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A NOTE ON THE LARGE SIEVE INEQUALITY FOR MODULI GENERATED BY A QUADRATIC

C. C. CORRIGAN

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Abstract

We develop a generalisation of the square sieve of Heath-Brown and use it to give an alternate proof of one of the large sieve inequalities in our previous paper ['A large sieve inequality for characters to quadratic moduli', Preprint, https://web.maths.unsw.edu.au/~ccorrigan/preprint6.pdf].

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1. Introduction

Applications in the literature of the large sieve inequality for sparse sets of moduli are vast, particularly in the cases where the sparse set is well distributed in the residue classes. In these cases, the set of moduli is usually generated by some function $f : \mathbb{N} \hookrightarrow \mathbb{N}$ with certain nice arithmetical properties. The classical large sieve inequality of Davenport and Halberstam [6] trivially gives

$$\sum_{q \leqslant Q} \sum_{\substack{a \leqslant f(q) \\ (a, f(q))=1}} \bigg| \sum_{n \leqslant N} z_n \mathbf{e}\Big(\frac{an}{f(q)}\Big) \bigg|^2 \ll \min(Qf(Q) + QN, f(Q)^2 + N) \sum_{n \leqslant N} |z_n|^2 \tag{1.1}$$

for any strictly increasing function $f : \mathbb{N} \hookrightarrow \mathbb{N}$. Here and in the remainder of this article, we write $\mathbf{e}(\alpha) = \exp(2\pi i \alpha)$ and we suppose that $Q, N \geqslant 3$ are large and that $(z_n)_{n \leqslant N}$ is an arbitrary nonzero sequence of complex numbers.

The case of square moduli was first studied by Zhao [9], using techniques from harmonic analysis. Later, following a combinatorial argument, Baier [1] showed that

$$\sum_{q \leqslant Q} \sum_{\substack{a \leqslant q^2 \\ (a,q)=1}} \bigg| \sum_{n \leqslant N} z_n \mathbf{e}\Big(\frac{an}{q^2}\Big) \bigg|^2 \ll_{\varepsilon} Q^{\varepsilon}(Q^3 + N + Q^2\sqrt{N}) \sum_{n \leqslant N} |z_n|^2, \tag{1.2}$$

which was further improved by Baier and Zhao [3]. Now, following the arguments of Baker [4], a bound analogous to (1.2), pertaining to monomials f of degree two, can easily be established. The general case, however, requires more consideration. In [5], adapting the combinatorial argument of Baier [1], we showed that

$$\sum_{q\leqslant Q}\sum_{\substack{a\leqslant f(q)\\(a,f(q))=1}}\Big|\sum_{n\leqslant N} z_n\mathbf{e}\Big(\frac{an}{f(q)}\Big)\Big|^2 \ll_{\varepsilon} f(Q)^{\varepsilon}(Qf(Q)+N+f(Q)\sqrt{N})\sum_{n\leqslant N}|z_n|^2, \tag{1.3}$$

where $f:\mathbb{N}\hookrightarrow\mathbb{N}$ is an arbitrary monotonic polynomial of degree two. In [2], Baier showed that (1.2) can be established using the square sieve of Heath-Brown [7] and some classical techniques from harmonic analysis. Our objective in this article is to show that this approach can also be used to establish (1.3). To this end, we first require a generalisation of the square sieve.

Remark on notation. In the following, we will denote by $\omega(n)$ and $\tau(n)$ the number of prime divisors and positive divisors of n, respectively. Additionally, ε will be used to denote an arbitrarily small positive constant, and may vary in value throughout.

2. Preliminary lemmata

In this section, we shall prove two simple results which will be the main tools used for our proof of (1.3) in the following section. First, we have the following generalisation of the square sieve of Heath-Brown [7].

LEMMA 2.1. *Suppose that $f:\mathbb{N}\hookrightarrow\mathbb{N}$ is a strictly increasing polynomial of degree two, with leading coefficient A and discriminant Δ_f. Moreover, suppose that $\mathscr{P}$ is a set of $P\geqslant 1$ primes and that $\phi:\mathbb{N}\to\mathbb{R}_+$ satisfies $\phi(n)=0$ whenever n is such that $4Af(n)+\Delta_f>e^P$. Then, we have the majorisation*

$$\sum_n \phi\circ f(n) \ll P^{-1}\sum_n\phi(n) + P^{-2}\sum_{\substack{p,p'\in\mathscr{P}\\p\neq p'}}\Big|\sum_n\phi(n)\Big(\frac{4An+\Delta_f}{pp'}\Big)\Big|,$$

where the implied constant is absolute.

PROOF. Suppose that n is a natural number satisfying $4Af(n)+\Delta_f\leqslant e^P$. Since $4Af(n)+\Delta_f=f'(n)^2$ for all natural numbers n,

$$\sum_{p\in\mathscr{P}}\Big(\frac{4Af(n)+\Delta_f}{p}\Big) = \sum_{\substack{p\in\mathscr{P}\\p\nmid 4Af(n)+\Delta_f}} 1 \geqslant P-\omega(4Af(n)+\Delta_f)\gg P,$$

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by virtue of the fact that $\omega(q) = o(\log q)$. Hence,

$$\begin{aligned}\sum_n \phi \circ f(n) &\ll P^{-2} \sum_n \phi(n)\Big(\sum_{p \in \mathscr{P}} \Big(\frac{4An + \Delta_f}{p}\Big)\Big)^2 \\ &= P^{-2} \sum_{p,p' \in \mathscr{P}} \sum_n \phi(n)\Big(\frac{4An + \Delta_f}{pp'}\Big) \\ &= P^{-2}\Big(\sum_{p \in \mathscr{P}} \sum_{\substack{n \\ p \nmid 4An+\Delta_f}} \phi(n) + \sum_{\substack{p,p' \in \mathscr{P} \\ p \neq p'}} \sum_n \phi(n)\Big(\frac{4An + \Delta_f}{pp'}\Big)\Big),\end{aligned}$$

from which the assertion follows immediately. □

Second, we have the following result pertaining to sums of values of the Jacobi symbol in an arithmetic progression, twisted by additive characters.

LEMMA 2.2. *Suppose that $M, N \geqslant 3$ are coprime and let χ be the Jacobi symbol modulo M. Moreover, suppose that B and C each belong to one of the primitive residue classes modulo M. Then, for any integer Δ,*

$$\sum_{r \leqslant MN} \chi(Br + \Delta)\mathbf{e}\Big(\frac{Cr}{MN}\Big) = \delta_{N|C}\mathbf{e}\Big(\frac{-C\bar{B}\Delta}{MN}\Big)\chi(CBN)\tau(\chi)N,$$

where $\mathbf{e}(B\bar{B}/M) = \mathbf{e}(1/M)$, and $\tau(\chi)$ denotes the Gauß sum of the character χ.

PROOF. It suffices to consider the case where $\Delta = 0$, for from this, the remaining cases follow by applying the translation $r \mapsto r - \bar{B}\Delta$. So, on noting that

$$\{mN + nM : m \leqslant M, n \leqslant N\}/(MN\mathbb{Z}) = \{r \leqslant MN\}/(MN\mathbb{Z})$$

whenever $(M, N) = 1$, we see that

$$\begin{aligned}\sum_{r \leqslant MN} \chi(Br)\mathbf{e}\Big(\frac{Cr}{MN}\Big) &= \sum_{m \leqslant M} \sum_{n \leqslant N} \chi(BmN)\mathbf{e}\Big(\frac{C(mN + nM)}{MN}\Big) \\ &= \chi(CBN) \sum_{m \leqslant M} \chi(Cm)\mathbf{e}\Big(\frac{Cm}{M}\Big) \sum_{n \leqslant N} \mathbf{e}\Big(\frac{Cn}{N}\Big),\end{aligned}$$

from which the assertion immediately follows. □

Having established our two main preliminaries, we shall now move on to the demonstration of (1.3). Our approach will closely follow the work of Baier [2], so we will keep brief our treatment of the lesser details.

3. Demonstration

Similarly to [5], we start by breaking the sum over $q \leqslant Q$ in (1.3) into $O(\log f(Q))$ intervals of the form $\mathscr{Q}_f(M) = \{q \leqslant Q : f(q) \sim M\}$, where $1 \ll M \ll f(Q)$. Let $\mathscr{A}$ be the set of Farey fractions with denominator in $\mathscr{Q}_f(M)$ and, for any real α and any small $\delta > 0$, define

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$$P_\delta(\alpha) = \#\{\alpha' \in \mathscr{A} : |\alpha - \alpha'| \leqslant \delta\}.$$

Following the standard procedure laid down by Wolke [8], we see that to establish (1.3), it suffices to show that the bound

$$\sup_{\alpha\in\mathbb{R}} P_\delta(\alpha) \ll_\varepsilon f(Q)^\varepsilon M\sqrt{\delta} \tag{3.1}$$

holds whenever $M^{-2} \leqslant \delta \leqslant M^{-1}$. As in [5], we note here that, in the case where δ is outside of the aforementioned range, (1.3) is contained in the trivial bound (1.1). Now, in the following, we may assume that α does not belong to the set

$$\mathfrak{M} = \bigcup_{r\leqslant 1/(128M\delta)} \bigcup_{\substack{b\leqslant r\\(b,r)=1}} B(b/r, 1/(8Mr))$$

of major arcs, for if $\alpha \in B(b/r, 1/(8Mr))$, then since $\delta \leqslant 1/(128Mr)$, we must have

$$P_\delta(\alpha) \leqslant \#\{\alpha \in \mathscr{A} : |b/r - \alpha| \leqslant 1/(4Mr)\} \leqslant \#\{a/f(q) \in \mathscr{A} : |bf(q) - ar| \leqslant \tfrac{1}{2}\},$$

which is clearly zero, since $r \leqslant (1/128)M < f(q)$. Hence, in the remainder of this section, we shall assume that α belongs to the set $\mathfrak{m} = [0,1]\backslash\mathfrak{M}$ of minor arcs.

By Dirichlet's approximation theorem, we see that for all $\alpha \in [0,1]$, there exists a Farey fraction b/r with $r \leqslant 128M$ such that $|b/r - \alpha| \leqslant 1/(128Mr)$. If $r \leqslant 1/(128M\delta)$, we must have $\alpha \in \mathfrak{M}$, so we may assume that this is not the case. Hence, for all $\alpha \in \mathfrak{m}$, there exists a Farey fraction b/r such that $|b/r - \alpha| < \delta$, and thus, we must have $P_\delta(\alpha) \leqslant P_{2\delta}(b/r)$.

We now fix a Farey fraction b/r, and suppose that $\phi, \psi : \mathbb{R} \to \mathbb{R}_+$ are two infinitely differentiable functions having support on $[\frac{1}{2}, \frac{5}{2}]$ and $[-\frac{9}{2}, \frac{9}{2}]$, respectively. Moreover, we suppose that ϕ and ψ are bounded below by 1 on the intervals $[1,2]$ and $[-4,4]$, respectively. Then, we have the majorisation

$$P_{2\delta}(b/r) \ll \sum_{q\in\mathbb{Z}}\sum_{a\in\mathbb{Z}} \phi\Big(\frac{f(q)}{M}\Big)\psi\Big(\frac{ar - bf(q)}{Mr\delta}\Big). \tag{3.2}$$

Now, supposing that $R > f(Q)^\varepsilon$, we note that the set $\mathscr{P} = \{p \sim R : p \nmid 4rA\}$ has cardinality $P \sim R/\log R$, and thus applying Lemma 2.1 to (3.2) yields

$$\begin{aligned} P_{2\delta}(b/r) \ll P^{-1} &\sum_{q\in\mathbb{Z}}\sum_{a\in\mathbb{Z}} \phi\Big(\frac{q}{M}\Big)\psi\Big(\frac{ar - bq}{Mr\delta}\Big) \\ &+ P^{-2} \sum_{\substack{p,p'\in\mathscr{P}\\ p\neq p'}} \Big|\sum_{q\in\mathbb{Z}}\sum_{a\in\mathbb{Z}} \Big(\frac{4Aq + \Delta_f}{pp'}\Big)\phi\Big(\frac{q}{M}\Big)\psi\Big(\frac{ar - bq}{Mr\delta}\Big)\Big|. \end{aligned} \tag{3.3}$$

For the sake of brevity, we shall denote by Σ_1 and Σ_2 the inner double sums of the first and second terms, respectively, on the right-hand side of (3.3).

To treat the sum Σ_1, we first note that if $a \in \mathbb{Z}$ is such that $|ar - bq| \leqslant \frac{9}{2}Mr\delta$ for some $q \in \mathbb{Z}$ satisfying $\frac{1}{2}M \leqslant q \leqslant \frac{5}{2}M$, then $|a/q - \alpha| \leqslant |b/r - \alpha| + 9\delta \leqslant 11\delta$. If, moreover, $(a,q) > 320M^2\delta$, then $q/(a,q) < 1/(128M\delta)$ and thus $11\delta < (a,q)/(8Mq)$.

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In this case, we clearly have $\alpha \in B(a/q, (a,q)/(8Mq)) \subset \mathfrak{M}$, which contradicts our assumption that $\alpha \in \mathfrak{m}$. Hence, we see that the a, q in the double sum Σ_1 must all satisfy $(a,q) \leqslant 320M^2\delta$, and thus

$$\Sigma_1 \leqslant \sum_{\frac{1}{2}M\leqslant q\leqslant \frac{5}{2}M} \sum_{qb/r-\frac{9}{2}M\delta\leqslant a\leqslant qb/r+\frac{9}{2}M\delta} 1 \leqslant \sum_{m\leqslant 320M^2\delta} \sum_{\frac{1}{2}M/m\leqslant q\leqslant \frac{5}{2}M/m} \sum_{\substack{qb/r-\frac{9}{2}M\delta\leqslant a\leqslant qb/r+\frac{9}{2}M\delta \\ (a,q)=1}} 1.$$

On noting that $|a/q - a'/q'| \geqslant 1/(qq') \asymp m^2/M^2$ for any distinct Farey fractions a/q and a'/q' satisfying $q, q' \asymp M/m$, we derive the bound

$$\Sigma_1 \ll \sum_{m\leqslant 320M^2\delta} \Big(1 + \frac{M^2\delta}{m^2}\Big) \ll M^2\delta, \tag{3.4}$$

which completes our treatment of Σ_1.

To treat the double sum Σ_2, we first split the outer sum into subsums over the residue classes modulo $pp'r$, and twice apply the Poisson summation formula to obtain the transformation

$$\begin{aligned}\Sigma_2 &= \sum_{m\leqslant pp'r} \Big(\frac{4Am+\Delta_f}{pp'}\Big) \sum_{q\in\mathbb{Z}} \sum_{a\in\mathbb{Z}} \phi\Big(\frac{m+pp'rq}{M}\Big)\psi\Big(\frac{ar-b(m+pp'rq)}{Mr\delta}\Big) \\ &= \frac{M^2\delta}{pp'r} \sum_{m\leqslant pp'r} \Big(\frac{4Am+\Delta_f}{pp'}\Big) \sum_{q\in\mathbb{Z}} \sum_{a\in\mathbb{Z}} \hat{\phi}\Big(\frac{M(q+bapp')}{pp'r}\Big)\hat{\psi}(M\delta a)\mathbf{e}\Big(\frac{mq}{pp'r}\Big) \\ &= \frac{M^2\delta}{pp'r} \sum_{q\in\mathbb{Z}} \sum_{a\in\mathbb{Z}} \hat{\phi}\Big(\frac{Mq}{pp'r}\Big)\hat{\psi}(M\delta a) \sum_{m\leqslant pp'r} \Big(\frac{4Am+\Delta_f}{pp'}\Big)\mathbf{e}\Big(\frac{m(q-bapp')}{pp'r}\Big).\end{aligned}$$

Hence, by Lemma 2.2 and the standard bound for Gauß sums,

$$\Sigma_2 \ll \frac{M^2\delta}{\sqrt{pp'}} \sum_{q\in\mathbb{Z}} \sum_{a\in\mathbb{Z}} \delta_{r|(q-bapp')}\hat{\phi}\Big(\frac{Mq}{pp'r}\Big)\hat{\psi}(M\delta a), \tag{3.5}$$

which, when averaged over distinct primes $p, p' \in \mathscr{P}$, will suffice for our argument. So, first, on combining (3.4) and (3.5) with (3.3), we obtain the majorisation

$$P_{2\delta}(b/r) \ll \frac{M^2\delta}{P} + \frac{M^2\delta}{P^2R} \sum_{\substack{p,p'\in\mathscr{P} \\ p\neq p'}} \sum_{\substack{q\in\mathbb{Z} \\ r|(q-bapp')}} \sum_{a\in\mathbb{Z}} \hat{\phi}\Big(\frac{Mq}{pp'r}\Big)\hat{\psi}(M\delta a). \tag{3.6}$$

Now, since ϕ and ψ are supported only on singular closed intervals, we see that their respective Fourier transforms must satisfy the property that $\hat{\phi}(x)$ and $\hat{\psi}(x)$ are, for any $C > 0$, majorised by $(1+|x|)^{-C}$. Consequently, we may truncate the triple sum on the right-hand side of (3.6) to derive the bound

$$\sum_{\substack{p,p'\in\mathscr{P} \\ p\neq p'}} \sum_{\substack{q\in\mathbb{Z} \\ r|(q-bapp')}} \sum_{a\in\mathbb{Z}} \hat{\phi}\Big(\frac{Mq}{pp'r}\Big)\hat{\psi}(M\delta a) \ll_{\varepsilon} f(Q)^{-8} + \sum_{R^2<m\leqslant 4R^2} \sum_{\substack{|q|<R^2rf(Q)^{\varepsilon}/M \\ |a|<f(Q)^{\varepsilon}/(M\delta) \\ r|(q-bam)}} \sum 1,$$

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by virtue of the fact that $M \ll f(Q)$. Hence, on noting that

$$\begin{aligned}\sum_{R^2<m\leqslant 4R^2}\ \sum_{\substack{|q|<R^2rf(Q)^\varepsilon/(M)\\ |a|<f(Q)^\varepsilon/(M\delta)\\ r|(q-bam)}}\sum 1 &\ll \Big(R^2+\frac{R^4f(Q)^\varepsilon}{M}\Big)+\sum_{|q|<R^2rf(Q)^\varepsilon/M}\ \sum_{\substack{R^2<m\leqslant 4R^2\\ a<f(Q)^\varepsilon/(M\delta)\\ r|(q-bam)}}\sum 1\\ &\ll \Big(R^2+\frac{R^4f(Q)^\varepsilon}{M}\Big)+\sum_{|q|<R^2rf(Q)^\varepsilon/M}\ \sum_{\substack{m<R^2f(Q)^\varepsilon/(M\delta)\\ r|(q-bm)}}\tau(m)\\ &\ll_\varepsilon \Big(R^2+\frac{R^4f(Q)^\varepsilon}{M}\Big)+R^\varepsilon f(Q)^\varepsilon\Big(1+\frac{R^2r}{M}\Big)\Big(1+\frac{R^2}{Mr\delta}\Big)\\ &\ll_\varepsilon R^\varepsilon f(Q)^\varepsilon\Big(R^2+\frac{R^4}{M^2\delta}\Big),\end{aligned}$$

we obtain from (3.6) the majorisation

$$\sup_{\alpha\in\mathfrak{m}} P_\delta(\alpha) \ll_\varepsilon R^\varepsilon f(Q)^\varepsilon\Big(\frac{M^2\delta}{R}+R\Big). \tag{3.7}$$

Taking $R = M\sqrt{\delta}$ in (3.7) yields (3.1), and thus the assertion (1.3) follows.

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C. C. CORRIGAN, School of Mathematics and Statistics,
University of New South Wales, Sydney, NSW 2052, Australia
e-mail: c.corrigan@student.unsw.edu.au