tradition initiated by al-Karağī and developed by his successors would provide us with new insights. On the one hand, one should not underestimate the circulation of texts such as these epistles among algebraists. While it is true that al-Karağī and his students are primarily practitioners and were not involved in philosophical debates, the analysis of their texts shows that they do address questions related to the nature of their mathematical objects, as well as to the status of algebra.[39] Questions such as what numbers are, or what the object of algebra is, do form part of their investigations, and their formulation of those questions shows the influence of the philosophical approaches which placed numbers at the core of the inquiry. Therefore, just like the notion of distance, other concepts, methods and demonstrative procedures of the Neo-Pythagorean tradition were also part of the arithmetical-algebraic approach.

On the other hand, the possible impact of the spread of algebraic practices within philosophical circles such as the Brethren in Purity still needs to be fully clarified. This kind of research would therefore further highlight the multilinearity of the circulation of traditions other than Euclidean within Islamicate mathematics.

2. AL-KARAĞĪ AND THE MENTION OF BOOK II IN AL-FAḪRĪ

Historiographical studies on Arabic algebraic works have repeatedly emphasized the presence of a dynamic of back-and-forth between algebra and the broader field of arithmetic. This dynamic is already present in the works of 9th-century algebraists such as al-Ḫwārizmī and Abū Kāmil, and it characterizes al-Karağī's work as well.[40] These scholars aimed to show that arithmetical problems of various types can all be solved through an equation, and thus become algebraic problems. In or-

[39] See in particular certain passages of *Al-badīᶜ* in which these topics are discussed, such as Anbouba, *L'algèbre Al-badīᶜ d'al-Karagī*, p. 46–47 of the critical edition, where al-Karağī discusses what algebra is (compared to geometry), and what its object is. With regard to al-Samawʾal, see the chapter "On the art of algebra as a part of the art of analysis" in Rashed, *L'algèbre arithmétique au xiiᵉ siècle*, p. 63–64.

[40] See Roshdi Rashed, *Al-Khwārizmī: The beginnings of Algebra* (London: Saqi, 2009); Rashed, *Abū Kāmil*; and Albrecht Heeffer, "A Conceptual Analysis of Early Arabic Algebra," in S. Rahman, T. Street and H. Tahiri (ed.), *The Unity of Science in the Arabic Tradition: Science, Logic, Epistemology and their Interactions* (Springer, 2008).

https://doi.org/10.1017/S0957423924000092 Published online by Cambridge University Press

der to provide the basic tools that are necessary for the solution of these problems, al-Karağī systematizes the rules for computations in the form of an arithmetic of algebraic unknowns.

In this arithmetic of unknowns, the equation (i. e. the co-equality of two algebraic expressions)[41] constitutes the final step of the procedure. Therefore, unlike Abū Kāmil, who begins his algebraic treatise with the study of the six forms of quadratic equations, al-Karağī focuses rather upon the notion of arithmetical operations, and upon the application of computational rules. In this way, the study of the equation loses its place of priority and instead occupies just one chapter among others in the book. Besides the rules for operating with algebraic quantities, al-Karağī's arithmetic of unknowns required additional theoretical tools in order to more firmly establish the correspondence between calculating within the framework of arithmetic and calculating within the framework of algebra. To that end, al-Karağī lists a certain number of propositions and theorems that he considers indispensable if one wants to solve and justify algebraic problems.

Within this framework, geometry does not disappear from the investigation, but it no longer constitutes a research subject in itself. For example, contrary to al-H̱wārizmī and Abū Kāmil, al-Karağī and his school of arithmeticians-algebraists do not include in their algebraic treatises any geometrical problems that can be solved via algebraic methods.

In his *Al-faḫrī*, al-Karağī introduces the topics of his treatise in the following order. First, he explains the elementary arithmetic operations (plus the extraction of the square root) applied to algebraic unknowns. Thereafter, two chapters deal with theorems and propositions of various origins which should help the algebraists in the solution of algebraic problems. Al-Karağī then continues by listing the "six algebraic problems" (i. e. the six forms of quadratic equations), and concludes the first part of the treatise with a short chapter on *istiqrāʾ*.[42] The second part of the book contains a collection of 254 arithmetical problems which are

[41] The idea of co-equal polynomials has been discussed in Albrecht Heeffer and Maarten Van Dyck (ed.), *Philosophical Aspects of Symbolic Reasoning in Early Modern Mathematics* (London: College Publications, 2010).

[42] The term *istiqrāʾ* refers to the study of indeterminate equations and of the problems that can be solved with them. It is thus a form of indeterminate analysis which, in an equation such as $x^2+bx+c=y^2$, aims to replace y^2 with a perfect square in x. See on this subject Roshdi Rashed, *Histoire de l'analyse diophantienne classique: D'Abū Kāmil à Fermat* (De Gruyter, 2013) and, defending an opposite thesis in relation to the interpretation of what *istiqrāʾ* is, Christianidis and Oaks, *The Arithmetica of Diophantus*.

organized into five sections and quantitatively amount to more than half of the entire treatise.

Among the theorems and propositions which al-Karağī selected on the grounds that knowledge of them "serves to solve the difficulties [of the problems],"[43] we can identify the counterpart of propositions 1, 5, 6, and 4 of Book II of the *Elements*, together with two special cases of propositions 4 and 5, respectively. Since, according to al-Karağī, numbers are the objects of algebra, our author imports these Euclidean propositions into his arithmetical-algebraic account, replacing straight lines with integers, rectangles with products, and geometric squares with numerical squares. Therefore, he develops an arithmetical reading of the Euclidean text which will also characterize the rest of the theorems and propositions mentioned in these chapters. In order to better understand this approach, let us examine two examples. The first is the proposition corresponding to *Elements* II, 4. Al-Karağī states it as follows:

> Two numbers are given, one is the double of the other. If you add twice their product to the sum of their squares (*murabbaᶜīn*), the total is a square, and its root is the sum of the two numbers. If you subtract twice the product of one by the other from the sum of their squares, the remainder is a square, and its root is the difference between the two numbers. An example is three and six: the sum of their squares is forty-five, and twice the product of one by the other is thirty-six. If you subtract it from forty-five, the remainder will be nine, i. e. a square. If you add it to forty-five, you will obtain eighty-one, i. e. a square.[44]

We can observe that al-Karağī considers two numbers, let them be a and $2a$, and claims the following relation:

$$a^2 + (2a)^2 \pm 2(a \cdot 2a) = (a \pm 2a)^2.$$

He replaces the geometrical demonstration which would follow the Euclidean statement of the proposition with a numerical example. Indeed, by taking $a = 3$ and $2a = 6$, he computes

$$3^2 + 6^2 = 45 \text{ and } 2 \cdot 3 \cdot 6 = 36.$$

Hence, he obtains the two square numbers

$$45 - 36 = 9 \text{ and } 45 + 36 = 81.$$

[43] Trans. from al-Karağī, *Al-faḫrī*, ed. Saᶜīdān, p. 141. The translation of this title is also mentioned in Woepcke, *Extrait du Fakhrî*, p. 62. The original sentence is: ممّا يستعان بمعرفته على إخراج الشكل.

[44] Trans. from al-Karağī, *Al-faḫrī*, ed. Saᶜīdān, p. 143.

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Al-Karağī remarks that this reasoning is also valid when considering a number and its triple, a number and its quadruple, or even a number and any of its multiples. Consequently, he is able to generalize the property of the proposition. Indeed, he writes:

> If one of the two is three times the other, or four times the other, or some other multiples, then the rule will be valid for them: the product of each one by itself is such that, if you subtract twice the product of one of the two by the other from it, the remainder will be a square. If you add twice the product of one of the two by the other to it, the total will be a square.[45]

We can write the previous equality as follows:

$$a^2 + (na)^2 \pm 2(a \cdot na) = (a \pm na)^2$$

where na is a multiple of the number a.

It is at this point that, in order to justify the validity (*siḥḥa*) of the proposition, al-Karağī explicitly mentions Euclid:

> And Euclid has shown the validity of what we have mentioned in the second book, in the specific case (*šakl*) that one says is a demonstration of: two numbers (*aᶜdād*) are such that, if you divide them into two parts, if you square each part and multiply the two parts one by the other twice, then the result will be equal to the square of the divisor term (*al-ḫaṭṭ al-maqsūm*). And if you subtract twice the product of the two numbers from their sum, the remainder will be a square.[46]

As we can see, similarly to the account of the Brethren in Purity, al-Karağī's reading of the Euclidean proposition is purely arithmetical: the straight lines which were originally involved in the Euclidean account are now replaced by numbers. A special case follows the proposition. Al-Karağī first considers a square number, added to (or subtracted from) two of its roots plus one unit: the result is the root of the square plus (or minus) one unit. Thereafter, he considers a square number, plus (or minus) any number of its roots, plus (or minus) half that number of roots. This means that he moves from the equality

$$a^2 \pm 2a + 1 = (a \pm 1)^2$$

to the equality

$$a^2 \pm na + \left(\frac{n}{2}\right)^2 = \left(a \pm \frac{n}{2}\right)^2$$

and shows again that he aims to obtain a more general form of the proposition.

[45] Trans. from al-Karağī, *Al-faḫrī*, ed. Saᶜīdān, p. 143.

[46] Trans. from al-Karağī, *Al-faḫrī*, ed. Saᶜīdān, p. 143–144

In all these propositions, he joins to the statement a numerical example, which at the same time exemplifies and validates the proposition.[47] We summarize it as follows:

$$196 \pm 10 \cdot \sqrt{196} + \left(\frac{10}{2}\right)^2 = (14 \pm 5)^2.$$

The same strategy is applied to the proposition corresponding to *Elements* II, 5:

> Know that, given a number, if we divide it into two halves, then into two different parts, the product of one of the two different parts by the other, together with the square of the difference between one of these two parts and half the number, will be equal to half the number by itself.
>
> An example: you divide ten into two parts, let them be seven and three, and you multiply one part by the other. The result is twenty-one. Thereafter, you take the difference between half of ten and seven, or three. The result is two. You multiply it by itself, it becomes four, and you add it to twenty-one. The result is twenty-five, which is equal to five [multiplied] by five. This proposition has already been shown by Euclid in his book.[48]

It is important to note that the two propositions we have just examined are the only propositions in the book which al-Karağī explicitly attributes to Euclid. Given the fact that these arithmetician-algebraists rarely cite their sources, this mention is even more meaningful.

Finally, one can point out another emblematic aspect of this arithmetical reading, namely the role played by numerical examples. Al-Karağī reports propositions without mentioning any proofs: no con-

[47] This case-study corresponds to the seventh proposition of the chapter on certain propositions and theorems that are considered to be useful for algebraic problems. The proposition may be translated as follows: "A square number is such that, if you add to it two of its roots *and* one unit, it will become a square. And if you subtract from it two of its roots less one unit, the remainder will be a square as well. But if you add to a square number any number of roots, together with half the number of these roots by itself, then the result will be a square. Its root is equal to the root of the *square* (*māl*) with half the number of roots. If you subtract from it any number of roots less half the number of these roots by itself, the remainder will be a square. Its root is the root of the number less half the number of subtracted roots. An example of this: we add to one hundred and ninety-six ten of its roots *and* half the number of its roots by itself, i. e. twenty-five. The result is three hundred and sixty-one, and this is a square. If we subtract from it ten of its roots less twenty-five, the remainder is eighty-one, and this is also a square." Trans. from al-Karağī, *Al-faḫrī*, ed. Saᶜīdān, p. 144. As in the previous quotation, I shall henceforth write "and" and "square" in italics when they designate respectively an aggregation of algebraic terms (which will later correspond to the sign "+") and the algebraic square *māl*.

[48] Trans. from al-Karağī, *Al-faḫrī*, ed. Saᶜīdān, p. 142–143. Again we see that the numerical example always consists in dividing 10 into 7 and 3.

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struction of the geometric figure is given in order to represent the proposition; no succession of equalities according to a deductive structure is provided. Within this general framework, the example becomes the element that guarantees the validity of the proposition. It thus acquires a dual function: illustrative and argumentative. This dual status of the numerical example was already a key feature of several arithmetical practices well known to al-Karağī. Indeed, when the master of calculations wanted to show that the rule of computation he has just stated was valid, he offered several generic examples, which could provide evidence for the fact that the procedure will not generate an impossible operation.[49] Consequently, we can see that there are two notions of rigor, which are in some sense opposite, and yet both accepted and applied within this text: the rigour generated by the mention of Euclid (and which relies on the axiomatic-deductive structure that constitutes one of the crucial features of the *Elements*), and that which characterizes the practices of the *ḥussāb*. This co-existence is in complete accord with algebraic practices of the time, insofar as these constituted an art characterized by a flexible and unifying framework within which multiple – even opposite – approaches could be adopted.

Finally, we can observe that, in the previous chapters of *Al-faḫrī* dealing with elementary arithmetical operations applied to algebraic powers, several examples are formulated for the same rule. By contrast, for the propositions analyzed above one example is enough to convince the reader that the proposition is valid. The reference to Euclid will do the rest.

3. AL-KARAĞĪ AND THE MENTION OF BOOK II IN *AL-BADĪᶜ*

Equally focused on the computational aspects of algebra, the treatise *Al-badīᶜ* is a rather different kind of text compared to *Al-faḫrī*. It consists of three books (*maqāla*), which cover several subjects, among

[49] This practice is attested in treatises such as those on what Arab scholars called “Indian arithmetic” (*ḥisāb al-Hindī*). See in particular the writings of al-Uqlīdīsī and Kūšyār b. Labbān (10th century), which have been translated respectively in Aḥmad Salīm Saᶜīdān, *The arithmetic of al-Uqlīdisī: The story of Hindu-Arabic arithmetic as told in Kitāb al-fuṣūl fī al-ḥisāb al-Hindī* (Springer, 1978) and in Martin Levey and Martin Petruck, *Principles of Hindu reckoning* (Madison-Milwaukee, The University of Wisconsin Press, 1965). The same use of examples is attested in the texts on “aerial calculation” (*ḥisāb al-hawaʾī*). A typical example is the “Book on what is necessary from the size of arithmetic for scribes and administrators” written by the mathematician al-Buzğānī (10th century). The text is edited in Aḥmad Salīm Saᶜīdān, *Abū al-Wafāʾ al-Būzjānī: Kitāb al-manāzil al-sabᶜ* (Amman, 1971).

https://doi.org/10.1017/S0957423924000092 Published online by Cambridge University Press

which one can identify a detailed study of irrational numbers, of the operations with these numbers, as well as of the operation of extraction of square root applied to composite algebraic expressions. In contrast with *Al-faḫrī*, the target audience of *Al-badīᶜ* is an experienced reader who no longer needs the long collections of problems in order to get used to algebraic methods. In the first book of *Al-badīᶜ*, al-Karağī begins by recalling the definitions of Book VII, especially those of unit and number. He then summarizes most of the propositions of the arithmetical books of the *Elements*, namely Books VII, VIII and IX. He concludes his compendium with an arithmetical reading of the propositions of Book X.

Between the summary of Book VII and that of Book VIII, al-Karağī presents a group of remarkable identities with different origins. Among these identities, one can easily identify propositions 4, 5 and 6 of Book II of the *Elements* – already introduced in *Al-faḫrī* – as well as propositions 7, 8, 9 and 10. All told, then, our author reproduces a considerable part of Book II (the latter consisting of a total of fourteen propositions). Compared to *Al-faḫrī*, the statement of the propositions in this second book is streamlined, and the mathematical objects which are now considered are different: al-Karağī no longer considers "a number (*ᶜadad*) divided into two parts," but rather a magnitude (*miqdār*), and directly applies the property of the proposition to squares (*murabbaᶜ*) and roots (*jiḏr*).[50]

With regard to the methods of verification, the validity of the propositions is now deduced by means of a reasoning which is composed of several steps, each one stating an equivalence between non-instantiated numbers. In order to see more clearly the difference with *Al-faḫrī*, we will compare the propositions corresponding to II, 4 and II, 5.

[50] It has been noticed that the term *miqdār* is applied in many arithmetical treatises with the sense of numerical quantity. See for instance Jeffrey A. Oaks, "Polynomials and Equations in Arabic Algebra," *Archive for History of Exact Sciences*, 63 (2009), p. 169–203. This use is also attested in al-Karağī's arithmetical treatise *Al-kāfī*. In this sense, *miqdār* and *ᶜadad* prove to be interchangeable, and the use of the word "magnitude" outside the semantic field of geometry would allow us to translate it as "quantity." However, one should not forget that, in this text, al-Karağī is specifically presenting a compendium of the *Elements*, namely of a book in which the distinction between "magnitude" and "number," is of crucial importance. Therefore, in this specific context, I believe it is important to continue to translate *miqdār* as "magnitude." However, in order to distinguish the numerical meaning from the purely geometrical one, I will write it in italics. This convention allow us to emphasize the complexity and variety of the medieval reading of the *Elements*, as well as its contribution to the development of the lexicon of algebra (and of *ḥisāb*).

3.1. The mention of Elements *II, 4*

The proposition which corresponds to *Elements* II, 4 plays a fundamental role in *Al-badīᶜ*. Indeed, once proved, it will be applied within the demonstration of other propositions of the same group. Al-Karağī states it in the following way:

> Given two different squares (*murabbaᶜ*), if you add to their sum the product of their sides twice, or if you subtract it, then, after the addition or the subtraction, the result will be a square (*murabbaᶜ*).[51]

Here and in the other propositions of the group, the justification is introduced with the term *li-ajli an*, which I translate as "indeed." Al-Karağī splits the proof into two parts. The first part corresponds to the proper justification of the proposition:

> Indeed, the square of the greater [side] is equal to: the square of the smaller [side], *and* twice the product of the smaller by the difference between the two, *and* the square of the difference between the two. (The sum of the two squares is thus equal to: two squares of the smaller *and* twice the difference between the two, multiplied by the smaller, *and* the square of the difference between the two.) If you subtract from this expression twice the product that is equal to the square of the smaller *and* once the product of the smaller by the difference between the two, the result is the square of the difference between the two. And if you add it, the total becomes: four times the square of the smaller, *and* four times the product of the smaller by the difference between the two, *and* the square of the difference between the two, and this is a square.[52]

This means that, let b and c be two sides such that $b > c$, we obtain $b = c + (b - c)$. Consequently,

$$b^2 = c^2 + 2c(b-c) + (b-c)^2$$

and

$$b^2 + c^2 = 2c^2 + 2c(b-c) + (b-c)^2.$$

Either we subtract

$$2c^2 + 2c(b-c) + (b-c)^2 - 2[(c^2 + c(b-c)] = (b-c)^2$$

[51] Trans. from Anbouba, *L'algèbre Al-badīᶜ d'al-Karagī*, p. 18, fol. 21v. It should be noted that there exists a French translation of the text. See Christophe Hebeisen, "L'algèbre 'Al-badī' d'al-Karağī," thèse de doctorat (École Polytechnique Fédérale de Lausanne, 2009). I have chosen to provide the reader with my own translation of the text because, with regard to certain terms, I have preferred to adopt more literal translations which in my opinion are closer to the meaning of the original Arabic text.

[52] Trans. from Anbouba, *L'algèbre Al-badīᶜ d'al-Karagī*, p. 18, fol. 21v.

which is a square, and thus the proposition is proved; or we add:

$$2c^2 + 2c(b-c) + (b-c)^2 + 2[c^2 + c(b-c)] = 4c^2 + 4c(b-c) + (b-c)^2$$

and this solution is also a square. More precisely, it is the square of $2c + (b-c)$.

Al-Karağī wants now to show that $4c^2 + 4c(b-c) + (b-c)^2$ is precisely $b^2 + c^2 + 2bc$ where $2bc = 2[c^2 + c(b-c)]$. He writes:

> Indeed, the square of the greater [side] is equal to: the square of the smaller [side] *and* the square of the difference between the two *and* twice the product of the difference by the smaller. If you remove this from that (i. e. the previous) expression, there remains three times the square of the smaller *and* twice the product of the smaller by the difference. Thereafter, you apply to this the square of the greater [side] in compensation for what has been taken away, since they are equal. You already know that the square of the smaller together with its product by the difference one single time is equal to the product of the smaller by the greater. Therefore, if you take away this product twice, and if you replace it with twice the plane obtained by the smaller and the greater, the expression becomes: the square of the greater *and* the square of the smaller *and* twice the product of the smaller by the greater, and this is a square. What we have claimed is thus proved.[53]

In this passage al-Karağī again points out that $b^2 = c^2 + (b-c)^2 + 2c(b-c)$. By removing this expression from the square he has obtained, namely $4c^2 + 4c(b-c) + (b-c)^2$, he finds

$$4c^2 + 4c(b-c) + (b-c)^2 - [c^2 + (b-c)^2 + 2c(b-c)] = 3c^2 + 2c(b-c).$$

He adds to this result b^2 (which is equal to what he has previously subtracted). Since $c^2 + c(b-c) = cb$ (an equality which is not proved, but rather taken for granted), he can obtain

$$3c^2+2c(b-c)+b^2-2[c^2+c(b-c)] = 3c^2+2c(b-c)+b^2-2c^2-2c(b-c) = c^2+b^2$$

to which he adds $2bc$.

Therefore, $b^2 + c^2 + 2bc$ is a square and it is equal to $4c^2 + 4c(b-c) + (b-c)^2$.

In this more advanced work, al-Karağī uses neither the tools of geometry nor precise numerical examples; he opts rather to employ a demonstrative form of reasoning which relies exclusively on co-equal non-instantiated expressions. In this specific case, he decomposes the squares, and applies a reasoning that is similar to a proof by recurrence.

[53] Trans. from Anbouba, *L'algèbre Al-badī^c d'al-Karagī*, p. 17, fol. 21v-22r.

It is interesting to note that another attempt to develop this kind of reasoning can be found elsewhere in the work of al-Karağī, (and would be elaborated even more clearly by his successor al-Zanğānī) in another passage when he engages with the *Elements*, namely when he reinterprets the so-called Euclidean algorithm (Book X) in terms of numbers, therefore providing a way of understanding divisibility.[54]

3.2. The mention of Elements *II, 5*

In modern mathematical language, Euclid's proposition II, 5 states that, if $a = b + c$, then $bc + (\frac{a}{2} - c)^2 = (\frac{a}{2})^2$. In his account of the proposition, al-Karağī conceives the proposition for *magnitudes*, which become quantities involved in a computation. He writes:

> If you divide a *magnitude* (*miqdār*) into two different parts, the product of one by the other with the square of the difference between half the divided *magnitude* and one of the two parts is equal to half the term by itself.
>
> Indeed, half the divided *magnitude* is deficient from one of its two parts by a quantity equal to that obtained when the other is subtracted from it. If you posit the greater of the two parts equal to half the *magnitude and* one thing (*šayʾ*), and the other part half the divided magnitude less one thing, and if you multiply them, the result will be the square of half the divided less a *square* (*māl*).[55]

Hence, the first step of the demonstration consists in pointing out that, when $b > c$, $b - \frac{a}{2} = \frac{a}{2} - c$. Unlike the proposition that we have previously analyzed, al-Karağī adopts here the terminology that is typical of algebraic unknowns: he designates as the objects of computation the thing (*šayʾ*) and the algebraic square (*māl*). By positing b "half the divided magnitude *and* one thing," and c "half the divided magnitude less a thing," he obtains $bc = (\frac{a}{2} + x)(\frac{a}{2} - x) = (\frac{a}{2})^2 - x^2$. From that equality, he derives that:

> The *square* (*māl*) is the square (*murabbaʿ*) obtained by the difference between one of the two parts and half the size.[56]

which means that either $x^2 = (b - \frac{a}{2})^2$, or $x^2 = (\frac{a}{2} - c)^2$.

Thus, within the demonstration, al-Karağī transposes the proposition to algebraic unknowns, and translates into algebra a procedure which was previously conceived for geometric entities – in the original Euclid – or for numbers – in the reading that characterizes *Al-faḫrī*. As

[54] Anbouba, *L'algèbre Al-badīʿ d'al-Karagī*, p. 31, fol. 45r-v.

[55] Trans. from Anbouba, *L'algèbre Al-badīʿ d'al-Karagī*, p. 18, fol. 23r.

[56] Trans. from Anbouba, *L'algèbre Al-badīʿ d'al-Karagī*, p. 18, fol. 23r.

shown by the proposition corresponding to II, 4, the arithmetical reading does not disappear from *Al-badī*[c]. However, in this book al-Karağī's account is structured according to a different approach. First of all, the proof is now independent from both the geometrical figure and the evidence ensured by the numerical example. Indeed, al-Karağī uses here the term "magnitude" (*miqdār*) and plays on the double meaning acquired by this term in arithmetical sources. Second, his demonstrations rely exclusively on properties (distributivity, associativity, other Euclidean propositions, etc.) that are part of the argumentative toolbox conceived for algebra.

Later on in the text, this procedure will also be found in the chapter on incommensurability, in which al-Karağī transposes the classification of irrational lines presented in Book X to numbers.[57] More precisely, al-Karağī begins there by recalling that Euclid distinguished three categories of simple lines (*ḫaṭṭ*): those that are commensurable in length (*bi'l-iṭlāq*), those that are commensurable in power (*bi'l-quwwa*) and the medials (*muwassiṭ*). A few lines later, he replaces the term "line" with "magnitude" (*miqdār*).[58] Finally, he clarifies that his aim is to transpose this classification to "numbers" (*a*[c]*dād*).[59] In order to show that each proposition of the book can be applied to numbers, he formulates several numerical examples and develops a purely arithmetical reasoning. Euclidean binomials thus become numbers such as $10+\sqrt{75}$, $\sqrt{12}+3$, $\sqrt{20}+\sqrt{15}$, $6+\sqrt{28}$, and $\sqrt{24}+4$. The same applies to medials and apotomes. He also refers to "rational numbers" ([c]*adad munṭiq*) and "deaf magnitudes" (*miqdār aṣṣam*).[60]

It is curious to note that al-Karağī combines numbers spelled out as words with numbers written in Indo-Arabic numerals, as if different arithmetical traditions were merging into one in his work. There is another place in *Al-badī*[c] where we can see this use of notations combining Indo-Arabic numerals and written numbers. It is in the second book, in the chapter dealing with the method of extraction of the square

[57] Anbouba, *L'algèbre Al-badī*[c] *d'al-Karagī*, p. 29–31 and Hebeisen, "L'algèbre 'Al-badī' d'al-Karağī," p. 81–87.

[58] See in particular Anbouba, *L'algèbre Al-badī*[c] *d'al-Karagī*, p. 29, fol. 42r, especially the sentence: وكلّ واحد من هده المقادير لا ينقسم إلّا باسميه.

[59] Al-Karağī writes: "I shall [now] explain to you how to transpose these denominations to numbers, and I shall develop them. Indeed, if we wish to go further in the science of *ḥisāb*, we cannot be satisfied with that (i. e. with the denominations only)." Trans. from Hebeisen, "L'algèbre 'Al-badī' d'al-Karağī," p. 82. See also the Arabic edition in Anbouba, *L'algèbre Al-badī*[c] *d'al-Karagī*, p. 29, fol. 42r.

[60] Anbouba, *L'algèbre Al-badī*[c] *d'al-Karagī*, p. 32, fol. 44r and p. 42, fol. 63r.

root of an algebraic expression composed of several added and/or subtracted terms. Al-Karağī shows how to solve the operation through tabular methods that are typical of “Indian arithmetic.” It is precisely in practices involving “Indian arithmetic” that we find a widespread use of Indo-Arabic numerals, since they are perfectly suited to a tabular arrangement of computations.[61]

To conclude, in *Al-badīᶜ*, al-Karağī’s study of incommensurability and his treatment of propositions from Book II proceed in precisely the same way: he refers to propositions that are originally part of a geometric book; he formulates an arithmetic reading of them; and he includes this reading within an algebraic book, i. e. within a frame that is non-Euclidean by nature, but constitutes a unifying domain. For instance, algebra allows him to adopt within the same text different notions of number: the number that corresponds to the Euclidean definition (i. e. an integer, positive number greater or equal to two); the number involved in “Indian” arithmetical practices (which includes fractions and is written in Indo-Arabic numerals); the metrical number of Book II, as well as the irrational (deaf) number of Book X. As a result, the study of the reception of Book II within algebraic practices is by no means limited to the simple question of the transmission of Euclid’s *Elements*; instead, it necessarily involves an investigation into the different ways of conceiving the object “number” and it prompts broader reflections on the way in which the status of number is modified when algebra is introduced as a form of calculation with its own objects, rules, and argumentative tools.

4. AL-ZANĞĀNĪ AND THE MENTION OF BOOK II IN “BALANCE OF THE EQUATION:” LOOKING BACK IN ORDER TO LOOK FORWARD

Despite the temporal distance that separates them, the mathematician al-Zanğānī was deeply influenced by al-Karağī. Although he never mentions al-Karağī by name, a textual analysis of “Balance of the equation in the science of algebra and al-muqābala” reveals that both *Al-faḫrī* and *Al-badīᶜ* were important sources for his mathematical thought. This

[61] See the explanation of the operation of extraction of the square root in Anbouba, *L’algèbre Al-badīᶜ d’al-Karagī*, p. 52–53. On the use of Indo-Arabic numerals within the medieval mathematical context, see Charles Burnett, *Numerals and Arithmetic in the Middle Ages* (Routledge, 2010). It is important to remember that this mix of notations is also typical of al-Samawᵓal (who employs Indo-Arabic numerals within algebra to an even greater degree). On the other hand, it is absent from the writings of al-Zanğānī.

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can be seen in his choice of examples, in the formulation of certain computational rules, and in the selection of problems. Moreover, given that "Balance of the equation" represents the teaching activity of his author, we may conclude that algebra was still taught in the 13th century according to the precepts of al-Karağī. Al-Zanğānī's text thus provides evidence for the remarkable longevity of his predecessor's legacy.

The last part of this inquiry will take into account the way in which the propositions of Book II were read and applied to algebraic problems by al-Zanğānī. "Balance of the equation" is composed of ten chapters. After setting out the rules for arithmetical operations applied to algebraic unknowns (Chapters I-IV), al-Zanğānī includes a fifth chapter, entitled "On numbers in proportions," which contains a summary of the propositions of the Euclidean arithmetical books. This chapter therefore corresponds to the first part of *Al-badī*^{c}, as we saw above. After a sixth chapter dealing with the extraction of square roots, al-Zanğānī presents a collection of thirty-three propositions, which acts as a bridge between the arithmetic of unknowns and the collections of algebraic problems. The title of this chapter is "On several propositions, most of which are demonstrated in the book of Elements."[62] It includes the propositions deriving from Book II that were discussed by his predecessor, in addition to many others of non-Euclidean origin. Since the reader al-Zanğānī has in mind is a student who has not yet been initiated into algebra, when reporting any given Euclidean propositions he does not consider the – more advanced – version found in *Al-badī*^{c} but rather the version found in *Al-faḫrī*. Each proposition of the chapter is thus conceived for numbers (*a*^{c}*dād*), followed by a numerical example, and there is no deductive reasoning to ground the validity of the proposition. For instance, it is clear that the general statement of the proposition corresponding to *Elements* II, 4 is very similar to the one that we have already analysed in the case of *Al-faḫrī*:

> A given number is divided into two parts: the square of the whole number is equal to the squares of each of the two [parts] with twice their product, and the product of one [part] by the other is called "the two complementaries" (*mutammamīn*). The square of the two parts is greater than the complementaries of the square of the difference between the two parts, except for

[62] We suppose that the "Book of Elements" to which al-Zanğānī refers is the geometry book *Kitāb al-usūl al-handasa* ("Book of Elements of Geometry"), that he wrote and is currently preserved in one single manuscript copy (MS Baku B2520, 4280/1). See Ekmeleddin Ihsanoğlu and Boris A. Rosenfeld, *Mathematicians, Astronomers and Other Scholars of Islamic Civilization and their Works (7th–19th century)* (Istanbul, 2003), p. 207 and al-Zanğānī, *Balance de l'équation*, ed. Sammarchi, p. 51.

> [the case in which] the two parts are equal. In the latter case, the squares of the two parts are equal to the complementaries. If you add the complementaries to the squares of the two parts, the total is the square of the whole number. If you subtract [the complementaries] from both of them, if subtraction is possible, the remainder is the square of the difference between the two parts.[63]

As in *Al-faḫrī*, the statement is followed by the numerical example of 10 divided into 3 and 7. Therefore, according to the aforementioned quotation,

$$7^2 + 3^2 \pm 2 \cdot 7 \cdot 3 = (7 \pm 3)^2.$$

Two propositions later, al-Zanǧānī discusses the special case

$$a^2 \pm na + \left(\frac{n}{2}\right)^2 = \left(a \pm \frac{n}{2}\right)^2,$$

also considered by al-Karaǧī, and clarifies:

> Consequently, given three successive numbers, such as three, four and five, if you square half the mean, this is such that, if you add to it the greatest number, the result is a radicand (*majḏur*), and, if you subtract the smallest number from it, the result is also a radicand.[64]

This means that he applies the formula when $a = 1$ and n is even.

In the same vein, al-Zanǧānī's proposition 6 – which corresponds to proposition 5 of the *Elements* – states that:

> A number is divided into two different parts [...] The product of one of the two parts by the other with the square of half the difference between them is equal to the square of half the number. Therefore, the product of seven by three with the square of two is equal to the square of five.[65]

The corollary to the proposition is then introduced:

> It follows from this that, given a number that measures a number through another number, if you add the square of half the difference between the two numbers that are measuring to the number that is measured, then the total is the square of half the sum of the two numbers that are measuring it. If you subtract the number that is measured from the square of half [the sum] of the two measuring numbers, the remainder will be the square of half the difference between the two measuring numbers.[66]

[63] Al-Zanǧānī, *Balance de l'équation*, ed. Sammarchi, p. 145.

[64] Al-Zanǧānī, *Balance de l'équation*, ed. Sammarchi, p. 147. *Majḏur* literally means "that from which we take the square root," i. e. a perfect square. In these texts, it is used with the same sense of *māl* and *murabbaᶜ*.

[65] Trans. from al-Zanǧānī, *Balance de l'équation*, ed. Sammarchi, p. 144.

[66] Trans. from al-Zanǧānī, *Balance de l'équation*, ed. Sammarchi, p. 144.

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This means that, if $a = b \cdot c$, then $(\frac{b-c}{2})^2 + a = (\frac{b+c}{2})^2$ and $(\frac{b+c}{2})^2 - a = (\frac{b-c}{2})^2$.

In both the proposition and the corollary, one can find the same example which was previously formulated by al-Karaǧī, namely

- $12 = 3 \cdot 4$; therefore, $(\frac{4-3}{2})^2 = \frac{1}{4}$; $\frac{1}{4} + 12 = (\frac{7}{2})^2$; and $(\frac{7}{2})^2 - 12 = (\frac{4-3}{2})^2$;
- $12 = 6 \cdot 2$; therefore, $(\frac{6-2}{2})^2 = 4$; $12 + 4 = (\frac{8}{2})^2$; and $(\frac{8}{2})^2 - 12 = (\frac{6-2}{2})^2$.

Al-Zanǧānī explains that the proposition is also valid when considering other divisors of 12, be they integer or rational numbers.[67]

The examples taken into account are $12 = 12 \cdot 1$; $12 = 8 \cdot (1 + \frac{1}{2})$; $12 = 9 \cdot (1 + \frac{1}{3})$; and $12 = 10 \cdot (1 + \frac{1}{5})$.

Although deeply inspired by the arithmetical reading developed by his master, we can already observe certain elements of discontinuity: al-Zanǧānī opts for an elimination of all references to Euclid and composes a richer collection of arithmetical identities. Many of these propositions will be applied, starting from chapter VIII, to justify certain passages of the solution of algebraic problems.

4.1. A rather different justification by "the cause"

In Chapter VIII, whose title is "On the six algebraic problems," al-Zanǧānī sets out the six forms of quadratic equation, as well as the operations of *al-ǧabr* and *al-muqābala*. He imitates al-Karaǧī and places the study of equation right in the middle of his book. This chapter inaugurates the part of the treatise devoted to problems. It is an important chapter in regard to our investigation, since one can see that al-Zanǧānī develops here a justification of the algorithm for the solution of the three composite equations in a manner significantly different from that developed by his master. In order to understand the specific features of al-Zanǧānī's presentation, we first need to return to Abū Kāmil's justification by "the cause" (mentioned at the beginning of this article). In Abū Kāmil's treatise, the procedure for solving each form of quadratic

[67] This passage can be translated as follows: "An example: three measures twelve by four, and the square of half the difference between the two is a quarter. If you add it to twelve, the total is the square of half seven. If you subtract twelve from the square of half seven, the remainder is the square of half the difference between three and four. Similarly, [three] measures six by two, and the square of half the difference between the two is four. If you add it to twelve, the total is the square of half eight. If you subtract twelve from the square of half eight, the remainder is the square of half the difference between six and two. In the same way, [twelve] measures twelve by means of one, eight by one and a half, nine by one and a third, ten by one and a fifth and so on. There are no exceptions to what we have just mentioned." Trans. from al-Zanǧānī, *Balance de l'équation*, ed. Sammarchi, p. 145.

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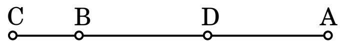

FIG. 3: Al-Karaǧī, the cause of the fourth algebraic problem (in Woepcke, *Extrait du Fakhrî*, p. 65)

equations is first stated in general terms, then applied to several numerical examples, and finally verified through one, or several, geometrical demonstrations ("the cause"). We have seen that propositions 5 and 6 are crucial parts of the justification, and yet they are used out of context and associated with computations of magnitudes. Abū Kāmil also relates the procedure to the construction of a geometric figure (represented in Fig.1 for the case of the first composite equation) which represents each step of the solution of the equation. The whole procedure is implemented when Abū Kāmil is looking for the numerical value of the root, as well as when he is directly looking for the numerical value of the algebraic square-*māl* (without considering the value of the root first).

For his part, al-Karaǧī suggests a significantly shortened version of this detailed account. Indeed, once he has presented the algorithm, he applies it to several numerical examples, and shows the strategy to obtain the numerical value of the algebraic square without considering the value of the root. In his account, the geometric figure is simplified to a line-segment, by means of which he establishes equalities between algebraic expressions supported by the reference to Euclid.

Al-Karaǧī also develops other examples, whose solution is justified via a geometrical representation. The latter involves the construction of rectangles in order to justify the procedure and is still simple and non-positional. Indeed, the focus is on the deduction of equalities, rather that on congruences. He concludes by demonstrating a purely arithmetical method, which he calls "the way of Diophantus."[68] To sum up, al-Karaǧī retains the essence of the geometric approach, but the constructive proof – which was so crucial to Euclid – is considerably less important in his presentation.

Building upon his predecessor, al-Zanǧānī begins by removing all geometric figures from his account. Furthermore, instead of referencing Euclid (and geometry), he directly justifies the "cause" of the composite equations via the arithmetical-algebraic propositions formulated in Chapter VII. In this way, all references to geometry (and to its rigor) are

[68] For an account of this arithmetical method of resolution, which is indeed typical of the Diophantine tradition, see Woepcke, *Extrait du Fakhrî*, p. 67–68 and Christianidis and Oaks, *The Arithmetica of Diophantus*, p. 147.

replaced by a proposition that is now framed in a different mathematical field. An example will clarify this point.

Since in this article I have mainly focused on the genesis of the proposition corresponding to *Elements* II, 5, I will consider now al-Zanǧānī's fifth composite algebraic problem, which requires precisely this proposition to be verified. I recall that the justification of the other two composite equations proceeded in a similar way but required proposition II, 6 for their verification. The fifth composite problem deals with equations of the form "squares and a number are equal to roots," which can be transcribed in modern mathematical terminology as

$$x^2 + c = bx.$$

This equation is the only quadratic equation which requires a diorism in order to exclude the cases which would end up with an impossible subtraction.[69] Al-Zanǧānī examines each of the three cases $c < (\frac{b}{2})^2$; $c = (\frac{b}{2})^2$; and $c > (\frac{b}{2})^2$. First, he formulates the algorithm for the solution of the equation when $c < (\frac{b}{2})^2$. The procedure is the same as the one developed by his predecessors, namely

$$\frac{b}{2} \to \left(\frac{b}{2}\right)^2 \to \left(\frac{b}{2}\right)^2 - c \to \sqrt{\left(\frac{b}{2}\right)^2 - c}.$$

Hence,

- either $\frac{b}{2} + \sqrt{(\frac{b}{2})^2 - c} = x_1$;
- or $\frac{b}{2} - \sqrt{(\frac{b}{2})^2 - c} = x_2$;

and both solutions are the root of the square.

Thereafter, al-Zanǧānī applies the procedure to the example "A square and twenty-nine units are equal to ten roots," i. e. $x^2 + 21 = 10x$. His justification by "the cause" is now formulated as follows:

> The cause of that corresponds to what we have explained in the propositions: given a number divided into two different parts, the product of one part by the other with the square of half their difference is equal to the square of half the number.[70]

[69] See on this subject Eleonora Sammarchi, "Additive and subtractive as relational entities in the algebra of al-Zanjānī (and his predecessors)," *Historia mathematica* (2024, online first), https://doi.org/10.1016/j.hm.2024.09.001.

[70] Trans. from al-Zanǧānī, *Balance de l'équation*, ed. Sammarchi, p. 161.

He applies the proposition to the aforementioned example:

> Here, ten is greater than the root because ten, multiplied by the root, becomes ten roots, which is equal to the product of the root by itself with twenty-one. You divide ten by the root and by the other number. The product of this number with the root – I mean the whole ten – by the root is the square with twenty-one. Therefore, if we remove from it the square that comes from the product of the root by the root, it will remain twenty-one, and this comes from the product of this number by the root. This root and the number are different, otherwise their product would have been equal to the square of half ten.[71]

Since $10x = x^2 + 21$, $10 > x$. Hence, $10 = x + a$, and $(x+a)x = x^2 + 21$. However, $ax = 21$, so $ax \neq (\frac{10}{2})^2$ which confirms $a \neq x$.

Al-Zanǧānī identifies the Euclidean property:

> So the product of this number and the root with the square of half the difference between the two is equal to the square of half ten.[72]

Hence, $xa + (\frac{x-a}{2})^2 = (\frac{x+a}{2})^2 = (\frac{10}{2})^2$. He concludes that:

> If we subtract twenty-one from the square of half ten, which corresponds to the product of that number by the root, there remains the square of half the difference. If we add its root to half ten, there results the greater part, and if we subtract it from it (i. e. the latter from the former), there remains the smaller part.[73]

In other words, since $(\frac{10}{2})^2 - 21 = (\frac{x-a}{2})^2 = 4$,

- either $\frac{10}{2} + \sqrt{4} = 7 = x_1$;
- or $\frac{10}{2} - \sqrt{4} = 3 = x_2$.

We can thus observe that al-Zanǧānī builds upon al-Karaǧī's work and develops a proof that is entirely conceived in arithmetical-algebraic terms. Finally, he shows that the other cases of the diorism do not require this lengthy proof, since

- if $c = (\frac{b}{2})^2$ then $\sqrt{c} = x$;
- if $c > (\frac{b}{2})^2$ then the problem would lead to an impossible subtraction

and would therefore be called "impossible" (*mustaḥil*).

Following the traditional account that dates back to al-Ḫwārizmī, he recalls the procedure for solving an equation of the form $ax^2 + c = bx$ where a is a multiple or fraction (i. e. "some squares" or "some parts of a square"). Therefore, he is able to show the validity of the procedure for all cases that he considers conceivable. Like al-Karaǧī, he concludes his account with the "Diophantine way."

[71] Trans. from al-Zanǧānī, *Balance de l'équation*, ed. Sammarchi, p. 162.
[72] Trans. from al-Zanǧānī, *Balance de l'équation*, ed. Sammarchi, p. 162.
[73] Trans. from al-Zanǧānī, *Balance de l'équation*, ed. Sammarchi, p. 162.

https://doi.org/10.1017/S0957423924000092 Published online by Cambridge University Press

Since al-Zanǧānī wrote this text in the middle of the 13th century, he benefited from a clearer organization of arithmetical knowledge, as well as from the development and systematization of algebraic methods for arithmetical and geometrical problems. Building upon the results of his predecessors, in these pages, al-Zanǧānī was able to elaborate the idea of an arithmetical "cause." This idea was already an intuition of al-Karaǧī and, in some sense, of Abū Kāmil as well. The "cause" is now totally internal to the algebraic book, and no longer needs any references to Euclidean authority.

5. CONCLUSION

The considerable number of quotations we have examined shows that textual analysis is a crucial component in the process of retracing the circulation of ancient texts. Our case studies add to the known examples of the Arabic indirect transmission of the *Elements* and, taken together, they highlight the complexity inherent in the study of this transmission. The Arabic reception of Euclidean works is far from being an unexplored field of investigation. However, we still lack much information regarding many of its actors and contributions. This urgent need of further studies is our preliminary conclusive remark. The philological-historical analysis of al-Karaǧī and of the authors related to him leads to the following conclusions.

When the Arabic versions of the *Elements* begun to circulate, algebraists such as Abū Kāmil viewed the mention of Euclid as a way to ground their own work and increase its cogency. This is especially true in relation to the use of propositions 5 and 6 of Book II within the study of equations. However, the Euclid that Abū Kāmil has in mind is already significantly different from the original one. The commentary of al-Nayrīzī – built upon Heron's commentary – and the reading of Book II developed by the Brethren in Purity – framed within a non-Euclidean tradition, such as the Neo-Pythagorean – are two other examples of a Euclid which was no longer Euclid. They provide evidence of the ability of the classics to inspire new results and to stimulated original contributions. As pointed out by Brentjes,

> It seems to be safe to assume that the medieval editors and commentators from the tenth century onwards relied upon more or less mixed and contaminated texts of the Arabic primary transmission of the Elements.[74]

[74] Brentjes, "Two comments on Euclid's *Elements*?" p. 50.

https://doi.org/10.1017/S0957423924000092 Published online by Cambridge University Press

It is very likely that this was also the case of al-Karağī. Despite our lack of information about his direct sources, we have identified striking similarities between this author and the earlier scholars mentioned just above. Just like them, al-Karağī refers the propositions of the *Elements* by radically changing their framework. Indeed, the properties originally identified for lines and surfaces are now transposed to numbers, the constructivism which characterizes Euclidean geometry is no longer relevant, and scholarly geometry becomes a toolbox for the development of arithmetical-algebraic practices. An additional nuance characterizes his later treatise *Al-badīᶜ*. Addressed to experienced algebraists, the book rephrases several parts of the *Elements*. We have seen that it is in this advanced study that al-Karağī considers the applicability of Euclidean propositions not only to arithmetical numbers, but also to algebraic entities such as the unknown quantity (*šayʾ*) and the algebraic square (*māl*). This shift to algebraic entities is explicitly stated only once in the text. However, it appears in an emblematic proposition for the study of equations.

Integrating Euclid within the existing arithmetical-algebraic tradition becomes an even more evident goal in the works of al-Zanğānī. The latter adopts the arithmetical reading of his master, but abandons the explicit reference to Euclid and grounds the whole justification of the equation on his own propositions. Consequently, the student-readers of "Balance of the equation" could find in a single algebraic work everything they needed to know in order to justify quadratic equations and solve all kinds of arithmetical problems via algebra.

Considering all this, we may conclude that the aim of the arithmetical-algebraic reading adopted by al-Karağī and his successor does not seem to be limited to the unification of geometrical magnitudes and numbers (i.e. to show that certain properties are valid for both magnitudes and numbers). It rather seems that the ultimate goal is to combine argumentative structures more broadly: that of scholarly Euclidean geometry (characterized by the analysis-synthesis distinction, by the theory of proportions, etc.) with that of computational practices (which relies on the generality of the procedure ensured by the selection of examples). According to this point of view, our investigation is closely related to the question of what constitutes a valid proof in the context of algebra.[75] It is precisely within this framework that we can understand the dual role of numerical examples which has emerged in our texts.

[75] See on this topic Karine Chemla, *The History of Mathematical Proof in Ancient Traditions* (Cambridge University Press, 2012).

Following the statement of the proposition, numerical examples replace the geometric figure and the deductive reasoning typical of geometric demonstration. They will thus at the same time exemplify and explain the rule (or the problem) for which they are formulated. This dual status was already present in texts on arithmetical practices; here, it becomes a way of dealing with scholarly content. Moreover, the same numerical examples are reproduced from one author to another. In the cases we have examined, the most frequently used example is that of the division of 10 into 7 and 3, with the conditions of the specific proposition being applied on these parts. Identical examples are also repeated from one text to another in order to present the six algebraic problems (i. e. the six forms of quadratic equations).

As I have already stated, the present investigation makes no claim to exhaustiveness. Its goal has been rather to highlight the plurality intrinsic in the name "Euclid" within medieval texts and to refine our understanding of the development and nature of algebra, while also opening up lines of future research into the role of examples, the notion of rigor, and the circulation of Greek traditions (certainly Euclidean, but also metrological, Neo-Pythagorean, Diophantine, etc.) in Islamicate mathematics.

Acknowledgements. This work has been inspired by the exchanges I had over the years with Pascal Crozet and Sabine Rommevaux-Tani, to whom I extend my gratitude. I also thank Vincenzo De Risi for his comments and remarks on the latest version of this paper.