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RESEARCH ARTICLE

A remark on Gibbs measures with log-correlated Gaussian fields

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Abstract
We study Gibbs measures with log-correlated base Gaussian fields on the d-dimensional torus. In the defocusing case, the construction of such Gibbs measures follows from Nelson's argument. In this paper, we consider the focusing case with a quartic interaction. Using the variational formulation, we prove nonnormalizability of the Gibbs measure. When $d = 2$, our argument provides an alternative proof of the nonnormalizability result for the focusing Φ^4_2-measure by Brydges and Slade (1996). Furthermore, we provide a precise rate of divergence, where the constant is characterized by the optimal constant for a certain Bernstein's inequality on $\mathbb{R}^d$. We also go over the construction of the focusing Gibbs measure with a cubic interaction. In the appendices, we present (a) nonnormalizability of the Gibbs measure for the two-dimensional Zakharov system and (b) the construction of focusing quartic Gibbs measures with smoother base Gaussian measures, showing a critical nature of the log-correlated Gibbs measure with a focusing quartic interaction.

Contents

1 Introduction 2
1.1 Log-correlated Gibbs measures . . . 2
1.2 Dynamical problems associated with the log-correlated Gibbs measures . . . 4
1.3 Nonnormalizability of the focusing Gibbs measure . . . 6
1.4 Gibbs measure with the cubic interaction . . . 8
2 Preliminary lemmas 10
2.1 Deterministic estimates . . . 11
2.2 Tools from stochastic analysis . . . 13
3 Nonnormalizability of the focusing Gibbs measure with the quartic interaction 15
3.1 Variational formulation . . . 15
3.2 Proof of Theorem 1.4 . . . 17
3.3 Proofs of the auxiliary lemmas . . . 25
4 Construction of the Gibbs measure with the cubic interaction 29
A On the Gibbs measure for the two-dimensional Zakharov system 31
B Focusing quartic Gibbs measures with smoother Gaussian fields 36
References 38

1. Introduction

1.1. Log-correlated Gibbs measures

In this paper, we study the Gibbs measure ρ on the d-dimensional torus on $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$, formally written as[1]

$$d\rho(u) = Z^{-1} \exp\left(\frac{\lambda}{k} \int_{\mathbb{T}^d} u^k dx\right) d\mu(u), \tag{1.1}$$

where $k \geq 3$ is an integer and the coupling constant $\lambda \in \mathbb{R} \setminus \{0\}$ denotes the strength of interaction, which is repulsive (i.e., defocusing) when $\lambda < 0$ and k is even, and is attractive (i.e., focusing) when $\lambda > 0$ or k is odd.[2] Here, μ is the log-correlated Gaussian free field on $\mathbb{T}^d$, formally given by

$$d\mu = Z^{-1} e^{-\frac{1}{2}\|u\|^2_{H^{d/2}}} du = Z^{-1} \prod_{n \in \mathbb{Z}^d} e^{-\frac{1}{2}\langle n \rangle^d |\widehat{u}(n)|^2} d\widehat{u}(n), \tag{1.2}$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$ and $\widehat{u}(n)$ denotes the Fourier coefficient of u. When $d = 2$, μ corresponds to the massive Gaussian free field on $\mathbb{T}^2$. Recall that this Gaussian measure μ is nothing but the induced probability measure under the map:[3]

$$\omega \in \Omega \longmapsto u(\omega) = \sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)}{\langle n \rangle^{\frac{d}{2}}} e_n, \tag{1.3}$$

where $e_n = e^{in\cdot x}$ and $\{g_n\}_{n \in \mathbb{Z}^d}$ is a sequence of mutually independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ conditioned that $g_{-n} = \overline{g_n}$.[4] See Remark 1.1. It is well known that a typical function u in the support of μ is merely a distribution and thus a renormalization on the potential energy $\frac{\lambda}{k} \int_{\mathbb{T}^d} u^k dx$ is required for the construction of the Gibbs measure ρ.

Our main goal in this paper is to study the Gibbs measure ρ in (1.1) in the focusing case. In particular, we prove the nonnormalizability of the focusing Gibbs measure ρ with the quartic interaction ($\lambda > 0$ and $k = 4$). See Theorem 1.4. We also present a brief discussion on the construction of the Gibbs measure with the cubic interaction. See Theorem 1.9.

Before proceeding further, let us first go over the defocusing case: $\lambda < 0$ and $k \geq 4$ is an even integer. When $d = 2$, the defocusing Gibbs measure ρ in (1.1) corresponds to the well-studied Φ^k_2-measure whose construction follows from the hypercontractivity of the Ornstein–Uhlenbeck semigroup (see Lemma 2.3) and Nelson's estimate [39]. See [60, 26, 21, 47]. For a general dimension $d \geq 1$, the same argument allows us to construct the defocusing Gibbs measure ρ in (1.1) for any $\lambda < 0$ and any even integer $k \geq 4$. Let us briefly go over the procedure.

Given $N \in \mathbb{N}$, we define the frequency projector[5] π_N by

$$\pi_N f = \sum_{|n| \leq N} \widehat{f}(n) e_n. \tag{1.4}$$

[1]In this introduction, we keep our discussion at a formal level and do not worry about renormalizations. While we keep the following discussion only to the real-valued setting, our results also hold in the complex-valued setting, where $k \geq 4$ is an even integer and u^k in (1.1) is replaced by $|u|^k$. See Footnote 6.
Hereafter, we use Z, Z_N, etc. to denote various normalization constants whose values may change line by line.
[2]In this paper, by 'focusing', we mean 'nondefocusing'. Namely, $\lambda > 0$ or k is odd in (1.1).
[3]By convention, we endow $\mathbb{T}^d$ with the normalized Lebesgue measure $dx_{\mathbb{T}^d} = (2\pi)^{-d} dx$ so that we do not need to worry about the factor 2π in various places. For simplicity of notation, we use dx to denote the standard Lebesgue measure $\mathbb{R}^d$ and the normalized Lebesgue measure on $\mathbb{T}^d$ in the following.
[4]In particular, g_0 is a standard real-valued Gaussian random variable. When $n \in \mathbb{N}$, Re g_n and Im g_n are real-valued Gaussian random variables with mean 0 and variance $\frac{1}{2}$.
[5]We may also proceed with regularization via mollification.

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For u as in (1.3), set $u_N = \pi_N u$. Then, for each fixed $x \in \mathbb{T}^d$, $u_N(x)$ is a mean-zero real-valued Gaussian random variable with variance

$$\sigma_N = \mathbb{E}\big[u_N^2(x)\big] = \sum_{|n| \le N} \frac{1}{\langle n \rangle^d} \sim \log N \longrightarrow \infty, \tag{1.5}$$

as $N \longrightarrow \infty$. Note that σ_N is independent of $x \in \mathbb{T}^d$ in the current translation invariant setting. We then define the renormalized power (= Wick power) $:u_N^k:$ by setting

$$:u_N^k(x): \stackrel{\text{def}}{=} H_k(u_N(x); \sigma_N), \tag{1.6}$$

where $H_k(x;\sigma)$ is the Hermite polynomial of degree k with a variance parameter σ defined through the following generating function:[6]

$$F(t,x;\sigma) \stackrel{\text{def}}{=} e^{tx - \frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma). \tag{1.7}$$

For readers' convenience, we write out the first few Hermite polynomials:

$$H_0(x;\sigma) = 1, \quad H_1(x;\sigma) = x, \quad H_2(x;\sigma) = x^2 - \sigma, \quad H_3(x;\sigma) = x^3 - 3\sigma x.$$

See, for example, [32], for further properties of the Hermite polynomials. We then define the following renormalized truncated potential energy:

$$R_N(u) = \frac{\lambda}{k} \int_{\mathbb{T}^d} :u_N^k: \, dx, \tag{1.8}$$

where the coupling constant $\lambda < 0$ denotes the strength of repulsive interaction. A standard computation allows us to show that $\{R_N\}_{N \in \mathbb{N}}$ forms a Cauchy sequence in $L^p(\mu)$ for any finite $p \ge 1$, thus converging to some random variable $R(u)$:

$$\lim_{N \to \infty} R_N(u) = R(u) \tag{1.9}$$

in $L^p(\mu)$ and almost surely See, for example, Proposition 1.1 in [47].[7]

Define the renormalized truncated Gibbs measure ρ_N by

$$d\rho_N(u) = Z_N^{-1} e^{R_N(u)} d\mu(u).$$

Then, a standard application of Nelson's estimate[8] yields the following uniform exponential integrability of the density; given any finite $p \ge 1$, there exists $C_{p,d} > 0$ such that

$$\sup_{N \in \mathbb{N}} \left\| e^{R_N(u)} \right\|_{L^p(\mu)} \le C_{p,d} < \infty. \tag{1.10}$$

See, for example, Proposition 1.2 in [47]. Then, the uniform bound (1.10) together with softer convergence in measure (as a consequence of (1.9)) implies the following L^p-convergence of the density:

[6]In the complex-valued setting (with even k), we use the Laguerre polynomial $c_k L_{\frac{k}{2}}(|u_N|^2; \sigma_N)$ to define the Wick renormalization. See [47].

[7]The claimed almost sure convergence follows form the $L^p(\Omega)$-convergence in [47, Proposition 1.1] together with the Borel–Cantelli lemma.

[8]One may also prove the uniform exponential integrability bound (1.10) via the variational approach as in [3], using the Boué–Dupuis variational formula (Lemma 3.1). When k is large, however, the combinatorial complexity for the variational approach may be cumbersome, while there is no such combinatorial issue in the approach of [21, 47].

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$$\lim_{N\to\infty} e^{R_N(u)} = e^{R(u)} \qquad \text{in } L^p(\mu).$$

See, for example, Remark 3.8 in [66]. This allows us to construct the defocusing Gibbs measure:

$$d\rho(u) = Z^{-1} e^{R(u)} d\mu(u)$$

as a limit of the truncated defocusing Gibbs measure ρ_N.

As mentioned above, our main goal is to study the Gibbs measure ρ with the log-correlated Gaussian field μ in the focusing case ($\lambda > 0$). Before doing so, we present a brief discussion on dynamical problems associated with these Gibbs measures in Subsection 1.2. We then present the nonnormalizability of the focusing log-correlated Gibbs measure with the quartic interaction (Theorem 1.4) and the construction of the focusing log-correlated Gibbs measure with the cubic interaction (Theorem 1.9).

Remark 1.1. Recall from [2, (4,2)] that the Green's function $G_{\mathbb{R}^d}$ for $(1-\Delta)^{\frac{d}{2}}$ on $\mathbb{R}^d$ satisfies

$$G_{\mathbb{R}^d}(x) = -c_d \log|x| + o(1) \tag{1.11}$$

as $x \longrightarrow 0$ for some $c_d > 0$. Here, in view of the translation invariance, we view G as a function of one variable through $G(x) \equiv G(x,0)$. It is a smooth function on $\mathbb{R}^d \setminus \{0\}$ and decays exponentially as $|x| \longrightarrow \infty$; see [28, Proposition 1.2.5].

Now, let G be the Green's function for $(1-\Delta)^{\frac{d}{2}}$ on $\mathbb{T}^d$. Then, we have

$$G \overset{\text{def}}{=} (1-\Delta)^{-\frac{d}{2}}\delta_0 = \sum_{n\in\mathbb{Z}^d} \frac{1}{\langle n\rangle^d} e_n = \lim_{N\to\infty} \sum_{|n|\le N} \frac{1}{\langle n\rangle^d} e_n. \tag{1.12}$$

Recall the Poisson summation formula ([27, Theorem 3.2.8]):

$$\sum_{n\in\mathbb{Z}^d} \mathcal{F}_{\mathbb{R}^d}(f)(n) e_n(x) = \sum_{m\in\mathbb{Z}^d} f(x+2\pi m), \qquad x\in\mathbb{R}^d, \tag{1.13}$$

for any function f on $\mathbb{R}^d$ such that $|f(x)| \lesssim \langle x\rangle^{-d-\delta}$ for some $\delta > 0$ and $\sum_{n\in\mathbb{Z}^d} |\mathcal{F}_{\mathbb{R}^d}(f)(n)| < \infty$. The Poisson summation formula (1.13) is a typical tool to pass information from $\mathbb{R}^d$ to a periodic torus $\mathbb{T}^d$; see [4, 50, 5] for example. Here, $\mathcal{F}_{\mathbb{R}^d}(f)(n)$ denotes the Fourier transform of f on $\mathbb{R}^d$ given by

$$\mathcal{F}_{\mathbb{R}^d}(f)(n) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(x) e_{-n}(x) dx, \tag{1.14}$$

where $dx = dx_{\mathbb{R}^d}$ is the standard Lebesgue measure on $\mathbb{R}^d$. Then, by applying (1.13) (with a frequency truncation π_N and taking $N \longrightarrow \infty$) together with the asymptotics (1.11), we conclude that there exists a smooth function R such that

$$G(x) = -c_d \log|x| + R(x) \tag{1.15}$$

for any $x \in \mathbb{T}^d \setminus \{0\}$. See [44, Section 2] for a related discussion. Finally, from (1.3), (1.12) and (1.15), we obtain

$$\mathbb{E}_\mu\big[u(x)u(y)\big] = G(x-y) = -c_d \log|x-y| + R(x-y)$$

for any $x, y \in \mathbb{T}^d$ with $x \neq y$.

1.2. Dynamical problems associated with the log-correlated Gibbs measures

From the viewpoint of mathematical physics such as Euclidean quantum field theory, the construction of the Gibbs measures ρ in (1.1) is of interest in its own right. In this subsection, we briefly discuss some

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examples of dynamical problems associated with these log-correlated Gibbs measures. These examples show the importance of studying the log-correlated Gibbs measure ρ in (1.1) from the (stochastic) partial differential equation (PDE) point of view.

The associated energy functional[9] for the Gibbs measure ρ in (1.1) is given by

$$E(u) = \frac{1}{2}\int_{\mathbb{T}^d} |(1-\Delta)^{\frac{d}{4}}u|^2 dx - \frac{\lambda}{k}\int_{\mathbb{T}^d} u^k dx. \tag{1.16}$$

The study of the Gibbs measures for Hamiltonian PDEs, initiated by [25, 33, 8, 38, 11], has been an active field of research over the last decade. We first list examples of the Hamiltonian PDEs generated by this energy functional $E(u)$ in (1.16) along with the references.

(i) fractional nonlinear Schrödinger equation (for complex-valued u):

$$i\partial_t u + (1-\Delta)^{\frac{d}{2}}u - \lambda|u|^{k-2}u = 0. \tag{1.17}$$

Equation (1.17) corresponds to the nonlinear half-wave equation (also known as the semirelativistic nonlinear Schrödinger equation (NLS)) when $d = 1$, to the well-studied cubic NLS when $d = 2$ ([11, 47, 23]), and to the biharmonic NLS when $d = 4$.

In Appendix A, we also provide a brief discussion on the Gibbs measure for the Zakharov system when $d = 2$.

(ii) fractional nonlinear wave equation (NLW):[10]

$$\partial_t^2 u + (1-\Delta)^{\frac{d}{2}}u - \lambda u^{k-1} = 0. \tag{1.18}$$

Equation (1.18) corresponds to the NLW equation (or the nonlinear Klein–Gordon equation) when $d = 2$ ([48]), and to the nonlinear beam equation when $d = 4$.

(iii) generalized Benjamin–Ono equation (with $d = 1$):[11]

$$\partial_t u + \mathcal{H}\partial_x^2 u - \lambda\partial_x(u^{k-1}) = 0, \tag{1.19}$$

where $\mathcal{H}$ denotes the Hilbert transform defined by $\widehat{\mathcal{H}f}(n) = -i\operatorname{sgn}(n)\widehat{f}(n)$ with the understanding that $\widehat{\mathcal{H}f}(0) = 0$. Equation (1.19) is known as the Benjamin–Ono equation when $k = 3$ ([67, 22]) and the modified Benjamin–Ono equation when $k = 4$.

Next, we list stochastic PDEs associated with the Gibbs measure ρ in (1.1).

(iv) parabolic stochastic quantization equation [52]:

$$\partial_t u + (1-\Delta)^{\frac{d}{2}}u - \lambda u^{k-1} = \sqrt{2}\xi. \tag{1.20}$$

Here, ξ denotes the space-time white noise on $\mathbb{T}^d \times \mathbb{R}_+$. When $d = 2$ and $\lambda < 0$, (1.20) corresponds to the standard parabolic Φ_2^k-model ([20, 56, 65]).

(v) canonical stochastic quantization equation [57]:

$$\partial_t^2 u + \partial_t u + (1-\Delta)^{\frac{d}{2}}u - \lambda u^{k-1} = \sqrt{2}\xi. \tag{1.21}$$

[9]Once again, we do not worry about renormalizations in this formal discussion.

[10]For (1.18), we need to add $\frac{1}{2}\int_{\mathbb{T}^d}(\partial_t u)^2 dx$ to the energy functional $E(u)$ in (1.16).

[11]For (1.19), the coefficient of the potential energy in (1.16) is slightly different. Thanks to the conservation of the spatial mean $\int_{\mathbb{T}} u\,dx$ under the generalized Benjamin–Ono (1.19), we can work on the mean-zero functions. In this case, we consider the Gibbs measure associated with the massless log-correlated Gaussian field by replacing $(1-\partial_x^2)^{\frac{1}{4}}$ in (1.16) with $(-\partial_x^2)^{\frac{1}{4}}$.

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Equation (1.21) corresponds to the stochastic damped NLW when $d = 2$ ([29, 30, 63]), and to the stochastic damped nonlinear beam equation when $d = 4$.

When $d = 2$, the conservative stochastic Cahn–Hilliard equation is known to (formally) preserve the Gibbs measure ρ in (1.1) ([55]).

For the equations listed above, once we establish local well-posedness almost surely with respect to the Gibbs measure initial data, Bourgain's invariant measure argument [8, 11] allows us to construct almost sure global dynamics and to prove invariance of the Gibbs measure. However, since functions on the support of the log-correlated Gibbs measure ρ in (1.1) almost surely belong to the L^p-based Sobolev spaces $W^{s,p}(\mathbb{T}^d) \setminus L^p(\mathbb{T}^d)$ only for $s < 0$ with any $1 \le p \le \infty$, there are only a handful of the well-posedness results [11, 22, 48, 29, 23] for the Hamiltonian PDEs mentioned above (including (1.21)).

Remark 1.2. We point out that as long as we can construct the Gibbs measure, a compactness argument with invariance of the truncated Gibbs measures and Skorokhod's theorem allows us to construct (nonunique) global-in-time dynamics along with invariance of the Gibbs measure in some mild sense. See [1, 19, 14, 47, 43]. In our current setting, this almost sure global existence result holds for (i) the defocusing case ($\lambda < 0$ and even $k \ge 4$; see the discussion in Subsection 1.1) and (ii) the quadratic nonlinearity (i.e., $k = 3$). See Theorem 1.9 for the latter case.

Remark 1.3. Given $\delta > 0$, consider the intermediate long wave equation (ILW) on $\mathbb{T}$:

$$\partial_t u - \mathcal{G}_\delta \partial_x^2 u - \partial_x (u^2) = 0, \tag{1.22}$$

where the dispersion operator $\mathcal{G}_\delta$ is given by

$$\widehat{\mathcal{G}_\delta f}(n) = -i\Big(\coth(\delta n) - \frac{1}{\delta n}\Big)\widehat{f}(n), \qquad n \in \mathbb{Z}. \tag{1.23}$$

Equation (1.22) models the internal wave propagation of the interface in a stratified fluid of finite depth $\delta > 0$, providing a natural connection between the Benjamin–Ono regime ($\delta = \infty$) and the Korteweg–de Vries (KdV) regime ($\delta = 0$). Indeed, there are results establishing convergence of ILW to the Benjamin–Ono equation (and the KdV equation) as $\delta \to \infty$ (and $\delta \to 0$, respectively); see [34, 15, 16] and the references therein. While it is not obvious from the rather complicated dispersive symbol in (1.23), the Gibbs measure associated to ILW is indeed log-correlated, and the results in this paper apply to the Gibbs measure associated to the generalized ILW (where the nonlinearity $\partial_x(u^2)$ in (1.22) is replaced by $\lambda \partial_x(u^{k-1})$). Furthermore, as $\delta \to \infty$ (and $\delta \to 0$), the Gibbs measure for the (generalized) ILW converges to that for the (generalized) Benjamin–Ono equation (and the (generalized) KdV equation, respectively) in an appropriate sense. See a recent work [35] for a further discussion. See also [17] for the construction and convergence of invariant measures for ILW associated with higher order conservation laws.

1.3. Nonnormalizability of the focusing Gibbs measure

We now turn our attention to the focusing case. In this subsection, we study the Gibbs measure ρ in (1.1) with the focusing quartic interaction ($\lambda > 0$ and $k = 4$). In this case, we prove the following nonnormalizability of the (renormalized) focusing Gibbs measure ρ.

Theorem 1.4. *Let $\lambda > 0$ and $k = 4$. Then, given any $K > 0$, we have*

$$\sup_{N \in \mathbb{N}} Z_{K,N} \stackrel{\text{def}}{=} \sup_{N \in \mathbb{N}} \mathbb{E}_\mu \Big[\mathbf{1}_{\{|\int_{\mathbb{T}^d} :u_N^2: dx| \le K\}} e^{R_N(u)} \Big] = \infty, \tag{1.24}$$

where R_N is the renormalized potential energy defined in (1.8) with $k = 4$. Moreover, the divergence rate of $Z_{K,N}$ is given by

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$$\log Z_{K,N} = \lambda \frac{C_B}{4} N^d \sigma_N^2 (1 + o(1)) \sim N^d (\log N)^2, \tag{1.25}$$

as $N \to \infty$. Here, C_B is the optimal constant in Bernstein's inequality:

$$\|Pf\|^4_{L^4(\mathbb{R}^d, \frac{dx}{(2\pi)^d})} \le C_B \|f\|^4_{L^2(\mathbb{R}^d, \frac{dx}{(2\pi)^d})}, \tag{1.26}$$

where P is the sharp Fourier projection onto the unit ball:

$$\widehat{Pf}(\xi) = \mathbf{1}_{\{|\xi|\le 1\}} \widehat{f}(\xi),$$

and σ_N is defined in (1.5). Moreover, we have

$$Z_K \stackrel{\text{def}}{=} \mathbb{E}_\mu \Big[\mathbf{1}_{\{|\int_{\mathbb{T}^d} :u^2: dx| \le K\}} e^{R(u)} \Big] = \infty, \tag{1.27}$$

where $R(u)$ is the limit of $R_N(u)$ defined in (1.9). In particular, the focusing Gibbs measure (even with a Wick-ordered L^2-cutoff) cannot be defined as a probability measure.

When $d = 2$, Theorem 1.4 provides an alternative proof of the nonnormalizability result for of the focusing Φ^4_2-measure due to Brydges and Slade [13] whose proof is based on analysis of a model closely related to the Berlin–Kac spherical model. Furthermore, Theorem 1.4 provides a precise rate (1.25) of divergence of the partition function $Z_{K,N}$. Our strategy for proving the divergence rate (1.25) is straightforward and thus is expected to be applicable to a wide range of models.

Our proof of Theorem 1.4 is based on the variational approach due to Barashkov and Gubinelli [3]. More precisely, we will rely on the Boué–Dupuis variational formula [7, 68]; see Lemma 3.1. Our main task is to construct a drift term which achieves the desired divergence (1.24). Our argument is inspired by recent works by the third author with Weber [64] and by the first and third authors with Okamoto [41, 42]. In particular, our presentation closely follows but refines that in [41], where an analogous nonnormalizability is shown for focusing Gibbs measures on $\mathbb{T}^3$ with a quartic interaction of Hartree-type. We point out that the argument in [41] shows nonnormalizability only for large $K \gg 1$ and thus we need to refine the argument to prove the divergence (1.24) for *any* $K > 0$. The main new ingredient (as compared to [41]) is the construction a drift term which approximates a blowup profile, such that the Wick-ordered L^2-cutoff does not exclude this blowup profile for *any* cutoff size $K > 0$. See, in particular, Lemma 3.4 and the proof of (3.42). We also mention related works [33, 13, 53, 12, 46, 54] on the nonnormalizability (and other issues) for focusing Gibbs measures.

Remark 1.5. As a direct consequence of (1.24), we have

$$\begin{aligned} \sup_{N\in\mathbb{N}} \mathbb{E}_\mu \Big[e^{R_N(u)} \Big] &\ge \sup_{N\in\mathbb{N}} \mathbb{E}_\mu \Big[\mathbf{1}_{\{\int_{\mathbb{T}^d} :u_N^2: dx \le K\}} e^{R_N(u)} \Big] \\ &\ge \sup_{N\in\mathbb{N}} \mathbb{E}_\mu \Big[\mathbf{1}_{\{|\int_{\mathbb{T}^d} :u_N^2: dx| \le K\}} e^{R_N(u)} \Big] = \infty. \end{aligned}$$

Remark 1.6. In the one-dimensional setting studied in [33, 46], the sharp Gagliardo–Nirenberg inequality on $\mathbb{R}$ plays an important role in determining (non-)normalizability of the focusing Gibbs measure with a sextic interaction. In our current problem with a quartic interaction, Bernstein's inequality (1.26) on $\mathbb{R}^d$, which is essentially a frequency-localized version of Sobolev's inequality, plays a crucial role in determining the precise divergence rate (1.25). We point out that this particular form of Bernstein's inequality appears due to the form of the regularization we use for our problem (namely, the sharp frequency truncation onto the frequencies $\{|n| \le N\}$). In the current singular setting where a renormalization is required, we need to start with a regularized problem. However, there are different ways to regularize a problem, and different regularizations lead to different divergence rates. For example, if we instead use a smooth frequency truncation, we would obtain a divergence rate with a different constant (while the essential rate $N^d(\log N)^2$ in (1.25) remains the same).

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Remark 1.7. (i) An analogous nonnormalizability result holds for a focusing Gibbs measure with the quartic interaction even if we endow it with taming by the Wick-ordered L^2-norm. See Remark 1.12.

(ii) By controlling combinatorial complexity, we can extend the nonnormalizability result in Theorem 1.4 to the higher-order interactions $k \geq 5$ in the focusing case (i.e., either k is odd or $\lambda > 0$ when k is even).

(iii) In terms of dynamical problems, Theorem 1.4 states that Gibbs measures associated with the equations listed in Subsection 1.2 do not exist for (i) $\lambda > 0$ and $k \geq 4$ or (ii) odd $k \geq 5$. This list in particular includes

- the focusing L^2-(super)critical fractional NLS (1.17) (including the focusing (super)cubic NLS on $\mathbb{T}^2$),
- the focusing L^2-(super)critical fractional NLW (1.18) (including the focusing (super)cubic NLW on $\mathbb{T}^2$ and the focusing (super)cubic nonlinear beam equation on $\mathbb{T}^4$),
- the focusing modified Benjamin–Ono equation (1.19) (and the focusing generalized Benjamin–Ono equation with $k \geq 5$).

See also Appendix A for a brief discussion on the two-dimensional Zakharov system.

Remark 1.8. In a recent work [46], the first and third authors with Okamoto studied the construction of the Φ^3_3-measure on $\mathbb{T}^3$ (i.e., (1.1) with $d = 3$ and $k = 3$) and established the following phase transition: normalizability in the weakly nonlinear regime ($|\lambda| \ll 1$) and nonnormalizability in the strongly nonlinear regime ($|\lambda| \gg 1$), where the latter result was obtained based on the strategy in the current paper. In particular, in view of the nonnormalizability of the Φ^3_3-measure in the strongly nonlinear regime, we expect that the same approach would yield nonnormalizability of the focusing Φ^k_3-measure for $k \geq 4$ (namely, (i) for even $k \geq 4$ with $\lambda > 0$ or (ii) for odd $k \geq 5$ with $\lambda \neq 0$).

1.4. *Gibbs measure with the cubic interaction*

Let us first go over the focusing Gibbs measure construction in the two-dimensional setting. In [10], Bourgain reported Jaffe's construction of a Φ^3_2-measure endowed with a Wick-ordered L^2-cutoff:

$$d\rho(u) = Z^{-1}\mathbf{1}_{\{\int_{\mathbb{T}^2} :u^2:\,dx \leq K\}} e^{\int_{\mathbb{T}^2} :u^3:\,dx} d\mu(u). \tag{1.28}$$

Note that the measure in (1.28) is not suitable to generate any NLS / NLW / heat dynamics since (i) the renormalized cubic power $:u^3:$ makes sense only in the real-valued setting and hence is not suitable for the Schrödinger equation and (ii) NLW and the heat equation do not preserve the L^2-norm of a solution and thus are incompatible with the Wick-ordered L^2-cutoff. In [10], Bourgain instead proposed to consider the Gibbs measure of the form:[12].

$$d\rho(u) = Z^{-1} e^{\int_{\mathbb{T}^2} :u^3:\,dx - A\left(\int_{\mathbb{T}^2} :u^2:\,dx\right)^2} d\mu(u) \tag{1.29}$$

(for sufficiently large $A > 0$) in studying NLW dynamics on $\mathbb{T}^2$.[13]

We now extend the construction of the Gibbs measures in (1.28) and (1.29) to a general dimension $d \geq 1$. Given $N \in \mathbb{N}$, let

$$R^\diamond_N(u) = \frac{\lambda}{3}\int_{\mathbb{T}^d} :u^3_N:\,dx - A\left(\int_{\mathbb{T}^d} :u^2_N:\,dx\right)^2, \tag{1.30}$$

[12]The choice of the exponent $\gamma = 2$ in $A\big(\int_{\mathbb{T}^2} :u^2:\,dx\big)^\gamma$ (with $A \gg 1$) is optimal. See Remark 4.2

[13]For the NLW dynamics, we need to couple ρ on the u-component with the white noise measure μ_0 on the $\partial_t u$-component (which is independent from ρ). More precisely, the Gibbs measure is of the form $\vec{\rho} = \rho \otimes \mu_0$, where the Φ^3_2-measure ρ in (1.29) is on the u-component and the white noise measure μ_0 is on the $\partial_t u$-component.

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where the coupling constant $\lambda \in \mathbb{R} \setminus \{0\}$ denotes the strength of cubic interaction, and define the truncated renormalized Gibbs measure ρ_N by

$$d\rho_N(u) = Z_N^{-1} e^{R_N^\diamond(u)} d\mu(u). \tag{1.31}$$

Then, we have the following result for the focusing Gibbs measure with a cubic interaction.

Theorem 1.9. *Let $\lambda \in \mathbb{R} \setminus \{0\}$. Given any finite $p \geq 1$, there exists sufficiently large $A = A(\lambda, p) > 0$ such that $R_N^\diamond$ in (1.30) converges to some limit $R^\diamond$ in $L^p(\mu)$. Moreover, there exists $C_{p,d,A} > 0$ such that*

$$\sup_{N \in \mathbb{N}} \Big\| e^{R_N^\diamond(u)} \Big\|_{L^p(\mu)} \leq C_{p,d,A} < \infty. \tag{1.32}$$

In particular, we have

$$\lim_{N \to \infty} e^{R_N^\diamond(u)} = e^{R^\diamond(u)} \qquad \text{in } L^p(\mu). \tag{1.33}$$

As a consequence, the truncated renormalized Gibbs measure ρ_N in (1.31) converges, in the sense of (1.33), to the focusing Gibbs measure ρ given by

$$d\rho(u) = Z^{-1} e^{R^\diamond(u)} d\mu(u).$$

Furthermore, the resulting Gibbs measure ρ is equivalent to the log-correlated Gaussian field μ.

As for the convergence of $R_N^\diamond$, we omit details since the argument is standard. See, for example, [47, Proposition 1.1], [49, Proposition 3.1], [31, Lemma 4.1] and [41, Lemma 5.1] for related details. As mentioned in Subsection 1.1, the main task is to prove the uniform integrability bound (1.32). Once this is done, the rest follows from a standard argument. In Section 4, we establish the bound (1.32) by using the variational formulation.

Remark 1.10. Note that

$$\mathbf{1}_{\{|\cdot| \leq K\}}(x) \leq \exp\big(-A|x|^\gamma\big) \exp(AK^\gamma) \tag{1.34}$$

for any $K, A, \gamma > 0$. Then, the following uniform bound for the focusing cubic interaction:

$$\sup_{N \in \mathbb{N}} \Big\| \mathbf{1}_{\{|\int_{\mathbb{T}^d} :u^2: dx| \leq K\}} e^{R_N(u)} \Big\|_{L^p(\mu)} \leq C_{p,d,K} < \infty$$

for any $K > 0$ follows as a direct consequence of the uniform bound (1.32) and (1.34) with $\gamma = 2$, where R_N is as in (1.8) with $\lambda \in \mathbb{R} \setminus \{0\}$ and $k = 3$. This allows us to construct the log-correlated Gibbs measure with the cubic interaction (with a Wick-ordered L^2-cutoff):

$$d\rho(u) = Z^{-1} \mathbf{1}_{\{|\int_{\mathbb{T}^d} :u^2: dx| \leq K\}} e^{\frac{\lambda}{3} \int_{\mathbb{T}^d} :u^3: dx} d\mu(u)$$

as a limit of its truncated version (for any $\lambda \in \mathbb{R} \setminus \{0\}$ and $K > 0$).

Remark 1.11. In [67], Tzvetkov constructed the Gibbs measure (with a Wick-ordered L^2-cutoff) for the Benjamin–Ono equation (1.19) with $k = 3$. Theorem 1.9 and Remark 1.10 provide an alternative proof of the construction of the Gibbs measure for the Benjamin–Ono equation.

Remark 1.12. (i) It follows from Theorem 1.4 and (1.34) that an analogue of Theorem 1.9 fails for the quartic interaction ($k = 4$). More precisely, we have

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$$\sup_{N\in\mathbb{N}}\bigg\|\exp\bigg(\frac{\lambda}{4}\int_{\mathbb{T}^d}:u_N^4:dx - A\bigg|\int_{\mathbb{T}^d}:u_N^2:dx\bigg|^\gamma\bigg)\bigg\|_{L^p(\mu)} = \infty$$

for any $\lambda, A, \gamma > 0$.

(ii) If we consider a smoother base Gaussian measure μ_α, then we can prove the following uniform exponential integrability bound; given any $\lambda > 0$, $\alpha > \frac{d}{2}$ and finite $p \geq 1$, there exists sufficiently large $A = A(\lambda, \alpha, p) > 0$ and $\gamma = \gamma(\alpha) > 0$ such that

$$\sup_{N\in\mathbb{N}}\bigg\|\exp\bigg(\frac{\lambda}{4}\int_{\mathbb{T}^d}u_N^4\,dx - A\Big(\int_{\mathbb{T}^d}u_N^2\,dx\Big)^\gamma\bigg)\bigg\|_{L^p(\mu_\alpha)} \leq C_{p,d,A} < \infty. \tag{1.35}$$

Here, μ_α denotes the Gaussian measure with a formal density

$$d\mu_\alpha = Z^{-1}e^{-\frac{1}{2}\|u\|_{H^\alpha}^2}du. \tag{1.36}$$

See Appendix B for the proof of (1.35). The bound (1.35) allows us to construct the focusing Gibbs measure with a focusing quartic interaction of the form:

$$d\rho_\alpha = Z^{-1}e^{\frac{\lambda}{4}\int_{\mathbb{T}^d}u^4dx - A\left(\int_{\mathbb{T}^d}u^2dx\right)^\gamma}d\mu_\alpha. \tag{1.37}$$

Moreover, in view of (1.34), we can also construct the following focusing Gibbs measure with an L^2-cutoff:

$$d\rho_\alpha = Z^{-1}\mathbf{1}_{\{\int_{\mathbb{T}^d}|u|^2dx\leq K\}}e^{\frac{\lambda}{4}\int_{\mathbb{T}^d}|u|^4dx}d\mu_\alpha \tag{1.38}$$

for any $K > 0$.

In [61, 62], Sun and Tzvetkov recently studied the following fractional NLS on $\mathbb{T}$:

$$i\partial_t u + (1-\partial_x^2)^\alpha u - \lambda|u|^2u = 0 \tag{1.39}$$

in the defocusing case ($\lambda < 0$). They proved almost sure local well-posedness of (1.39) with respect to the Gaussian measure μ_α in (1.36) for $\alpha > \frac{31-\sqrt{233}}{28} \approx 0.562$ ($> \frac{1}{2}$),[14] which in turn yielded almost sure global well-posedness with respect to the defocusing Gibbs measure (namely, ρ_α in (1.38) without an L^2-cutoff) and invariance of the defocusing Gibbs measure. Since their local result also holds in the focusing case ($\lambda > 0$), our construction of the focusing Gibbs measure ρ_α in (1.38) implies almost sure global well-posedness of (1.39) with respect to the focusing Gibbs measure ρ_α in (1.38) and its invariance under the dynamics of (1.39) for the same range of α.

(iii) Theorem 1.4 and Part (ii) of this remark show that in the case of the focusing quartic interaction, there is no phase transition, depending on the value of $\lambda > 0$. Compare this with the situation in [41, 42], where such a phase transition (as described in Remark 1.8) was established in the critical case. It may be of interest to pursue the issue of a possible phase transition for a higher-order focusing interaction, in the nonsingular regime $\alpha > \frac{d}{2}$.

2. Preliminary lemmas

In this section, we recall basic definitions and lemmas used in this paper.

Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. We define the L^2-based Sobolev space $H^s(\mathbb{T}^d)$ by the norm:

$$\|f\|_{H^s} = \|\langle n\rangle^s\widehat{f}(n)\|_{\ell_n^2}.$$

[14]See also a recent preprint [36], where the authors covered the range $\alpha > \frac{1}{2}$.

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We also define the L^p-based Sobolev space $W^{s,p}(\mathbb{T}^d)$ by the norm:

$$\|f\|_{W^{s,p}} = \big\|\mathcal{F}^{-1}[\langle n\rangle^s \widehat{f}(n)]\big\|_{L^p}.$$

When $p = 2$, we have $H^s(\mathbb{T}^d) = W^{s,2}(\mathbb{T}^d)$.

2.1. Deterministic estimates

We first recall the following interpolation and fractional Leibniz rule. As for the second estimate (2.1), see [29, Lemma 3.4].

Lemma 2.1. *The following estimates hold.*
(i) (interpolation) *For* $0 < s_1 < s_2$, *we have*

$$\|u\|_{H^{s_1}} \le \|u\|_{H^{s_2}}^{\frac{s_1}{s_2}} \|u\|_{L^2}^{\frac{s_2-s_1}{s_2}}.$$

(ii) (fractional Leibniz rule) *Let* $0 \le s \le 1$. *Suppose that* $1 < p_j, q_j, r < \infty$, $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, $j = 1, 2$. *Then, we have* [15]

$$\|\langle\nabla\rangle^s(fg)\|_{L^r(\mathbb{T}^d)} \lesssim \Big(\|f\|_{L^{p_1}(\mathbb{T}^d)}\|\langle\nabla\rangle^s g\|_{L^{q_1}(\mathbb{T}^d)} + \|\langle\nabla\rangle^s f\|_{L^{p_2}(\mathbb{T}^d)}\|g\|_{L^{q_2}(\mathbb{T}^d)}\Big), \tag{2.1}$$

where $\langle\nabla\rangle = \sqrt{1-\Delta}$.

The next lemma states almost optimal Bernstein's inequality on $\mathbb{T}^d$.

Lemma 2.2. *Given* $N \in \mathbb{N}$, *let* π_N *be the frequency projector as in (1.4). Then, we have*

$$\|\pi_N f\|_{L^4(\mathbb{T}^d)}^4 \le C_B N^d(1 + o(1))\|f\|_{L^2(\mathbb{T}^d)}^4$$

as $N \to \infty$, *where* C_B *is the optimal constant for Bernstein's inequality (1.26) on* $\mathbb{R}^d$.

Proof. Given $N \in \mathbb{N}$, let $C_{B,N}$ be the optimal constant for the following inequality on $\mathbb{T}^d$:

$$\|\pi_N f\|_{L^4(\mathbb{T}^d)}^4 \le C_{B,N} N^d \|\pi_N f\|_{L^2(\mathbb{T}^d)}^4, \tag{2.2}$$

and let f_N be an optimizer for (2.2) with $\|f_N\|_{L^2(\mathbb{T}^d)} = 1$ and $\pi_N f_N = f_N$. In particular, we have

$$\|f_N\|_{L^4(\mathbb{T}^d)}^4 = C_{B,N} N^d. \tag{2.3}$$

Note that such an optimizer exists since the set $\{f_N : \|f_N\|_{L^2(\mathbb{T}^d)} = 1, \pi_N f_N = f_N\}$ is compact. Moreover, by Sobolev's inequality on the torus, we have

$$C_{B,N} \lesssim 1, \tag{2.4}$$

uniformly in $N \in \mathbb{N}$. Then, in view of (2.3), it suffices to show that

$$\limsup_{N\to\infty} N^{-d}\|f_N\|_{L^4(\mathbb{T}^d)}^4 \le C_B. \tag{2.5}$$

Fix small $\varepsilon > 0$. Let $\chi_\varepsilon \in C_c^\infty(\mathbb{R}^d; [0, 1])$ be a smooth bump function which is compactly supported on $[-\pi, \pi)^d \cong \mathbb{T}^d$ such that $\chi_\varepsilon \equiv 1$ on $[-\pi+c_0\varepsilon, \pi-c_0\varepsilon]^d$ for some small $c_0 = c_0 > 0$ to be chosen later. Recalling that $dx_{\mathbb{T}^d} = (2\pi)^{-d}dx$ is the normalized Lebesgue measure on $\mathbb{T}^d$, we see that $\|f_N\|_{L^4(\mathbb{T}^d)}^4$ is

[15] We use the convention that the symbol $\lesssim$ indicates that inessential constants are suppressed in the inequality.

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the average of $|f(x)|^4$ on $\mathbb{T}^d$. Hence, by suitably translating f_N (that does not affect its optimality) and choosing $c_0 = c_0 > 0$ sufficiently small (independent of small $\varepsilon > 0$ and $N \in \mathbb{N}$), we have

$$\|\chi_\varepsilon^2 f_N\|_{L^4(\mathbb{T}^d)} \geq (1-\varepsilon)\|f_N\|_{L^4(\mathbb{T}^d)}, \tag{2.6}$$

uniformly in $N \in \mathbb{N}$. In the following, when we view f_N as a function on $\mathbb{R}^d$, we simply view it as a periodic function: $f(x) = f(x + 2\pi m)$, $m \in \mathbb{Z}^d$.

Let $\theta \in C_c^\infty(\mathbb{R}^d; [0,1])$ be a smooth radial bump function on $\mathbb{R}^d$ such that $\theta(\xi) = 1$ for $|\xi| \leq 1$ and $\theta(\xi) = 0$ for $|\xi| > 2$. Given $M > 0$, set $\theta_M(\xi) = \theta\big(\frac{\xi}{M}\big)$. Now, we set

$$\chi_{\varepsilon,M} = \mathbf{Q}_M(\chi_\varepsilon) := \mathcal{F}^{-1}_{\mathbb{R}^d}(\theta_M) * \chi_\varepsilon, \tag{2.7}$$

where $\mathcal{F}^{-1}_{\mathbb{R}^d}$ is the inverse Fourier transform on $\mathbb{R}^d$. Namely, $\chi_{\varepsilon,M}$ is the frequency-localized version of χ_ε onto the frequencies $\{\xi \in \mathbb{R}^d : |\xi| \leq 2M\}$. Then, by choosing $M = M(\varepsilon, N) > 0$ sufficiently large, we have

$$\|\chi_\varepsilon - \chi_{\varepsilon,M}\|_{L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)} = \|(\mathrm{Id} - \mathbf{Q}_M)\chi_\varepsilon\|_{L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)} \ll \varepsilon N^{-\frac{d}{4}}. \tag{2.8}$$

Since χ_ε is a Schwartz function, we have $M(\varepsilon, N) = o(N)$ for each fixed $\varepsilon > 0$.

By the definition (2.7) of $\chi_{\varepsilon,M}$ and choosing $M = M(\varepsilon, N) = o(N)$ possibly larger, we have

$$\begin{aligned}
\|\chi_{\varepsilon,M}(\,\cdot + 2\pi m)\|_{L^\infty([-\pi,\pi)^d)} &= \sup_{x\in[-\pi,\pi)^d} M^d \int_{\mathbb{R}^d} \chi_\varepsilon(x + 2\pi m - y)\mathcal{F}^{-1}_{\mathbb{R}^d}(\theta)(My)dy \\
&\lesssim \frac{M^d}{\langle Mm\rangle^{2d+1}} \ll \frac{\varepsilon}{\langle m\rangle^d},
\end{aligned} \tag{2.9}$$

uniformly in $m \in \mathbb{Z}^d \setminus \{0\}$, where the penultimate step follows from $\operatorname{supp}\chi_\varepsilon \subset [-\pi,\pi)^d$ and the fact that θ is a Schwartz function. Then, from the periodicity of f_N, $\operatorname{supp}\chi_\varepsilon \subset [-\pi,\pi)^d$, (2.8) and (2.9), we obtain

$$\begin{aligned}
&\|(\chi_\varepsilon - \chi_{\varepsilon,M})f_N\|_{L^4(\mathbb{R}^d, \frac{dx}{(2\pi)^d})} \\
&= \bigg(\frac{1}{(2\pi)^d}\sum_{m\in\mathbb{Z}^d}\int_{[-\pi,\pi)^d}(\chi_\varepsilon - \chi_{\varepsilon,M})^4(x+2\pi m)|f_N(x)|^4 dx\bigg)^{\frac14} \\
&\leq \|f_N\|_{L^4(\mathbb{T}^d)}\bigg(\|\chi_\varepsilon - \chi_{\varepsilon,M}\|^4_{L^\infty([-\pi,\pi)^d)} + \sum_{m\in\mathbb{Z}^d\setminus\{0\}}\|\chi^4_{\varepsilon,M}(\,\cdot + 2\pi m)\|^4_{L^\infty([-\pi,\pi)^d)}\bigg)^{\frac14} \\
&\ll \varepsilon\|f_N\|_{L^4(\mathbb{T}^d)}.
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
&\|(\chi_\varepsilon^2 - \chi_{\varepsilon,M}^2)f_N\|_{L^4(\mathbb{R}^d,\frac{dx}{(2\pi)^d})\cap L^2(\mathbb{R}^d,\frac{dx}{(2\pi)^d})} \\
&\leq \|\chi_\varepsilon - \chi_{\varepsilon,M}\|_{L^\infty(\mathbb{R}^d)\cap L^4(\mathbb{R}^d,\frac{dx}{(2\pi)^d})} \\
&\quad\times\Big(2\|\chi_\varepsilon f_N\|_{L^4(\mathbb{R}^d,\frac{dx}{(2\pi)^d})} + \|(\chi_\varepsilon - \chi_{\varepsilon,M})f_N\|_{L^4(\mathbb{R}^d,\frac{dx}{(2\pi)^d})}\Big) \\
&\ll \varepsilon N^{-\frac{d}{4}}\|f_N\|_{L^4(\mathbb{T}^d)}.
\end{aligned} \tag{2.10}$$

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Hence, from (2.6) and (2.10), we have

$$(1-2\varepsilon)\|f_N\|_{L^4(\mathbb{T}^d)} \le \|\chi^2_{\varepsilon,M} f_N\|_{L^4(\mathbb{R}^d,\frac{dx}{(2\pi)^d})} \le (1+\varepsilon)\|f_N\|_{L^4(\mathbb{T}^d)} \tag{2.11}$$

for any small $\varepsilon > 0$, uniformly in $N \in \mathbb{N}$.

Define the function $g_N, g_{N,M} : \mathbb{R}^d \to \mathbb{C}$ by setting

$$g_N(x) = \frac{1}{N^{\frac{d}{2}}}\chi^2_\varepsilon\Big(\frac{x}{N}\Big) f_N\Big(\frac{x}{N}\Big) \qquad \text{and} \qquad g_{N,M}(x) = \frac{1}{N^{\frac{d}{2}}}\chi^2_{\varepsilon,M}\Big(\frac{x}{N}\Big) f_N\Big(\frac{x}{N}\Big). \tag{2.12}$$

Then, from (2.11) and (2.12), we have

$$N^{\frac{d}{4}}\|g_{N,M}\|_{L^4(\mathbb{R}^d,\frac{dx}{(2\pi)^d})} = \|\chi^2_{\varepsilon,M} f_N\|_{L^4(\mathbb{R}^d,\frac{dx}{(2\pi)^d})} \ge (1-2\varepsilon)\|f_N\|_{L^4(\mathbb{T}^d)}. \tag{2.13}$$

By Hölder's inequality and (2.10) with (2.3) and (2.4), we have

$$\begin{aligned}\|g_N - g_{N,M}\|_{L^2(\mathbb{R}^d,\frac{dx}{(2\pi)^d})} &= \|(\chi^2_\varepsilon - \chi^2_{\varepsilon,M}) f_N\|_{L^2(\mathbb{R}^d,\frac{dx}{(2\pi)^d})}\\ &\ll \varepsilon.\end{aligned}$$

Noting that $\|g_N\|_{L^2(\mathbb{R}^d,\frac{dx}{(2\pi)^d})} = \|\chi^2_\varepsilon f_N\|_{L^2(\mathbb{T}^d)} \le 1$, we then obtain

$$\|g_{N,M}\|_{L^2(\mathbb{R}^d,\frac{dx}{(2\pi)^d})} \le (1+\varepsilon). \tag{2.14}$$

Finally, recalling that the Fourier support of $f_N = \pi_N f$ (as a function on $\mathbb{T}^d$) is contained in $\{n \in \mathbb{Z}^d |n| \le N\}$ and the Fourier support of $\chi_{\varepsilon,M}$ (as a function on $\mathbb{R}^d$ is contained in $\{\xi \in \mathbb{R}^d : |\xi| \le 2M\}$ and that $M(\varepsilon, N) = o(N)$, it follows from (2.12) that

$$\operatorname{supp}(\widehat{g}_{N,M}) \subset \Big\{\xi \in \mathbb{R}^d : |\xi| \le \frac{N+2M}{N}\Big\} \subset \big\{\xi \in \mathbb{R}^d : |\xi| \le 1 + o(1)\big\}. \tag{2.15}$$

Therefore, from (2.13) and (the scaled version of) (1.26) with (2.15) followed by (2.14), we conclude that

$$\begin{aligned} N^{-d}\|f_N\|^4_{L^4(\mathbb{T}^d)} &\le (1-2\varepsilon)^{-4}\|g_{N,M}\|^4_{L^4(\mathbb{R}^d,\frac{dx}{(2\pi)^d})}\\ &\le (1-2\varepsilon)^{-4} 4C_B(1+o(1))\|g_{N,M}\|^4_{L^2(\mathbb{R}^d,\frac{dx}{(2\pi)^d})}\\ &\le \Big(\frac{1+\varepsilon}{1-2\varepsilon}\Big)^4 C_B(1+o(1)). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, by taking the lim sup as $N \to \infty$, we obtain (2.5). □

2.2. *Tools from stochastic analysis*

Next, we recall the Wiener chaos estimate (Lemma 2.3). For this purpose, we first recall basic definitions from stochastic analysis; see [6, 59]. Let (H, B, ν) be an abstract Wiener space. Namely, ν is a Gaussian measure on a separable Banach space B with $H \subset B$ as its Cameron–Martin space. Given a complete orthonormal system $\{e_j\}_{j\in\mathbb{N}} \subset B^*$ of $H^* = H$, we define a polynomial chaos of order k to be an element of the form $\prod_{j=1}^\infty H_{k_j}(\langle x, e_j\rangle)$, where $x \in B$, $k_j \ne 0$ for only finitely many j's, $k = \sum_{j=1}^\infty k_j$, H_{k_j} is the Hermite polynomial of degree k_j as in (1.7), and $\langle\cdot,\cdot\rangle = {}_B\langle\cdot,\cdot\rangle_{B^*}$ denotes the B-B^* duality pairing. We then denote the closure of polynomial chaoses of order k under $L^2(B, \nu)$ by $\mathcal{H}_k$. The elements in $\mathcal{H}_k$

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are called homogeneous Wiener chaoses of order k. We also set

$$\mathcal{H}_{\le k} = \bigoplus_{j=0}^{k} \mathcal{H}_j$$

for $k \in \mathbb{N}$.

Let $L = \Delta - x \cdot \nabla$ be the Ornstein–Uhlenbeck operator.[16] Then, it is known that any element in $\mathcal{H}_k$ is an eigenfunction of L with eigenvalue $-k$. Then, as a consequence of the hypercontractivity of the Ornstein–Uhlenbeck semigroup $U(t) = e^{tL}$ due to Nelson [39], we have the following Wiener chaos estimate [60, Theorem I.22].

Lemma 2.3. *Let $k \in \mathbb{N}$. Then, we have*

$$\|X\|_{L^p(\Omega)} \le (p-1)^{\frac{k}{2}} \|X\|_{L^2(\Omega)}$$

for any $p \ge 2$ and any $X \in \mathcal{H}_{\le k}$.

Lemma 2.4. *Let ν_N be the law of $I_N \overset{\text{def}}{=} \int_{\mathbb{T}^d} :u_N^2(x): dx$, where u is as in (1.3) and $u_N = \pi_N u$. Then, for every $N \in \mathbb{N}$, ν_N is absolutely continuous with respect to the Lebesgue measure λ on $\mathbb{R}$. Moreover, we have*

$$\left\| \frac{d\nu_N}{d\lambda} \right\|_{L^\infty(\mathbb{R})} \lesssim 1, \tag{2.16}$$

uniformly in $N \in \mathbb{N}$. As a consequence, we have

$$\mu\bigg(\int_{\mathbb{T}^d} :u^2(x): dx = K \bigg) = 0 \tag{2.17}$$

for any $K \in \mathbb{R}$, where μ is the log-correlated Gaussian free field defined in (1.2).

Proof. By the definition (1.6) of $:u_N^2:$ with (1.3), we have

$$\begin{aligned}
\int_{\mathbb{T}^d} :u_N^2(x): dx &= \sum_{0 \le |n| \le N} \frac{|g_n|^2 - 1}{\langle n \rangle^d} \\
&= \sum_{0 \le |n| \le 1} \frac{|g_n|^2 - 1}{\langle n \rangle^d} + \sum_{2 \le |n| \le N} \frac{|g_n|^2 - 1}{\langle n \rangle^d} \\
&=: A_1 + A_{2,N}
\end{aligned}$$

with the understanding that $A_{2,N} = 0$ when $N = 1$. Because of independence of the Gaussians $\{g_n\}_{|n|>2}$ from g_0 and g_1, the random variables A_1 and $A_{2,N}$ are independent. Note that the law μ_1 of A_1 (and $\mu_{2,N}$ of $A_{2,N}$ when $N \ge 2$, respectively) is absolutely continuous with respect to the Lebesgue measure λ on $\mathbb{R}$. Thus, we have $d\mu_1 = \sigma_1 d\lambda$ for some $\sigma_1 \in L^1(\mathbb{R})$ (and $d\mu_{2,N} = \sigma_{2,N} d\lambda$ for some $\sigma_{2,N} \in L^1(\mathbb{R})$ when $N \ge 2$, respectively).

We have

$$\sum_{0 \le |n| \le 1} \frac{|g_n|^2 - 1}{\langle n \rangle^d} = (g_0^2 - 1) + 2^{1-\frac{d}{2}} (|g_1|^2 - 1).$$

Letting σ_{10} (and σ_{11}) be the density for $g_0^2 - 1$ (and $2^{1-\frac{d}{2}}(|g_1|^2 - 1)$, respectively), we have

$$\sigma_1 = \sigma_{10} * \sigma_{11}.$$

[16]For simplicity, we write the definition of the Ornstein–Uhlenbeck operator L when $B = \mathbb{R}^d$.

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Note that g_0^2 is a chi-square distribution of one degree of freedom and thus the density σ_{10} for $g_0^2 - 1$ is unbounded.[17] On the other hand, $2|g_1|^2 = 2(\operatorname{Re} g_1)^2 + 2(\operatorname{Im} g_1)^2$ is a chi-square distribution of two degrees of freedom and thus the density σ_{11} for $2^{1-\frac{d}{2}}(|g_1|^2 - 1)$ is bounded. Hence, by Young's inequality, we have

$$\|\sigma_1\|_{L^\infty(\mathbb{R})} = \|\sigma_{10} * \sigma_{11}\|_{L^\infty(\mathbb{R})} \le \|\sigma_{10}\|_{L^1(\mathbb{R})}\|\sigma_{11}\|_{L^\infty(\mathbb{R})} < \infty,$$

which proves (2.16), when $N = 1$. Next, we consider the case $N \ge 2$. Denoting by σ_{2n} the density for $2\langle n\rangle^{-d}(|g_n|^2 - 1)$, by Young's inequality, we have

$$\|\sigma_{2,N}\|_{L^1(\mathbb{R})} = \|\sigma_{22} * \sigma_{23} * \cdots \sigma_{2N}\|_{L^1(\mathbb{R})} \le \prod_{n=2}^{N} \|\sigma_{2n}\|_{L^1(\mathbb{R})} = 1, \tag{2.18}$$

where the last equality holds since σ_{2n} is a density of a probability measure. Hence, by Young's inequality with (2.18), we have

$$\begin{aligned}
\left\|\frac{d\nu_N}{d\lambda}\right\|_{L^\infty(\mathbb{R})} &= \left\|\frac{d\,\mathrm{Law}(A_1 + A_{2,N})}{d\lambda}\right\|_{L^\infty(\mathbb{R})} \\
&= \|\sigma_1 * \sigma_{2,N}\|_{L^\infty(\mathbb{R})} \\
&\le \|\sigma_1\|_{L^\infty(\mathbb{R})}\|\sigma_{2,N}\|_{L^1(\mathbb{R})} \\
&= \|\sigma_1\|_{L^\infty(\mathbb{R})} \\
&\lesssim 1,
\end{aligned}$$

uniformly in $N \ge 2$. This proves (2.16).

Let $I_\infty = \int_{\mathbb{T}^d} :u^2(x):\, dx$. Since I_N converges to I_∞ in law (see, for example, [47, Proposition 1.1]), it follows from the Portmanteau theorem and (2.16) that

$$\begin{aligned}
\mathbb{P}(I_\infty = K) \le \mathbb{P}\big(I_\infty \in (K-\varepsilon, K+\varepsilon)\big) &\le \liminf_{N\to\infty} \mathbb{P}\big(I_N \in (K-\varepsilon, K+\varepsilon)\big) \\
&= \liminf_{N\to\infty} \nu_N\big((K-\varepsilon, K+\varepsilon)\big) \\
&\le \sup_{N\in\mathbb{N}} \left\|\frac{d\nu_N}{d\lambda}\right\|_{L^\infty(\mathbb{R})} \cdot \lambda\big((K-\varepsilon, K+\varepsilon)\big) \\
&\lesssim \varepsilon
\end{aligned}$$

for any $\varepsilon > 0$. Since the choice of $\varepsilon > 0$ was arbitrary, we then conclude (2.17). □

3. Nonnormalizability of the focusing Gibbs measure with the quartic interaction

In this section, we present the proof of the nonnormalizability of the log-correlated Gibbs measure with the focusing quartic interaction (Theorem 1.4).

3.1. *Variational formulation*

In order to prove (1.24) and (1.27), we use a variational formula for the partition function as in [64, 41]. Let us first introduce some notations. Fix a a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $W(t)$ be a cylindrical

[17] In particular, (2.16) is false when $N = 0$.

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Brownian motion in $L^2(\mathbb{T}^d)$. Namely, we have

$$W(t) = \sum_{n \in \mathbb{Z}^d} B_n(t) e_n, \tag{3.1}$$

where $\{B_n\}_{n \in \mathbb{Z}^d}$ is a sequence of mutually independent complex-valued[18] Brownian motions such that $\overline{B_n} = B_{-n}$, $n \in \mathbb{Z}^d$. Then, define a centered Gaussian process $Y(t)$ by

$$Y(t) = \langle \nabla \rangle^{-\frac{d}{2}} W(t). \tag{3.2}$$

Note that we have $\mathrm{Law}(Y(1)) = \mu$, where μ is the log-correlated Gaussian measure in (1.2). By setting $Y_N = \pi_N Y$, we have $\mathrm{Law}(Y_N(1)) = (\pi_N)_* \mu$, that is, the pushforward of μ under π_N. In particular, we have $\mathbb{E}[Y_N^2(1)] = \sigma_N$, where σ_N is as in (1.5). Here, the expectation $\mathbb{E}$ is with respect to the underlying probability measure $\mathbb{P}$.

Next, let $\mathbb{H}_a$ denote the space of drifts, which are progressively measurable[19] processes belonging to $L^2([0,1]; L^2(\mathbb{T}^d))$, $\mathbb{P}$-almost surely. We now state the Boué–Dupuis variational formula [7, 68]; in particular, see Theorem 7 in [68].

Lemma 3.1. *Let Y be as in (3.2). Fix $N \in \mathbb{N}$. Suppose that $F : C^\infty(\mathbb{T}^d) \to \mathbb{R}$ is measurable such that $\mathbb{E}\big[|F(\pi_N Y(1))|^p\big] < \infty$ and $\mathbb{E}\big[|e^{-F(\pi_N Y(1))}|^q\big] < \infty$ for some $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have*

$$-\log \mathbb{E}\Big[e^{-F(\pi_N Y(1))}\Big] = \inf_{\theta \in \mathbb{H}_a} \mathbb{E}\bigg[F(\pi_N Y(1) + \pi_N I(\theta)(1)) + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2_x}^2 dt\bigg], \tag{3.3}$$

where $I(\theta)$ is defined by

$$I(\theta)(t) = \int_0^t \langle \nabla \rangle^{-\frac{d}{2}} \theta(t') dt' \tag{3.4}$$

and the expectation $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$ is an expectation with respect to the underlying probability measure $\mathbb{P}$.

In the following, we construct a drift θ depending on Y and the Boué–Dupuis variational formula (Lemma 3.1) is suitable for this purpose since an expectation in (3.3) is taken with respect to the underlying probability measure $\mathbb{P}$. Compare this with the variational formula in [31], where an expectation is taken with respect to a shifted measure.

Before proceeding to the proof of Theorem 1.4, we state a lemma on the pathwise regularity bounds of $Y(1)$ and $I(\theta)(1)$.

Lemma 3.2. (i) *Let $\varepsilon > 0$. Then, given any finite $p \geq 1$, we have*

$$\mathbb{E}\Big[\|Y_N(1)\|_{W^{-\varepsilon,\infty}}^p + \|:\!Y_N^2(1)\!:\|_{W^{-\varepsilon,\infty}}^p + \big\|:\!Y_N^3(1)\!:\big\|_{W^{-\varepsilon,\infty}}^p\Big] \leq C_{\varepsilon,p} < \infty, \tag{3.5}$$

uniformly in $N \in \mathbb{N}$.

(ii) *For any $\theta \in \mathbb{H}_a$, we have*

$$\|I(\theta)(1)\|_{H^{\frac{d}{2}}}^2 \leq \int_0^1 \|\theta(t)\|_{L^2}^2 dt. \tag{3.6}$$

Before proceeding to the proof of Lemma 3.2, recall the following orthogonality result [40, Lemma 1.1.1]; let f and g be jointly Gaussian random variables with mean zero and variances σ_f and σ_g. Then,

[18] By convention, we normalize B_n such that $\mathrm{Var}(B_n(t)) = t$. In particular, B_0 is a standard real-valued Brownian motion.

[19] Namely, the map $(t, \omega) \in [0,1] \times \Omega \mapsto \theta(t,\omega) \in L^2(\mathbb{T}^d)$ is $\mathcal{B}_{[0,t]} \otimes \mathcal{F}_t$-measurable, where $\mathcal{B}_{[0,t]}$ denotes the Borel sets in $[0,t]$ and $\{\mathcal{F}_t\}_{0 \leq t \leq 1}$ denotes the filtration induced by the process Y.

we have

$$\mathbb{E}\big[H_k(f;\sigma_f)H_\ell(g;\sigma_g)\big] = \delta_{k\ell}k!\big\{\mathbb{E}[fg]\big\}^k, \tag{3.7}$$

where $H_k(x,\sigma)$ denotes the Hermite polynomial of degree k with variance parameter σ.

Proof. Part (i) is a direct consequence of pathwise regularities of the log-correlated Gaussian process Y (and its Wick powers) whose law at time $t = 1$ is given by μ in (1.2). See, for example, [48, Proposition 2.3] and [29, Proposition 2.1] for related results when $d = 2$. For readers' convenience, we present details. Given $\varepsilon > 0$ and finite $p \geq 1$, let $r \geq p$ such that $\varepsilon r > 2d$. Then, from the Sobolev embedding theorem and Minkowski's integral inequality, we have

$$\begin{aligned}\Big\| \| :Y_N^k(1): \|_{W^{-\varepsilon,\infty}} \Big\|_{L^p(\Omega)} &\lesssim \Big\| \| :Y_N^k(1): \|_{W^{-\frac{\varepsilon}{2},r}} \Big\|_{L^p(\Omega)} \\ &\leq \Big\| \| \langle\nabla\rangle^{-\frac{\varepsilon}{2}} :Y_N^k(1,x): \|_{L^p(\Omega)} \Big\|_{L^r_x}.\end{aligned} \tag{3.8}$$

On the other hand, from (1.6) and (3.7) with (3.1) and (3.2), we have

$$\begin{aligned}\mathbb{E}\big[:Y_N^k(1,x)::Y_N^k(1,y): \big] &= k!\big\{\mathbb{E}[Y_N(1,x)Y_N(1,y)]\big\}^k \\ &= k! \sum_{\substack{n_1,\dots,n_k\in\mathbb{Z}^d \\ |n_j|\leq N}} \prod_{j=1}^k \frac{1}{\langle n_j\rangle^d} e_{n_1+\cdots+n_k}(x-y).\end{aligned}$$

By applying the Bessel potentials $\langle\nabla\rangle_x^{-\frac{\varepsilon}{2}}$ and $\langle\nabla\rangle_y^{-\frac{\varepsilon}{2}}$ of order $-\frac{\varepsilon}{2}$ and then setting $x = y$, we have

$$\mathbb{E}\Big[|\langle\nabla\rangle^{-\frac{\varepsilon}{2}} :Y_N^k(1,x):|^2\Big] = k! \sum_{\substack{n_1,\dots,n_k\in\mathbb{Z}^d \\ |n_j|\leq N}} \prod_{j=1}^k \frac{1}{\langle n_j\rangle^d \langle n_1+\cdots+n_k\rangle^\varepsilon} \lesssim 1, \tag{3.9}$$

uniformly in $N \in \mathbb{N}$. Then, (3.5) follows from (3.8), Lemma 2.3 and (3.9).

As for Part (ii), the estimate (3.6) follows from (3.4), Minkowski's inequality and Cauchy–Schwarz's inequality. See the proof of Lemma 4.7 in [31] . □

3.2. *Proof of Theorem 1.4*

In this subsection, we present the proof of Theorem 1.4. Let us first discuss the divergence (1.27) for any $K > 0$. Given $K, L > 0$ and $N \in \mathbb{N}$, define $Z_{K,L,N}$ and $Z_{K,L}$ by

$$Z_{K,L,N} = \mathbb{E}_\mu\Big[\exp\big(\min(R_N(u),L)\big)\cdot \mathbf{1}_{\{|\int_{\mathbb{T}^d}:u_N^2:dx|\leq K\}}\Big]$$

and

$$Z_{K,L} = \mathbb{E}_\mu\Big[\exp\big(\min(R(u),L)\big)\cdot \mathbf{1}_{\{|\int_{\mathbb{T}^d}:u^2:dx|\leq K\}}\Big].$$

Then, by the monotone convergence theorem, we have

$$Z_K = \lim_{L\to\infty} Z_{K,L}.$$

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Moreover, by the dominated convergence theorem together with the almost sure convergence[20] of $R_N(u)$ (and $\int_{\mathbb{T}^d} :u_N^2: dx$) to $R(u)$ (and $\int_{\mathbb{T}^d} :u^2: dx$, respectively) and Lemma 2.4 (which guarantees almost sure convergence of $\mathbf{1}_{\{|\int_{\mathbb{T}^d} :u_N^2: dx| \le K\}}$ to $\mathbf{1}_{\{|\int_{\mathbb{T}^d} :u^2: dx| \le K\}}$), we obtain

$$Z_{K,L} = \lim_{N\to\infty} Z_{K,L,N}.$$

Therefore, (1.27) follows once we prove the following divergence:

$$\lim_{L\to\infty} \liminf_{N\to\infty} Z_{K,L,N} = \infty, \tag{3.10}$$

where $R_N(u)$ is as in (1.8) with $\lambda > 0$ and $k = 4$.

Noting that

$$Z_{K,L,N} \ge \mathbb{E}_\mu\Big[\exp\Big(\min\big(R_N(u), L\big)\cdot \mathbf{1}_{\{|\int_{\mathbb{T}^d} :u_N^2: dx| \le K\}}\Big)\Big] - 1, \tag{3.11}$$

the divergence (3.10) (and thus (1.24)) follows once we prove

$$\lim_{L\to\infty} \liminf_{N\to\infty} \mathbb{E}_\mu\Big[\exp\Big(\min\big(R_N(u), L\big)\cdot \mathbf{1}_{\{|\int_{\mathbb{T}^d} :u_N^2: dx| \le K\}}\Big)\Big] = \infty. \tag{3.12}$$

By the Boué–Dupuis variational formula (Lemma 3.1), we have

$$\begin{aligned}
&-\log \mathbb{E}_\mu\Big[\exp\Big(\min\big(R_N(u), L\big)\cdot \mathbf{1}_{\{|\int_{\mathbb{T}^d} :u_N^2: dx| \le K\}}\Big)\Big]\\
&\quad = \inf_{\theta \in \mathbb{H}_a} \mathbb{E}\bigg[-\min\big(R_N(Y(1) + I(\theta)(1)), L\big)\\
&\qquad\qquad \times \mathbf{1}_{\{|\int_{\mathbb{T}^d} :(\pi_N Y(1))^2: + 2(\pi_N Y(1))(\pi_N I(\theta)(1)) + (\pi_N I(\theta)(1))^2 dx| \le K\}}\\
&\qquad\qquad + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2_x}^2 dt\bigg],
\end{aligned} \tag{3.13}$$

where $Y(1)$ is as in (3.2). Here, $\mathbb{E}_\mu$ and $\mathbb{E}$ denote expectations with respect to the Gaussian field μ in (1.2) and the underlying probability measure $\mathbb{P}$, respectively. In the following, we show that the right-hand side of (3.13) tends to $-\infty$ as $N, L \to \infty$. The main idea is to construct a drift θ such that $I(\theta)$ looks like '$-Y(1)+$ a perturbation', where the perturbation term is bounded in $L^2(\mathbb{T}^d)$ but has a large L^4-norm.[21]

• **Part 1:** We first present several preliminary results. The proofs of Lemmas 3.3 and 3.4 are presented in Subsection 3.3. We first construct a perturbation term in the next lemma. Fix a large parameter $M \gg 1$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a real-valued Schwartz function with $\|f\|_{L^2(\mathbb{R}^d, \frac{dx}{(2\pi)^d})} = 1$ such that its Fourier transform $\widehat{f}$ is supported on $\{\xi \in \mathbb{R}^d : |\xi| \le 1\}$ with $\widehat{f}(0) = 0$. Define a function f_M on $\mathbb{T}^d$ by

$$f_M = M^{-\frac{d}{2}} \sum_{\substack{n \in \mathbb{Z}^d \\ |n| \le M}} \widehat{f}\Big(\frac{n}{M}\Big) e_n, \tag{3.14}$$

where $\widehat{f} = \mathcal{F}_{\mathbb{R}^d}(f)$ denotes the Fourier transform on $\mathbb{R}^d$ defined in (1.14). Then, a direct computation yields the following lemma.

[20]See, for example, [47, Proposition 1.1] together with the Borel–Cantelli lemma.

[21]While we do not make use of solitons in an explicit manner in this paper, one should think of this perturbation as something like a soliton or a finite blowup solution (at a fixed time) with a highly concentrated profile whose L^4-norm blows up while its L^2-norm remains bounded. See Lemma 3.3.

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Lemma 3.3. *Let $\alpha > 0$. Then, we have*

$$\int_{\mathbb{T}^d} f_M^2\, dx = 1 + O(M^{-\alpha}), \tag{3.15}$$

$$\int_{\mathbb{T}^d} f_M^4\, dx = M^d \|f\|^4_{L^4(\mathbb{R}^d, \frac{dx}{(2\pi)^d})} + O(M^{-\alpha}) \sim M^d, \tag{3.16}$$

$$\begin{aligned}\int_{\mathbb{T}^d} (\langle \nabla \rangle^{-\alpha} f_M)^2 dx &\le C(f) M^{-d-2+\max(d+2-2\alpha,0)} \\ &= \begin{cases} M^{-2\alpha}, & \text{for } \alpha \le \frac{d}{2}+1, \\ M^{-d-2}, & \text{for } \alpha > \frac{d}{2}+1. \end{cases}\end{aligned} \tag{3.17}$$

for any $M \gg 1$ and some constant $C(f) > 0$.

See Lemma 5.13 in [41] for an analogous result on the construction of a perturbation term. While Lemma 3.3 follows from a similar consideration, we present some details of the proof in Subsection 3.3.

In the next lemma, we construct an approximation ζ_M to Y in (3.2) by solving stochastic differential equations. Note that, in [41], such an approximation of $Y(1)$ was constructed essentially by (a suitable frequency truncation of) $Y(\frac{1}{2})$, which was sufficient to prove a divergence analogous to (3.12) for *large* $K \gg 1$. In order to prove the divergence (3.12) for *any* $K > 0$, we need to establish a more refined approximation argument. For simplicity, we denote $Y(1)$ and $\pi_N Y(1)$ by Y and Y_N, respectively, in the following.

Lemma 3.4. *Given $M \gg 1$, define ζ_M by its Fourier coefficients as follows. For $|n| \le M$, $\widehat{\zeta}_M(n,t)$ is a solution of the following differential equation:*

$$\begin{cases} d\widehat{\zeta}_M(n,t) = \langle n \rangle^{-\frac{d}{2}} M^{\frac{d}{2}} (\widehat{Y}(n,t) - \widehat{\zeta}_M(n,t))dt \\ \widehat{\zeta}_M|_{t=0} = 0, \end{cases} \tag{3.18}$$

and we set $\widehat{\zeta}_M(n,t) \equiv 0$ for $|n| > M$. Then, $\zeta_M(t)$ is a centered Gaussian process in $L^2(\mathbb{T}^d)$, which is frequency localized on $\{|n| \le M\}$, satisfying

$$\mathbb{E}\big[\zeta_M^2(x)\big] = \sigma_M(1 + o(1)) \sim \log M, \tag{3.19}$$

$$\mathbb{E}\bigg[2\int_{\mathbb{T}^d} Y_N \zeta_M\, dx - \int_{\mathbb{T}^d} \zeta_M^2\, dx\bigg] = \sigma_M(1 + o(1)) \sim \log M, \tag{3.20}$$

$$\mathbb{E}\bigg[\bigg|\int_{\mathbb{T}^d} :(Y_N - \zeta_M)^2:\, dx\bigg|^2\bigg] \lesssim M^{-d} \log M, \tag{3.21}$$

$$\mathbb{E}\bigg[\Big(\int_{\mathbb{T}^d} Y_N f_M\, dx\Big)^2\bigg] + \mathbb{E}\bigg[\Big(\int_{\mathbb{T}^d} \zeta_M f_M\, dx\Big)^2\bigg] \lesssim M^{-d}, \tag{3.22}$$

$$\mathbb{E}\bigg[\int_0^1 \Big\|\frac{d}{ds}\zeta_M(s)\Big\|^2_{H^{\frac{d}{2}}} ds\bigg] \lesssim M^d \tag{3.23}$$

for any $N \ge M \gg 1$, where $\zeta_M = \zeta_M|_{t=1}$ and

$$:(Y_N - \zeta_M)^2: = (Y_N - \zeta_M)^2 - \mathbb{E}\big[(Y_N - \zeta_M)^2\big]. \tag{3.24}$$

Here, (3.19) is independent of $x \in \mathbb{T}^d$.

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We now define $\alpha_{M,N}$ by

$$\alpha_{M,N} = \frac{\mathbb{E}\Big[2\int_{\mathbb{T}^d} Y_N \zeta_M\,dx - \int_{\mathbb{T}^d} \zeta_M^2\,dx\Big]}{\int_{\mathbb{T}^d} f_M^2\,dx} \tag{3.25}$$

for $N \geq M \gg 1$. Then, from (3.15) and (3.20), we have

$$\alpha_{M,N} = \sigma_M(1 + o(1)) \sim \log M \tag{3.26}$$

for any $N \geq M \gg 1$.

• **Part 2:** In this part, we prove the divergence (3.12). For $M \gg 1$, we set f_M, ζ_M and $\alpha_{M,N}$ as in (3.14), Lemma 3.4 and (3.25). For the minimization problem (3.13), we set a drift $\theta = \theta^0$ by

$$\theta^0(t) = \langle\nabla\rangle^{\frac{d}{2}}\bigg(-\frac{d}{dt}\zeta_M(t) + \sqrt{\alpha_{M,N}}\,f_M\bigg) \tag{3.27}$$

such that

$$\Theta^0 = I(\theta^0)(1) = \int_0^1 \langle\nabla\rangle^{-\frac{d}{2}}\theta^0(t)\,dt = -\zeta_M + \sqrt{\alpha_{M,N}}\,f_M. \tag{3.28}$$

We also define $Q(u)$ by

$$Q(u) = \frac{1}{4}\int_{\mathbb{T}^d} u^4 dx \qquad \text{and} \qquad Q_{\mathbb{R}^d}(v) = \frac{1}{4(2\pi)^d}\int_{\mathbb{R}^d} v^4 dx, \tag{3.29}$$

for $u \in L^4(\mathbb{T}^d)$ and $v \in L^4(\mathbb{R}^d)$, respectively.

Let us first make some preliminary computations. By Cauchy's inequality, we have

$$\begin{aligned} |\zeta_M(\sqrt{\alpha_{M,N}}\,f_M)^3| &\leq \frac{\delta}{4}\alpha_{M,N}^2 f_M^4 + \frac{1}{\delta}\alpha_{M,N}\zeta_M^2 f_M^2,\\ |\zeta_M^3\sqrt{\alpha_{M,N}}\,f_M| &\leq \frac{\delta}{4}\zeta_M^4 + \frac{1}{\delta}\alpha_{M,N}\zeta_M^2 f_M^2 \end{aligned} \tag{3.30}$$

for any $0 < \delta < 1$. Then, from (3.28), (3.29) and (3.30), we have

$$\begin{aligned} &Q(\Theta^0) - \alpha_{M,N}^2 Q(f_M)\\ &\quad = -\int_{\mathbb{T}^d} \zeta_M(\sqrt{\alpha_{M,N}}\,f_M)^3 dx + \frac{3}{2}\int_{\mathbb{T}^d} \zeta_M^2(\sqrt{\alpha_{M,N}}\,f_M)^2 dx\\ &\qquad - \int_{\mathbb{T}^d} \zeta_M^3\sqrt{\alpha_{M,N}}\,f_M\,dx + Q(\zeta_M)\\ &\quad \geq -\delta\alpha_{M,N}^2 Q(f_M) - C_\delta\alpha_{M,N}\int_{\mathbb{T}^d}\zeta_M^2 f_M^2\,dx + (1-\delta)Q(\zeta_M)\\ &\quad \geq -\delta\alpha_{M,N}^2 Q(f_M) - C_\delta\alpha_{M,N}\int_{\mathbb{T}^d}\zeta_M^2 f_M^2\,dx \end{aligned} \tag{3.31}$$

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for any $0 < \delta < 1$. From (3.26), (3.19) in Lemma 3.4 and (3.15) in Lemma 3.3, we have

$$\begin{aligned}\mathbb{E}\bigg[\alpha_{M,N}\int_{\mathbb{T}^d}\zeta_M^2 f_M^2 dx\bigg] &= \alpha_{M,N}\int_{\mathbb{T}^d}\mathbb{E}[\zeta_M^2(x)]f_M^2(x)dx\\ &\sim (\log M)^2\|f_M\|_{L^2}^2 \lesssim (\log M)^2\end{aligned}\tag{3.32}$$

for any $N \geq M \gg 1$. Therefore, it follows from (3.31), (3.32) and (3.26) with (3.29) and (3.16) that for any measurable set E with $\mathbb{P}(E) > 0$ and any $L \gg \lambda \cdot \alpha_{M,N}^2 Q(f_M)$, we have

$$\begin{aligned}\mathbb{E}\Big[\min\big(\gamma\lambda Q(\Theta^0), L\big)\cdot \mathbf{1}_E\Big] &\geq \gamma\lambda(1-\delta)\alpha_{M,N}^2 Q(f_M)\mathbb{P}(E) - \gamma C_\delta'(\log M)^2\\ &= \gamma\lambda(1-\delta)\sigma_M^2 M^d Q_{\mathbb{R}^d}(f)\mathbb{P}(E)(1+o(1))\end{aligned}\tag{3.33}$$

for any $N \geq M \gg 1$.

Recall from (3.14) that $\widehat{f}_M$ is supported on $\{|n| \leq M\}$. Then, by Lemma 3.2 (ii) with (3.28), (3.27), (3.23) in Lemma 3.4, (3.26) and (3.15) in Lemma 3.3, we have

$$\begin{aligned}\mathbb{E}\big[\|\Theta^0\|_{H^{\frac{d}{2}}}^2\big] &\leq \mathbb{E}\bigg[\int_0^1 \|\theta^0(t)\|_{L^2}^2 dt\bigg]\\ &\lesssim \mathbb{E}\bigg[\int_0^1 \Big\|\frac{d}{ds}\zeta_M(s)\Big\|_{H^{\frac{d}{2}}}^2 ds\bigg] + M^d\alpha_{M,N}\|f_M\|_{L^2}^2\\ &\lesssim M^d\log M.\end{aligned}\tag{3.34}$$

Lastly, recall the following identity (see [48, (1.18)]):

$$H_k(x+y;\sigma) = \sum_{\ell=0}^k \binom{k}{\ell} x^{k-\ell} H_\ell(y;\sigma),\tag{3.35}$$

which follows from a Taylor expansion with the differentiation rule [32, p. 159]: $H_k(x;\sigma) = kH_{k-1}(x;\sigma)$. Then, from (1.8) with $k = 4$ and (3.35), we have

$$\begin{aligned}R_N(Y+\Theta^0) &= \frac{\lambda}{4}\int_{\mathbb{T}^d} :Y_N^4:\, dx + \lambda\int_{\mathbb{T}^d} :Y_N^3:\,\Theta^0 dx + \frac{3\lambda}{2}\int_{\mathbb{T}^d} :Y_N^2:\,(\Theta^0)^2 dx\\ &\quad + \lambda\int_{\mathbb{T}^d} Y_N(\Theta^0)^3 dx + \frac{\lambda}{4}\int_{\mathbb{T}^d}(\Theta^0)^4 dx,\end{aligned}\tag{3.36}$$

where we used

$$\pi_N\Theta^0 = \Theta^0\tag{3.37}$$

for $N \geq M \geq 1$. We now state a lemma, controlling the second, third and fourth terms on the right-hand side of (3.36). We present the proof of this lemma in Subsection 3.3.

Lemma 3.5. *There exist small $\varepsilon > 0$ and a constant $c_0 = c_0(\varepsilon) > 0$ such that for any $\delta > 0$, we have*

$$\bigg|\int_{\mathbb{T}^d} :Y_N^3:\,\Theta^0 dx\bigg| \leq c(\delta)\|:Y_N^3:\|_{W^{-\varepsilon,\infty}}^2 + \delta\|\Theta^0\|_{H^{\frac{d}{2}}}^2,\tag{3.38}$$

$$\bigg|\int_{\mathbb{T}^d} :Y_N^2:\,(\Theta^0)^2 dx\bigg| \leq c(\delta)\|:Y_N^2:\|_{W^{-\varepsilon,\infty}}^4 + \delta\Big(\|\Theta^0\|_{H^{\frac{d}{2}}}^2 + \|\Theta^0\|_{L^4}^4\Big),\tag{3.39}$$

$$\bigg|\int_{\mathbb{T}^d} Y_N(\Theta^0)^3 dx\bigg| \leq c(\delta)\|Y_N\|_{W^{-\varepsilon,\infty}}^{c_0} + \delta\Big(\|\Theta^0\|_{H^{\frac{d}{2}}}^2 + \|\Theta^0\|_{L^4}^4\Big),\tag{3.40}$$

uniformly in $N \in \mathbb{N}$.

Fix small $\delta_0 > 0$. Then, from (3.36) and Lemma 3.5, we have

$$\begin{aligned} R_N(Y+\Theta^0) &\geq (1-\delta_0)\lambda Q(\Theta^0) \\ &\quad - c(\delta_0)\lambda\Big(\| :Y_N^3: \|_{W^{-\varepsilon,\infty}}^2 + \| :Y_N^2: \|_{W^{-\varepsilon,\infty}}^4 + \|Y_N\|_{W^{-\varepsilon,\infty}}^{c_0}\Big) \\ &\quad - c\delta_0\lambda\|\Theta^0\|_{H^{\frac{d}{2}}}^2 - |R_N(Y)|. \end{aligned} \tag{3.41}$$

We are now ready to put everything together. With (3.37) in mind, suppose that for any $K > 0$ and small $\delta_1 > 0$, there exists $M_0 = M_0(K, \delta_1) \geq 1$ such that

$$\mathbb{P}\bigg(\bigg|\int_{\mathbb{T}^d} (: Y_N^2 : + 2Y_N\Theta^0 + (\Theta^0)^2)dx\bigg| \leq K\bigg) \geq 1 - \delta_1, \tag{3.42}$$

uniformly in $N \geq M \geq M_0$. Then, it follows from (3.13), (3.41), (3.33), Lemma 3.2 (3.34), (1.9) (controlling $|R_N(Y)|$, uniformly in $N \in \mathbb{N}$), and (3.26) with (3.37) that there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned} &- \log \mathbb{E}_\mu\Big[\exp\Big(\min\big(R_N(u), L\big)\cdot \mathbf{1}_{\{|\int_{\mathbb{T}^d} :u_N^2: dx| \leq K\}}\Big)\Big] \\ &\leq \mathbb{E}\bigg[- \min\big(R_N(Y+\Theta^0), L\big) \\ &\qquad \times \mathbf{1}_{\{|\int_{\mathbb{T}^d}(:Y_N^2: + 2Y_N\Theta^0 + (\Theta^0)^2)dx| \leq K\}} + \frac{1}{2}\int_0^1 \|\theta^0(t)\|_{L_x^2}^2 dt\bigg] \\ &\leq \mathbb{E}\bigg[- \min\big((1-\delta_0)\lambda Q(\Theta^0), L\big)\cdot \mathbf{1}_{\{|\int_{\mathbb{T}^d}(:Y_N^2: + 2Y_N\Theta^0 + (\Theta^0)^2)dx| \leq K\}} \\ &\qquad + c(\delta_0)\lambda\Big(\| :Y_N^3: \|_{W^{-\varepsilon,\infty}}^2 + \| :Y_N^2: \|_{W^{-\varepsilon,\infty}}^4 + \|Y_N\|_{W^{-\varepsilon,\infty}}^{c_0}\Big) \\ &\qquad + c\delta_0\lambda\|\Theta^0\|_{H^{\frac{d}{2}}}^2 + |R_N(Y)| + \frac{1}{2}\int_0^1 \|\theta^0(t)\|_{L_x^2}^2 dt\bigg] \\ &\leq -(1-\delta_0)(1-\delta)(1-\delta_1)\lambda\alpha_{M,N}^2 M^d Q_{\mathbb{R}^d}(f)(1+o(1)) \\ &\quad + C_1(\delta_0,\lambda)M^d \log M + C_2(\delta_0,\lambda) \\ &= -(1-\delta_0)(1-\delta)(1-\delta_1)\lambda\sigma_M^2 M^d Q_{\mathbb{R}^d}(f)(1+o(1)) \end{aligned} \tag{3.43}$$

for any $N \geq M \geq M_0(K, \delta_1)$ and $L \gg \lambda\cdot\alpha_{M,N}^2 Q(f_M) \sim \lambda M^d(\log M)^2$. Therefore, we obtain

$$\begin{aligned} &\lim_{L\to\infty}\liminf_{N\to\infty}\mathbb{E}_\mu\Big[\exp\Big(\min\big(\lambda R_N(u), L\big)\Big)\cdot \mathbf{1}_{\{|\int_{\mathbb{T}^d} :u_N^2: dx| \leq K\}}\Big] \\ &\quad \geq \exp\Big((1-\delta_0)(1-\delta)(1-\delta_1)\lambda\sigma_M^2 M^d Q_{\mathbb{R}^d}(f)(1+o(1))\Big) \longrightarrow \infty, \end{aligned} \tag{3.44}$$

as $M \to \infty$. This proves (3.12) by assuming (3.42).

It remains to prove (3.42) for any $K > 0$ and small $\delta_1 > 0$. From (3.28), we have

$$\begin{aligned} &\mathbb{E}\bigg[\bigg|\int_{\mathbb{T}^d}\Big(: Y_N^2 : + 2Y_N\Theta^0 + (\Theta^0)^2\Big)dx\bigg|^2\bigg] \\ &= \mathbb{E}\bigg[\bigg|\int_{\mathbb{T}^d} : Y_N^2 : dx - 2\int_{\mathbb{T}^d} Y_N\zeta_M dx + \int_{\mathbb{T}^d}\zeta_M^2 dx + \alpha_{M,N}\int_{\mathbb{T}^d} f_M^2 dx \\ &\quad + 2\sqrt{\alpha_{M,N}}\int_{\mathbb{T}^d}(Y_N - \zeta_M)f_M dx\bigg|^2\bigg]. \end{aligned} \tag{3.45}$$

From (3.26) and (3.22) in Lemma 3.4, we have

$$\mathbb{E}\bigg[\bigg|\sqrt{\alpha_{M,N}}\int_{\mathbb{T}^d}(Y_N-\zeta_M)f_M\,dx\bigg|^2\bigg]\lesssim M^{-d}\log M. \tag{3.46}$$

On other hand, from (3.25) and (3.24), we have

$$\begin{aligned}\int_{\mathbb{T}^d} :Y_N^2:\,dx-2\int_{\mathbb{T}^d}Y_N\zeta_M\,dx+\int_{\mathbb{T}^d}\zeta_M^2\,dx+\alpha_{M,N}\int_{\mathbb{T}^d}f_M^2\,dx\\ =\int_{\mathbb{T}^d}(Y_N-\zeta_M)^2-\mathbb{E}\big[(Y_N-\zeta_M)^2\big]\,dx\\ =\int_{\mathbb{T}^d}:(Y_N-\zeta_M)^2:\,dx.\end{aligned} \tag{3.47}$$

Hence, from (3.45), (3.46) and (3.47) with (3.21) in Lemma 3.4, we obtain

$$\mathbb{E}\bigg[\bigg|\int_{\mathbb{T}^d}\Big(:Y_N^2:+2Y_N\Theta^0+(\Theta^0)^2\Big)dx\bigg|^2\bigg]\lesssim M^{-d}\log M.$$

Therefore, by Chebyshev's inequality, given any $K>0$ and small $\delta_1>0$, there exists $M_0=M_0(K,\delta_1)\geq 1$ such that

$$\mathbb{P}\bigg(\bigg|\int_{\mathbb{T}^d}(:Y_N^2:+2Y_N\Theta^0+(\Theta^0)^2)dx\bigg|>K\bigg)\leq C\frac{M^{-d}\log M}{K^2}<\delta_1$$

for any $M\geq M_0(K,\delta_1)$. This proves (3.42).

• **Part 3:** In this last part, we establish the exact divergence rate (1.25) of $Z_{K,N}$. From (3.44) with $M=N$, we already have

$$\log Z_{K,N}\geq(1-\delta_0)(1-\delta)(1-\delta_1)\lambda\sigma_N^2N^dQ_{\mathbb{R}^d}(f)(1+o(1)) \tag{3.48}$$

as $N\to\infty$, for any small $\delta,\delta_0,\delta_1>0$ and any Schwartz function f with $\|f\|_{L^2(\mathbb{R}^d,\frac{dx}{(2\pi)^d})}=1$, $\operatorname{supp}(\widehat{f})\subset\{|\xi|\leq 1\}$ and $\widehat{f}(0)=0$. Since Schwartz functions with $\operatorname{supp}(\widehat{f})\subset\{|\xi|\leq 1\}$ and $\widehat{f}(0)=0$ are dense in $L^2(\mathbb{R}^d)\cap\big\{f:\operatorname{supp}(\widehat{f})\subset\{|\xi|\leq 1\}\big\}$, there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of Schwartz functions with $\|f_n\|_{L^2(\mathbb{R}^d,\frac{dx}{(2\pi)^d})}=1$ and $\operatorname{supp}(\widehat{f_n})\subset\{|\xi|\leq 1\}$ which are almost optimizers for Bernstein's inequality (1.26) on $\mathbb{R}^d$, namely, we have

$$\lim_{n\to\infty}Q_{\mathbb{R}^d}(f_n)=\frac{C_B}{4}.$$

Therefore, by inserting f_n in (3.48) and taking $n\to\infty$ and $\delta,\delta_0,\delta_1\to 0$, we obtain

$$\liminf_{N\to\infty}\frac{\log Z_{K,N}}{\sigma_N^2N^d}\geq\lambda\frac{C_B}{4}. \tag{3.49}$$

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Hence, it remains to prove the upper bound. In view of (3.13), we have

$$\begin{aligned}
\log Z_{K,N} &\le \sup_{\theta\in\mathbb{H}_a} \mathbb{E}\bigg[R_N(Y+\Theta) \\
&\qquad \times \mathbf{1}_{\{|\int_{\mathbb{T}^d}(:Y_N^2:+2Y_N\Theta_N+\Theta_N^2)dx|\le K\}} - \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2_x}^2 dt\bigg] \\
&\le \sup_{\theta\in\mathcal{L}^2_{t,x}} \mathbb{E}\bigg[R_N(Y+\Theta) \\
&\qquad \times \mathbf{1}_{\{|\int_{\mathbb{T}^d}(:Y_N^2:+2Y_N\Theta_N+(\Theta_N)^2)dx|\le K\}} - \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2_x}^2 dt\bigg] \\
&\le \sup_{\Theta\in\mathcal{H}_x^{\frac{d}{2}}} \mathbb{E}\bigg[R_N(Y+\Theta)\cdot\mathbf{1}_{\{|\int_{\mathbb{T}^d}(:Y_N^2:+2Y_N\Theta_N+(\Theta_N)^2)dx|\le K\}} - \frac{1}{2}\|\Theta_N\|^2_{H^{\frac{d}{2}}}\bigg],
\end{aligned} \tag{3.50}$$

where $\Theta = I(\theta)(1)$ in the first two lines and $\Theta_N = \pi_N\Theta$. Here, the space $\mathcal{L}^2_{t,x}$ denotes the space of drifts, which are stochastic processes belonging to $L^2([0,1];L^2(\mathbb{T}^d))$ $\mathbb{P}$-almost surely (namely, they do not have be adapted), and the space $\mathcal{H}_x^{\frac{d}{2}}$ denotes the space of $H^{\frac{d}{2}}(\mathbb{T}^d)$-valued random variables.

For any $\Theta\in\mathcal{H}_x^{\frac{d}{2}}$, let $V = Y+\Theta$. Then, with $V_N = \pi_N V$, we have

$$\Theta_N = -Y_N + V_N, \tag{3.51}$$

and thus we see that

$$\int_{\mathbb{T}^d}(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2)dx \le K \quad \text{is equivalent to} \quad \int_{\mathbb{T}^d} V_N^2 dx \le K + \sigma_N, \tag{3.52}$$

where $\sigma_N = \mathbb{E}\big[Y_N^2\big]$ is as in (1.5). Hence, from (3.50), a change of variables $\Theta_N = -Y_N + V_N$, (3.52) and the almost optimal Bernstein inequality (Lemma 2.2), we have

$$\begin{aligned}
\log Z_{K,N} &\le \sup_{\Theta\in\mathcal{H}_x^{\frac{d}{2}}} \mathbb{E}\Big[R_N(Y+\Theta)\cdot\mathbf{1}_{\{|\int_{\mathbb{T}^d}(:Y_N^2:+2Y_N\Theta_N+(\Theta_N)^2)dx|\le K\}}\Big] \\
&\le \sup_{V_N\in\mathcal{H}_x^{\frac{d}{2}}} \mathbb{E}\Big[R_N(V)\cdot\mathbf{1}_{\{\int_{\mathbb{T}^d}V_N^2dx\le K+\sigma_N\}}\Big] \\
&\le \sup_{V_N\in\mathcal{H}_x^{\frac{d}{2}}} \mathbb{E}\Big[\lambda\frac{C_{\mathrm{B}}}{4}N^d(1+o(1))\|V_N\|_{L^2}^4\cdot\mathbf{1}_{\{\int_{\mathbb{T}^d}V_N^2dx\le K+\sigma_N\}}\Big] + O(\lambda\sigma_N^2) \\
&\le \lambda\frac{C_B}{4}N^d(1+o(1))(K+\sigma_N)^2 + O(\lambda\sigma_N^2) = \lambda\frac{C_B}{4}N^d\sigma_N^2(1+o(1))
\end{aligned} \tag{3.53}$$

as $N\to\infty$, where, in the third step, we used

$$R_N(V) = \frac{\lambda}{4}V_N^4 - \frac{3\lambda}{2}\sigma_N V_N^2 + \frac{3\lambda}{4}\sigma_N^2 \le \frac{\lambda}{4}V_N^4 + \frac{3\lambda}{4}\sigma_N^2.$$

Therefore, combining this with (3.49), we conclude (1.25).

Remark 3.6. The perturbation (at the level of Θ^0 in (3.28)) is given by f_M (modulo the logarithmic factor $\sqrt{\alpha_{M,N}}$). We point out that Lemma 3.3 shows that f_M looks like a highly concentrated profile

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whose L^4-norm (in fact, any L^p-norm for $p > 2$) blows up while its L^2-norm is $O(1)$ as $M \to \infty$. Note that the blowup of L^4-norm (3.16) was crucially used in (3.33), which led to the desired divergence rate $M^d(\log M)^2$ in (3.44). Moreover, the uniform (in M) bound (3.15) on the L^2-norm f_M played an essential role in (3.32) and (3.34) to guarantee that the terms in (3.32) and (3.34) grow at a slower rate than $M^d(\log M)^2$.

3.3. *Proofs of the auxiliary lemmas*

In this subsection, we present the proofs of Lemmas 3.3, 3.4 and 3.5.

We first briefly discuss the proof of Lemma 3.3.

Proof of Lemma 3.3. Define a function F_M on $\mathbb{R}^d$ by setting

$$F_M(x) = M^{\frac{d}{2}} f(Mx).$$

Then, from the Poisson summation formula (1.13) with (3.14), we have

$$f_M(x) = \sum_{m\in\mathbb{Z}^d} F_M(x + 2\pi m) = \sum_{m\in\mathbb{Z}^d} T_m f(x), \tag{3.54}$$

where $T_m f(x) = M^{\frac{d}{2}} f(M(x + 2\pi m))$.

Recall our convention of the normalized Lebesgue measure on $\mathbb{T}^d$. Since f is a Schwartz function, we have

$$\begin{aligned}\int_{\mathbb{T}^d} (T_0 f(x))^k dx &= \frac{M^{d(\frac{k}{2}-1)}}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbf{1}_{[-\pi M,\pi M)^d}(x) f^k(x) dx \\ &= \frac{M^{d(\frac{k}{2}-1)}}{(2\pi)^d} \int_{\mathbb{R}^d} f^k(x) dx + O(M^{-\alpha})\end{aligned} \tag{3.55}$$

for any $\alpha > 0$. On the other hand, from (3.54), for $k \in \mathbb{N}$, we have

$$\begin{aligned}\int_{\mathbb{T}^d} f_M^k(x) dx &= \int_{\mathbb{T}^d} \bigg(\sum_{m\in\mathbb{Z}^d} T_m f(x)\bigg)^k dx \\ &= \int_{\mathbb{T}^d} (T_0 f)^k(x) dx + \text{l.o.t.}.\end{aligned} \tag{3.56}$$

Here, l.o.t. consists of the sum of the terms of the form

$$\int_{\mathbb{T}^d} \prod_{j=1}^k T_{m_j} f(x) dx,$$

where $m_j \neq 0$ for at least one j. It follows from the fast decay of the Schwartz function f that, for any $\kappa > 0$, there exists $C > 0$ such that

$$|T_m f(x)| = M^{\frac{d}{2}} |f(M(x + 2\pi m))| \le C(Mm)^{-\kappa}$$

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for any $m \in \mathbb{Z}^d \setminus \{0\}$; see the proof of Lemma 5.13 in [41]. As a consequence, by summing over $m_j \in \mathbb{Z}^d$, $j = 1, \dots, k$ (not all zero), we obtain

$$|\,\text{l.o.t.}| \lesssim M^{-\alpha}. \tag{3.57}$$

Therefore, from (3.55), (3.56) and (3.57) with $\|f\|_{L^2(\mathbb{R}^d, \frac{dx}{(2\pi)^d})} = 1$, we conclude (3.15) and (3.16).

Next, we prove (3.17). Since f is a Schwartz function with $\widehat{f}(0) = 0$, it follows from the fundamental theorem of calculus that

$$|\widehat{f}(\xi)| = |\widehat{f}(\xi) - \widehat{f}(0)| \le C_f\,|\xi| \tag{3.58}$$

for any $\xi \in \mathbb{R}^d$. By Plancherel's identity with (3.14) and (3.58), we have

$$\begin{aligned}
\int_{\mathbb{T}^d} (\langle \nabla \rangle^{-\alpha} f_M)^2 dx &= M^{-d} \sum_{\substack{n \in \mathbb{Z}^d \\ |n| \le M}} \Big|\widehat{f}\Big(\frac{n}{M}\Big)\Big|^2 \frac{1}{\langle n \rangle^{2\alpha}} \\
&\le C_f^2 M^{-d-2} \sum_{\substack{n \in \mathbb{Z}^d \\ |n| \le M}} \frac{1}{\langle n \rangle^{2(\alpha-1)}} \\
&\lesssim C_f^2 M^{-d-2+\max(d+2-2\alpha,0)}.
\end{aligned}$$

This prove (3.17). □

Next, we present the proof of the approximation lemma (Lemma 3.4).

Proof of Lemma 3.4. Let

$$X_n(t) = \widehat{Y}_N(n,t) - \widehat{\zeta}_M(n,t), \qquad |n| \le M. \tag{3.59}$$

Then, from (3.2) and (3.18), we see that $X_n(t)$ satisfies the following stochastic differential equation:

$$\begin{cases} dX_n(t) = -\langle n \rangle^{-\frac{d}{2}} M^{\frac{d}{2}} X_n(t)dt + \frac{1}{\langle n \rangle^{\frac{d}{2}}} dB_n(t) \\ X_n(0) = 0 \end{cases}$$

for $|n| \le M$. By solving this stochastic differential equation, we have

$$X_n(t) = \frac{1}{\langle n \rangle^{\frac{d}{2}}} \int_0^t e^{-\langle n \rangle^{-\frac{d}{2}} M^{\frac{d}{2}}(t-s)} dB_n(s). \tag{3.60}$$

Then, from (3.59) and (3.60), we have

$$\widehat{\zeta}_M(n,t) = \widehat{Y}_N(n,t) - \frac{1}{\langle n \rangle^{\frac{d}{2}}} \int_0^t e^{-\langle n \rangle^{-\frac{d}{2}} M^{\frac{d}{2}}(t-s)} dB_n(s) \tag{3.61}$$

for $|n| \le M$. Hence, from (3.61), the independence of $\{B_n\}_{n \in \mathbb{Z}^d}$,[22] Ito's isometry and (3.2), we have

[22]Here, we are referring to the independence modulo the condition $\overline{B_n} = B_{-n}$, $n \in \mathbb{Z}^d$. Similar comments apply in the following.

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$$
\begin{aligned}
\mathbb{E}\big[|\zeta_M(x)|^2\big] &= \sum_{|n|\le M}\bigg(\mathbb{E}\big[|\widehat{Y}_N(n)|^2\big] - \frac{2}{\langle n\rangle^d}\int_0^1 e^{-\langle n\rangle^{-\frac{d}{2}}M^{\frac{d}{2}}(1-s)}ds \\
&\qquad + \frac{1}{\langle n\rangle^d}\int_0^1 e^{-2\langle n\rangle^{-\frac{d}{2}}M^{\frac{d}{2}}(1-s)}ds\bigg) \\
&= \sigma_M + O\Big(\sum_{|n|\le M}\frac{1}{\langle n\rangle^{\frac{d}{2}}}\cdot\frac{1}{M^{\frac{d}{2}}}\Big) \\
&= \sigma_M(1+o(1))
\end{aligned}
\tag{3.62}
$$

for any $M \gg 1$. This proves (3.19).

By Parseval's theorem, (3.61), (3.19) and proceeding as in (3.62), we have

$$
\begin{aligned}
\mathbb{E}\Big[2\int_{\mathbb{T}^d} Y_N\zeta_M\,dx - \int_{\mathbb{T}^d}\zeta_M^2\,dx\Big] &= \mathbb{E}\Big[2\sum_{|n|\le M}\widehat{Y}_N(n)\overline{\widehat{\zeta}_M(n)} - \sum_{|n|\le M}|\widehat{\zeta}_M(n)|^2\Big] \\
&= \mathbb{E}\bigg[\sum_{|n|\le M}|\widehat{\zeta}_M(n)|^2 + \sum_{|n|\le M}\bigg(\frac{2}{\langle n\rangle^{\frac{d}{2}}}\int_0^1 e^{-\langle n\rangle^{-\frac{d}{2}}M^{\frac{d}{2}}(1-s)}dB_n(s)\bigg)\overline{\widehat{\zeta}_M(n)}\bigg] \\
&= \sigma_M(1+o(1)) + O\Big(\sum_{|n|\le M}\frac{1}{\langle n\rangle^{\frac{d}{2}}}\cdot\frac{1}{M^{\frac{d}{2}}}\Big) \\
&= \sigma_M(1+o(1))
\end{aligned}
$$

for any $N \ge M \gg 1$. This proves (3.20).

Note that $\widehat{Y}(n) - \widehat{\zeta}_M(n)$ is a mean-zero Gaussian random variable. Then, from (3.61) and Ito's isometry, we have

$$
\begin{aligned}
\mathbb{E}\bigg[\Big(|\widehat{Y}_N(n) - \widehat{\zeta}_M(n)|^2 - \mathbb{E}\big[|\widehat{Y}(n) - \widehat{\zeta}_M(n)|^2\big]\Big)^2\bigg] &\lesssim \Big(\mathbb{E}\big[|\widehat{Y}_N(n) - \widehat{\zeta}_M(n)|^2\big]\Big)^2 \\
&= \frac{1}{\langle n\rangle^{2d}}\bigg(\int_0^1 e^{-2\langle n\rangle^{-\frac{d}{2}}M^{\frac{d}{2}}(1-s)}ds\bigg)^2 \sim \frac{1}{\langle n\rangle^d}\cdot\frac{1}{M^d}.
\end{aligned}
\tag{3.63}
$$

Hence, from Plancherel's identity, (3.24), the independence of $\{B_n\}_{n\in\mathbb{Z}^d}$, the independence of $\big\{|\widehat{Y}_N(n)|^2 - \mathbb{E}\big[|\widehat{Y}_N(n)|^2\big]\big\}_{M<|n|\le N}$ and

$$
\big\{|\widehat{Y}_N(n) - \widehat{\zeta}_M(n)|^2 - \mathbb{E}\big[|\widehat{Y}_N(n) - \widehat{\zeta}_M(n)|^2\big]\big\}_{|n|\le M},
$$

(3.2), and (3.63), we have

$$
\begin{aligned}
&\mathbb{E}\bigg[\Big|\int_{\mathbb{T}^d} :(Y_N - \zeta_M)^2:\,dx\Big|^2\bigg] \\
&\quad = \sum_{M<|n|\le N}\mathbb{E}\bigg[\Big(|\widehat{Y}_N(n)|^2 - \mathbb{E}\big[|\widehat{Y}_N(n)|^2\big]\Big)^2\bigg] \\
&\qquad + \sum_{|n|\le M}\mathbb{E}\bigg[\Big(|\widehat{Y}_N(n) - \widehat{\zeta}_M(n)|^2 - \mathbb{E}\big[|\widehat{Y}_N(n) - \widehat{\zeta}_M(n)|^2\big]\Big)^2\bigg] \\
&\quad \lesssim \sum_{M<|n|\le N}\frac{1}{\langle n\rangle^{2d}} + \sum_{|n|\le M}\frac{1}{\langle n\rangle^d}\frac{1}{M^d} \lesssim M^{-d}\log M.
\end{aligned}
$$

This proves (3.21).

From (3.17) and (3.2), we have

$$\begin{aligned}\mathbb{E}\bigg[\Big(\int_{\mathbb{T}^d} Y_N f_M\,dx\Big)^2\bigg] &= \mathbb{E}\bigg[\Big(\sum_{|n|\le M}\widehat{Y}_N(n)\overline{\widehat{f}_M(n)}\Big)^2\bigg] = \sum_{|n|\le M}\frac{1}{\langle n\rangle^d}|\widehat{f}_M(n)|^2\\ &\le \int_{\mathbb{T}^d}\big(\langle\nabla\rangle^{-\frac{d}{2}} f_M(x)\big)^2 dx \lesssim M^{-d}.\end{aligned}\tag{3.64}$$

From (3.60), Ito's isometry and (3.17), we have

$$\begin{aligned}\mathbb{E}\bigg[\Big(\sum_{|n|\le M} X_n(1)\overline{\widehat{f}_M(n)}\Big)^2\bigg] &= \mathbb{E}\bigg[\bigg|\sum_{|n|\le M}\bigg(\frac{1}{\langle n\rangle^{\frac{d}{2}}}\int_0^1 e^{-\langle n\rangle^{-\frac{d}{2}}M^{\frac{d}{2}}(1-s)}dB_n(s)\bigg)\widehat{f}_M(n)\bigg|^2\bigg]\\ &\lesssim M^{-\frac{d}{2}}\sum_{|n|\le M}\frac{1}{\langle n\rangle^{\frac{d}{2}}}|\widehat{f}_M(n)|^2\\ &\lesssim M^{-d}.\end{aligned}\tag{3.65}$$

Hence, (3.22) follows from (3.64) and (3.65) with (3.61).

Lastly, from (3.18), (3.59) and (3.60) and Ito's isometry, we have

$$\begin{aligned}\mathbb{E}\bigg[\int_0^1\Big\|\frac{d}{ds}\zeta_M(s)\Big\|_{H^{\frac{d}{2}}}^2 ds\bigg] &= M^d\,\mathbb{E}\bigg[\int_0^1\big\|\pi_M(Y_N(s))-\zeta_M(s)\big\|_{L^2}^2 ds\bigg]\\ &= M^d\,\mathbb{E}\bigg[\int_0^1\Big(\sum_{|n|\le M}|X_n(s)|^2\Big)ds\bigg]\\ &= M^d\sum_{|n|\le M}\frac{1}{\langle n\rangle^d}\int_0^1\int_0^s e^{-2\langle n\rangle^{-\frac{d}{2}}M^{\frac{d}{2}}(s-s')}ds'ds\\ &\lesssim M^d\sum_{|n|\le M}\frac{1}{\langle n\rangle^{\frac{d}{2}}}\cdot\frac{1}{M^{\frac{d}{2}}}\\ &\lesssim M^d,\end{aligned}$$

yielding (3.23). This completes the proof of Lemma 3.4. □

Finally, we present the proof of Lemma 3.5.

Proof of Lemma 3.5. From the duality and Cauchy's inequality, we have

$$\begin{aligned}\bigg|\int_{\mathbb{T}^d} :Y_N^3:\Theta^0 dx\bigg| &\le \|:Y_N^3:\|_{W^{-\varepsilon,\infty}}\|\Theta^0\|_{W^{\varepsilon,1}} \le \|:Y_N^3:\|_{W^{-\varepsilon,\infty}}\|\Theta^0\|_{H^{\frac{d}{2}}}\\ &\le c(\delta)\|:Y_N^3:\|_{W^{-\varepsilon,\infty}}^2 + \delta\|\Theta^0\|_{H^{\frac{d}{2}}}^2.\end{aligned}\tag{3.66}$$

This yields (3.38).

From the fractional Leibniz rule (Lemma 2.1 (ii)), we have

$$\begin{aligned}\bigg|\int_{\mathbb{T}^d} :Y_N^2:(\Theta^0)^2 dx\bigg| &\le \|:Y_N^2:\|_{W^{-\varepsilon,\infty}}\|(\Theta^0)^2\|_{W^{\varepsilon,1}}\\ &\le \|:Y_N^2:\|_{W^{-\varepsilon,\infty}}\|(\Theta^0)^2\|_{W^{\varepsilon,\frac{4}{3}}}\\ &\lesssim \|:Y_N^2:\|_{W^{-\varepsilon,\infty}}\|\Theta^0\|_{H^{\frac{d}{2}}}\|\Theta^0\|_{L^4}.\end{aligned}\tag{3.67}$$

Then, the second estimate (3.39) follows from Young's inequality.

Lastly, we consider (3.40). From the fractional Leibniz rule (Lemma 2.1 (ii)) (with $\frac{1}{1+\delta} = \frac{1}{2+\delta_0} + \frac{1}{4} + \frac{1}{4}$ for small $\delta, \delta_0 > 0$), Sobolev's inequality, and the interpolation (Lemma 2.1 (i)), we have

$$\begin{aligned}\bigg|\int_{\mathbb{T}^d} Y_N (\Theta_N^0)^3 dx\bigg| &\le \|Y_N\|_{W^{-\varepsilon,\infty}} \|\langle\nabla\rangle^{\varepsilon} (\Theta_N^0)^3\|_{L^{1+\delta}} \\ &\lesssim \|Y_N\|_{W^{-\varepsilon,\infty}} \|\Theta_N^0\|_{W^{\varepsilon,2+\delta_0}} \|\Theta_N^0\|_{L^4}^2 \\ &\lesssim \|Y_N\|_{W^{-\varepsilon,\infty}} \|\Theta_N^0\|_{H^{\frac{d}{2}}}^{\beta} \|\Theta_N^0\|_{L^4}^{3-\beta}\end{aligned} \tag{3.68}$$

for some small $\beta > 0$. Then, the third estimate (3.40) follows from Young's inequality since $\frac{\beta}{2} + \frac{3-\beta}{4} < 1$ for small $\beta > 0$. This completes the proof of Lemma 3.5. □

4. Construction of the Gibbs measure with the cubic interaction

In this section, we present the proof of Theorem 1.9. We prove the uniform exponential integrability (1.32) via the variational formulation. Since the argument is identical for any finite $p \ge 1$, we only present details for the case $p = 1$. Moreover, the precise value of $\lambda \in \mathbb{R} \setminus \{0\}$ does not play any role and thus we set $\lambda = 3$ in the following.

In view of the Boué–Dupuis formula (Lemma 3.1), it suffices to establish a lower bound on

$$\mathcal{W}_N(\theta) = \mathbb{E}\bigg[- R_N^{\diamond}(Y(1) + I(\theta)(1)) + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L_x^2}^2 dt\bigg], \tag{4.1}$$

uniformly in $N \in \mathbb{N}$ and $\theta \in \mathbb{H}_a$. We set $Y_N = \pi_N Y = \pi_N Y(1)$ and $\Theta_N = \pi_N \Theta = \pi_N I(\theta)(1)$.

From (1.30) and (3.35), we have

$$\begin{aligned}R_N^{\diamond}(Y + \Theta) = \int_{\mathbb{T}^d} :Y_N^3: dx + 3\int_{\mathbb{T}^d} :Y_N^2: \Theta_N dx + 3\int_{\mathbb{T}^d} Y_N \Theta_N^2 dx \\ + \int_{\mathbb{T}^d} \Theta_N^3 dx - A\bigg\{\int_{\mathbb{T}^d} \Big(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2\Big) dx\bigg\}^2.\end{aligned} \tag{4.2}$$

Hence, from (4.1) and (4.2), we have

$$\begin{aligned}\mathcal{W}_N(\theta) = \mathbb{E}\bigg[&- \int_{\mathbb{T}^d} :Y_N^3: dx - 3\int_{\mathbb{T}^d} :Y_N^2: \Theta_N dx - 3\int_{\mathbb{T}^d} Y_N \Theta_N^2 dx \\ &- \int_{\mathbb{T}^d} \Theta_N^3 dx + A\bigg\{\int_{\mathbb{T}^d} \Big(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2\Big) dx\bigg\}^2 \\ &+ \frac{1}{2}\int_0^1 \|\theta(t)\|_{L_x^2}^2 dt\bigg].\end{aligned} \tag{4.3}$$

In the following, we first state a lemma, controlling the terms appearing in (4.3). We present the proof of this lemma at the end of this section.

Lemma 4.1. (i) *There exist small $\varepsilon > 0$ and a constant $c > 0$ such that*

$$\bigg|\int_{\mathbb{T}^d} :Y_N^2: \Theta_N dx\bigg| \le c\| :Y_N^2: \|_{W^{-\varepsilon,\infty}}^2 + \frac{1}{100}\|\Theta_N\|_{H^{\frac{d}{2}}}^2, \tag{4.4}$$

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$$\left|\int_{\mathbb{T}^d} Y_N \Theta_N^2 \, dx\right| \le c\|Y_N\|_{W^{-\varepsilon,\infty}}^6 + \frac{1}{100}\Big(\|\Theta_N\|_{H^{\frac{d}{2}}}^2 + \|\Theta_N\|_{L^2}^4\Big), \tag{4.5}$$

$$\left|\int_{\mathbb{T}^d} \Theta_N^3 \, dx\right| \le \frac{1}{100}\|\Theta_N\|_{H^{\frac{d}{2}}}^2 + \frac{A}{100}\|\Theta_N\|_{L^2}^4 \tag{4.6}$$

for any sufficiently large $A > 0$, uniformly in $N \in \mathbb{N}$.

(ii) *Let $A > 0$. Given any small $\varepsilon > 0$, there exists $c = c(\varepsilon, A) > 0$ such that*

$$\begin{aligned} A\bigg\{\int_{\mathbb{T}^d}\Big(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2\Big)dx\bigg\}^2& \\ \ge \frac{A}{4}\|\Theta_N\|_{L^2}^4 - \frac{1}{100}\|\Theta_N\|_{H^{\frac{d}{2}}}^2 &- c\bigg\{\|Y_N\|_{W^{-\varepsilon,\infty}}^c + \bigg(\int_{\mathbb{T}^d} :Y_N^2: dx\bigg)^2\bigg\}, \end{aligned} \tag{4.7}$$

uniformly in $N \in \mathbb{N}$.

As in [3, 31, 45, 41], the main strategy is to establish a pathwise lower bound on $\mathcal{W}_N(\theta)$ in (4.3), uniformly in $N \in \mathbb{N}$ and $\theta \in \mathbb{H}_a$, by making use of the positive terms:

$$\mathcal{U}_N(\theta) = \mathbb{E}\bigg[\frac{A}{4}\|\Theta_N\|_{L^2}^4 + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L_x^2}^2 dt\bigg] \tag{4.8}$$

coming from (4.3) and (4.7). From (4.3) and (4.8) together with Lemmas 4.1 and 3.2, we obtain

$$\inf_{N\in\mathbb{N}} \inf_{\theta\in\mathbb{H}_a} \mathcal{W}_N(\theta) \ge \inf_{N\in\mathbb{N}} \inf_{\theta\in\mathbb{H}_a} \Big\{ -C_0 + \frac{1}{10}\mathcal{U}_N(\theta)\Big\} \ge -C_0 > -\infty. \tag{4.9}$$

Then, the uniform exponential integrability (1.32) follows from (4.9) and Lemma 3.1. This proves Theorem 1.9.

We conclude this section by presenting the proof of Lemma 4.1.

Proof of Lemma 4.1. (i) The estimate (4.4) follows from replacing $:Y_N^3:$ in (3.66) by $:Y_N^2:$.

With small $\delta > 0$, it follows from the fractional Leibniz rule (Lemma 2.1 (ii)) and Sobolev's inequality as in (3.68) that

$$\begin{aligned} \left|\int_{\mathbb{T}^d} Y_N \Theta_N^2 \, dx\right| &\le \|Y_N\|_{W^{-\varepsilon,\infty}}\|\Theta_N^2\|_{W^{\varepsilon,1+\delta}} \\ &\le \|Y_N\|_{W^{-\varepsilon,\infty}}\|\Theta_N\|_{H^\varepsilon}^2 \\ &\lesssim \|Y_N\|_{W^{-\varepsilon,\infty}}\|\Theta_N\|_{H^{\frac{d}{2}}}^\beta \|\Theta_N\|_{L^2}^{2-\beta} \end{aligned}$$

for some small $\beta > 0$. Then, the second estimate (4.5) follows from Young's inequality since $\frac{\beta}{2} + \frac{2-\beta}{4} < 1$.

As for the third estimate (4.6), it follows from Sobolev's inequality, Lemma 2.1 (i) and Cauchy's inequality that

$$\begin{aligned} \left|\int_{\mathbb{T}^d} \Theta_N^3 \, dx\right| &\le C\|\Theta_N\|_{H^{\frac{d}{6}}}^3 \le C\|\Theta_N\|_{H^{\frac{d}{2}}}\|\Theta_N\|_{L^2}^2 \\ &\le \frac{1}{100}\|\Theta_N\|_{H^{\frac{d}{2}}}^2 + \frac{A}{100}\|\Theta_N\|_{L^2}^4, \end{aligned}$$

where $A > 0$ is sufficiently large.

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(ii) The bound (4.7) follows from a slight modification of Lemma 5.8 in [41]. Noting that

$$(a+b+c)^2 \geq \frac{1}{2}c^2 - 2(a^2+b^2)$$

for any $a, b, c \in \mathbb{R}$, we have

$$\begin{aligned} A\bigg\{ \int_{\mathbb{T}^d} \Big(:Y_N^2: +2Y_N\Theta_N + \Theta_N^2 \Big) dx \bigg\}^2 \\ \geq \frac{A}{2}\bigg(\int_{\mathbb{T}^d} \Theta_N^2\, dx \bigg)^2 - 2A\bigg\{ \bigg(\int_{\mathbb{T}^d} :Y_N^2: dx \bigg)^2 + \bigg(\int_{\mathbb{T}^d} Y_N\Theta_N\, dx \bigg)^2 \bigg\}. \end{aligned} \tag{4.10}$$

From Lemma 2.1 (i) and Young's inequality, we have

$$\begin{aligned} \bigg| \int_{\mathbb{T}^d} Y_N\Theta_N\, dx \bigg|^2 &\leq \|Y_N\|_{W^{-\varepsilon,\infty}}^2 \|\Theta_N\|_{W^{\varepsilon,1}}^2 \leq \|Y_N\|_{W^{-\varepsilon,\infty}}^2 \|\Theta_N\|_{H^\varepsilon}^2 \\ &\lesssim \|Y_N\|_{W^{-\varepsilon,\infty}}^2 \|\Theta_N\|_{L^2}^{2-\frac{4\varepsilon}{d}} \|\Theta_N\|_{H^{\frac{d}{2}}}^{\frac{4\varepsilon}{d}} \\ &\leq c\|Y_N\|_{W^{-\varepsilon,\infty}}^{\frac{2d}{d-2\varepsilon}} + \frac{1}{8}\|\Theta_N\|_{L^2}^4 + \frac{1}{200A}\|\Theta_N\|_{H^{\frac{d}{2}}}^2. \end{aligned} \tag{4.11}$$

Hence, (4.7) follows from (4.10) and (4.11). □

Remark 4.2. In considering the construction of the Gibbs measure with the cubic interaction, it is possible to consider the following renormalized potential energy with a general power $\gamma > 0$ on the Wick-ordered L^2-norm:

$$R_N^{\diamond,\gamma}(u) = \frac{\lambda}{3}\int_{\mathbb{T}^d} :u_N^3: dx - A\bigg(\int_{\mathbb{T}^d} :u_N^2: dx \bigg)^\gamma, \tag{4.12}$$

where the coupling constant $\lambda \in \mathbb{R} \setminus \{0\}$ denotes the strength of cubic interaction as in (1.30). When $\gamma = 2$, $R_N^{\diamond,\gamma}(u)$ reduces to $R_N^\diamond(u)$ in (1.30).

In the following, let us briefly discuss the optimality of the power $\gamma = 2$ in Theorem 1.9. In view of (4.5) and (4.6), we need to control the term $\|\Theta_N\|_{L^2}^4$, which forces us to choose $\gamma \geq 2$ in (4.12). When $\gamma = 2$, it is also necessary to choose A sufficiently large because of (4.5). When $\gamma < 2$ or when $\gamma = 2$ and A is sufficiently small, the taming by the Wick-ordered L^2-norm in (4.12) is too weak to control the terms mentioned above, and thus we expect an analogous nonnormalizability result to hold by repeating the proof of Theorem 1.4.

A. On the Gibbs measure for the two-dimensional Zakharov system

In this appendix, we give a brief discussion on Gibbs measures for the following scalar Zakharov system on $\mathbb{T}^d$:

$$\begin{cases} i\partial_t u + \Delta u = uw \\ c^{-2}\partial_t^2 w - \Delta w = \Delta(|u|^2). \end{cases} \tag{A.1}$$

This is a coupled system of Schrödinger and wave equations. The unknown u for the Schrödinger part is complex-valued, while the unknown w for the wave part is real-valued. By introducing the velocity field $\vec{v}$:

$$\partial_t w = -c^2\nabla \cdot \vec{v},$$

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we can rewrite (A.1) as

$$\begin{cases} i\partial_t u + \Delta u = uw \\ \partial_t w = -c^2 \nabla \cdot \vec{v} \\ \partial_t \vec{v} = -\nabla w - \nabla(|u|^2). \end{cases} \tag{A.2}$$

Note that (A.2) is a Hamiltonian system with the Hamiltonian

$$H(u, w, \vec{v}) = \frac{1}{2}\int_{\mathbb{T}^d} \big(|\nabla u|^2 + |u|^2 w\big)dx + \frac{1}{4}\int_{\mathbb{T}^d} w^2 dx + \frac{c^2}{4}\int_{\mathbb{T}^d} |\vec{v}|^2 dx. \tag{A.3}$$

Moreover, the wave energy, namely, the L^2-norm of the Schrödinger component:

$$M(u) = \int_{\mathbb{T}^d} |u|^2 dx$$

is known to be conserved. See [18].

By setting $W = \frac{1}{\sqrt{2}} w$ and $\vec{V} = (V_1, \dots, V_d) = \frac{c}{\sqrt{2}}\vec{v}$, we can rewrite the Hamiltonian in (A.3) as

$$H(u, W, \vec{V}) = \frac{1}{2}\int_{\mathbb{T}^d} \big(|\nabla u|^2 + \sqrt{2}|u|^2 W\big)dx + \frac{1}{2}\int_{\mathbb{T}^d} W^2 dx + \frac{1}{2}\int_{\mathbb{T}^d} |\vec{V}|^2 dx. \tag{A.4}$$

Then, the Gibbs measure for the system (A.2) is formally given by

$$\begin{aligned} d\rho &= Z^{-1} e^{-H(u,W,\vec{V}) - \frac{1}{2}M(u)} du\, dW\, d\vec{V} \\ &= Z^{-1} e^{Q(u,W)} d\mu_1(u) d\mu_0(W) \prod_{j=1}^{d} d\mu_0(V_j), \end{aligned} \tag{A.5}$$

where the potential $Q(u, W)$ is given by

$$Q(u, W) = -\frac{1}{\sqrt{2}}\int_{\mathbb{T}^d} |u|^2 W dx, \tag{A.6}$$

the measure μ_1 denotes the complex-valued version of the massive Gaussian free field on $\mathbb{T}^d$ with the density formally given by

$$d\mu_1 = Z^{-1} e^{-\frac{1}{2}\|u\|_{H^1}^2} du = Z^{-1} \prod_{n \in \mathbb{Z}^d} e^{-\frac{1}{2}\langle n \rangle^2 |\widehat{u}(n)|^2} d\widehat{u}(n),$$

and μ_0 denotes the white noise measure defined as the pushforward measure $\mu_0 = (\langle \nabla \rangle^{\frac{d}{2}})_* \mu$, with μ as in (1.2). In view of the conservation of the Hamiltonian $H(u, W, \vec{V})$ and the wave energy $M(u)$, the Gibbs measure ρ in (A.5) expected to be invariant under the Zakharov dynamics.

As in the case of the focusing NLS, the main issue in constructing the Gibbs measure ρ in (A.5) comes from the focusing nature of the potential, that is, the potential $Q(u, W)$ is unbounded from above. In a seminal paper [33], Lebowitz, Rose and Speer constructed the Gibbs measure ρ when $d = 1$, by inserting a cutoff in terms of the conserved wave energy $M(u) = \|u\|_{L^2}^2$, which was then proved to be invariant under (A.2) on $\mathbb{T}$ (and thus (A.1)) by Bourgain [9].

Then, a natural question is to consider the construction of the Gibbs measure ρ in the two-dimensional setting.[23] Before doing this, let us recall the relation between the Zakharov system and the focusing

[23]In a recent work [58], the second author studied the construction of the Gibbs measure for the Zakharov–Yukawa system on $\mathbb{T}^2$ (i.e., Δ in (A.1) is replaced by $-(-\Delta)^\gamma$, $\gamma < 1$) and showed that the renormalized Gibbs measure is indeed normalizable when $\gamma < 1$. See [58] for details.

cubic NLS. By sending the wave speed c in (A.1) to ∞, the Zakharov system converges, at a formal level, to the focusing cubic NLS. See, for example, [51, 37] for rigorous convergence results on $\mathbb{R}^d$. When $d = 2$, Theorem 1.4 states that the (renormalized) Gibbs measure for the focusing cubic NLS on $\mathbb{T}^2$ is not normalizable, even with a Wick-ordered L^2-cutoff. This suggests that, when $d = 2$, the Gibbs measure ρ in (A.5) for the Zakharov system may not be constructible even with a Wick-ordered L^2-cutoff on the Schrödinger component u.

Given $N \in \mathbb{N}$, define the following renormalized truncated potential energy:

$$Q_N(u, W) = -\frac{1}{\sqrt{2}} \int_{\mathbb{T}^2} :|u_N|^2: W dx, \tag{A.7}$$

where $u_N = \pi_N u$ as in Subsection 1.1 and $:|u_N|^2: = |u_N|^2 - \sigma_N$. We then define the renormalized truncated Gibbs measure ρ_N on $\mathbb{T}^2$, endowed with a Wick-ordered L^2-cutoff, by

$$d\rho_N = Z_N^{-1} \mathbf{1}_{\{|\int_{\mathbb{T}^2} :|u_N|^2: dx| \le K\}} e^{Q_N(u,W)} d\mu_1(u) d\mu_0(W) \prod_{j=1}^{2} d\mu_0(V_j).$$

By integrating in (V_1, V_2) and then in W, we have

$$\begin{aligned}
&\iint \mathbf{1}_{\{|\int_{\mathbb{T}^2} :|u_N|^2: dx| \le K\}} e^{Q_N(u,W)} d\mu_0(W) \prod_{j=1}^{2} d\mu_0(V_j) \\
&\quad = \mathbf{1}_{\{|\int_{\mathbb{T}^2} :|u_N|^2: dx| \le K\}} \int \exp\bigg(-\frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}^2} \mathcal{F}(:|u_N|^2:)(n)\, \overline{\widehat{W}(n)} \bigg) d\mu_0(W) \\
&\quad = \mathbf{1}_{\{|\int_{\mathbb{T}^2} :|u_N|^2: dx| \le K\}} \int_{\mathbb{R}} \exp\bigg(-\frac{1}{\sqrt{2}} \mathcal{F}(:|u_N|^2:)(0)\, g_0 \bigg) \frac{e^{-\frac{1}{2}g_0^2}}{\sqrt{2\pi}} dg_0 \\
&\qquad \times \prod_{n\in\Lambda} \frac{1}{\pi} \int_{\mathbb{C}} \exp\Big(-\sqrt{2}\,\mathrm{Re}\, \Big(\mathcal{F}(|u_N|^2)(n)\, \overline{g_n} \Big) \Big) e^{-|g_n|^2} dg_n,
\end{aligned} \tag{A.8}$$

where $\{g_n\}_{n\in\mathbb{Z}^2}$ is as in (1.3)[24] and Λ denotes the index set given by $\Lambda = (\mathbb{Z} \times \mathbb{Z}_+) \cup (\mathbb{Z}_+ \times \{0\})$ such that $\mathbb{Z}^2 = \Lambda \cup (-\Lambda) \cup \{0\}$. Here, we used the fact that $\mathcal{F}(:|u_N|^2:)(n) = \mathcal{F}(|u_N|^2)(n)$ for $n \ne 0$. Then, recalling the moment generating function $\mathbb{E}[e^{tX}] = e^{\frac{1}{2}\sigma t^2}$ for $X \sim \mathcal{N}_{\mathbb{R}}(0, \sigma)$, we have

$$\begin{aligned}
(\text{A.8}) &= \mathbf{1}_{\{|\int_{\mathbb{T}^2} :|u_N|^2: dx| \le K\}} \exp\bigg(\frac{1}{4} \big(\mathcal{F}(:|u_N|^2:)(0) \big)^2 \bigg) \\
&\qquad \times \prod_{n\in\Lambda} \frac{1}{\pi} \int_{\mathbb{C}} \exp\bigg(-\sqrt{2}\,\mathrm{Re}\big(\mathcal{F}(|u_N|^2)(n)\big)\, \mathrm{Re}\, g_n \\
&\qquad\qquad - \sqrt{2}\,\mathrm{Im}\big(\mathcal{F}(|u_N|^2)(n)\big)\, \mathrm{Im}\, g_n \bigg) e^{-|g_n|^2} dg_n \\
&\ge \mathbf{1}_{\{|\int_{\mathbb{T}^2} :|u_N|^2: dx| \le K\}} \exp\bigg(\frac{1}{4} \big\| \pi_{\ne 0} |u_N|^2 \big\|_{L^2}^2 - CK^2 \bigg),
\end{aligned} \tag{A.9}$$

where $\pi_{\ne 0}$ is the projection onto nonzero frequencies.

---

[24]In particular, g_0 is a standard real-valued Gaussian random variables where $\mathrm{Re}\, g_n$ and $\mathrm{Im}\, g_n$, $n \in \Lambda$, are independent real-valued Gaussian random variables with mean 0 and variance $\frac{1}{2}$.

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Let $\{h_n\}_{n\in\mathbb{Z}^2}$ be a sequence of mutually independent standard complex-valued Gaussian random variables. Then, we have

$$\begin{aligned}\int \big\|\pi_{\neq 0}|u_N|^2\big\|_{L^2}^2 d\mu_1 &= \mathbb{E}\Bigg[\sum_{\substack{n_1-n_2+n_3-n_4=0\\ |n_j|\le N\\ n_1-n_2\neq 0}} \frac{h_{n_1}}{\langle n_1\rangle}\frac{\overline{h_{n_2}}}{\langle n_2\rangle}\frac{h_{n_3}}{\langle n_3\rangle}\frac{\overline{h_{n_4}}}{\langle n_4\rangle}\Bigg]\\ &= \sum_{|n_1|\le N}\frac{1}{\langle n_1\rangle^2}\sum_{\substack{|n_3|\le N\\ n_3\neq n_1}}\frac{1}{\langle n_3\rangle^2} \sim (\log N)^2 \longrightarrow \infty, \end{aligned} \tag{A.10}$$

as $N\to\infty$. Then, from (A.10), the interpolation of the L^p-spaces and Lemma 2.3, we have

$$\begin{aligned}\log N &\sim \Big\|\|\pi_{\neq 0}|u_N|^2\|_{L^2}\Big\|_{L^2(\mu_1)} \ge \Big\|\|\pi_{\neq 0}|u_N|^2\|_{L^2}\Big\|_{L^1(\mu_1)}\\ &\ge \frac{\Big\|\|\pi_{\neq 0}|u_N|^2\|_{L^2}\Big\|_{L^2(\mu_1)}^3}{\Big\|\|\pi_{\neq 0}|u_N|^2\|_{L^2}\Big\|_{L^4(\mu_1)}^2} \sim \log N. \end{aligned} \tag{A.11}$$

Also, from Lemma 2.3 and (1.5), we have

$$\Big\|\|u_N\|_{L^4_x}\Big\|_{L^2(\mu_1)} \lesssim \Big\|\|u_N\|_{L^2(\mu_1)}\Big\|_{L^4_x} \sim \sigma_N^{\frac12} \sim (\log N)^{\frac12}. \tag{A.12}$$

Hence, given sufficiently small $\varepsilon \gg \eta > 0$, it follows from Lemma 3.1, Cauchy's inequality, Sobolev's inequality, (A.11) and (A.12) that

$$\begin{aligned}&-\log\bigg(\int \exp\Big(-\eta\big\|\pi_{\neq 0}|u_N|^2\big\|_{L^2}\Big)d\mu_1(u)\bigg)\\ &\quad= \inf_{\theta\in\mathbb{H}_a}\mathbb{E}\bigg[\eta\big\|\pi_{\neq 0}|\pi_N Y(1)+\pi_N I(\theta)(1)|^2\big\|_{L^2} + \frac12\int_0^1\|\theta(t)\|_{L^2_x}^2dt\bigg]\\ &\quad\ge \inf_{\theta\in\mathbb{H}_a}\mathbb{E}\bigg[\eta\Big(\big\|\pi_{\neq 0}|\pi_N Y(1)|^2\big\|_{L^2} - 2\|\pi_N Y(1)\pi_N I(\theta)(1)\|_{L^2}\\ &\qquad\qquad - \|\pi_N I(\theta)(1)\|_{L^4}^2\Big) + \frac12\int_0^1\|\theta(t)\|_{L^2_x}^2dt\bigg]\\ &\quad\ge \inf_{\theta\in\mathbb{H}_a}\mathbb{E}\bigg[\eta\Big(\big\|\pi_{\neq 0}|\pi_N Y(1)|^2\big\|_{L^2} - \varepsilon\|\pi_N Y(1)\|_{L^4}^2\Big) + \frac14\int_0^1\|\theta(t)\|_{L^2_x}^2dt\bigg]\\ &\quad\gtrsim \eta(\log N).\end{aligned}$$

Therefore, we obtain

$$\int \exp\Big(-\eta\big\|\pi_{\neq 0}|u_N|^2\big\|_{L^2}\Big)d\mu_1(u) \le \exp(-c\eta\log N)$$

for some constant $c>0$. Then, by Chebyshev's inequality, we conclude that, for any $M>0$,

$$\mu_1\Big(\big\|\pi_{\neq 0}|u_N|^2\big\|_{L^2} > M\Big) \ge 1-\exp\big(\eta(M-c\log N)\big) \longrightarrow 1, \tag{A.13}$$

as $N\to\infty$.

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We also note that, given any $K > 0$, there exists a constant $c_K > 0$ such that

$$\mu_1\Big(\Big|\int_{\mathbb{T}^2} :|u_N|^2: dx\Big| \le K\Big) \ge c_K, \tag{A.14}$$

uniformly in $N \in \mathbb{N}$. Indeed, for $L = L(K) > 0$ (to be chosen later), as in (3.11), we have

$$\mathbb{E}_{\mu_1}\big[e^L \cdot \mathbf{1}_{\{|\int_{\mathbb{T}^2}:|u_N|^2:dx|\le K\}}\big] \ge \mathbb{E}_{\mu_1}\Big[\exp\big(L\cdot \mathbf{1}_{\{|\int_{\mathbb{T}^2}:|u_N|^2:dx|\le K\}}\big)\Big] - 1. \tag{A.15}$$

Now, by repeating the argument in Subsection 3.2, in particular, (3.34) and (3.42) with $M = M_0(K)$, we have

$$\begin{aligned} -\log \mathbb{E}_{\mu_1}&\Big[\exp\big(L\cdot \mathbf{1}_{\{|\int_{\mathbb{T}^2}:|u_N|^2:dx|\le K\}}\big)\Big] \\ &\le \mathbb{E}\Big[-L\cdot \mathbf{1}_{\{|\int_{\mathbb{T}^d}(:Y_N^2:+2Y_N\Theta^0+(\Theta^0)^2)dx|\le K\}} + \frac{1}{2}\int_0^1 \|\theta^0(t)\|_{L^2_x}^2 dt\Big] \\ &\le -\frac{1}{2}L + CM_0^d \log M_0 \le -\frac{1}{4}L \end{aligned} \tag{A.16}$$

by choosing $L = L(M_0) = L(K) \gg 1$ sufficiently large. From (A.15) and (A.16), we then obtain

$$\mu_1\Big(\Big|\int_{\mathbb{T}^2} :|u_N|^2: dx\Big| \le K\Big) \ge \frac{e^{\frac{1}{4}L}-1}{e^L} =: c_K,$$

yielding (A.14).

Therefore, from (A.8), (A.9), (A.13) and (A.14), we obtain, for any $K > 0$,

$$\begin{aligned} \lim_{N\to\infty} &\iiint \mathbf{1}_{\{|\int_{\mathbb{T}^2}:|u_N|^2:dx|\le K\}} e^{Q_N(u,W)} d\mu_1(u) d\mu_0(W) \prod_{j=1}^2 d\mu_0(V_j) \\ &\ge \liminf_{N\to\infty} \int \mathbf{1}_{\{|\int_{\mathbb{T}^2}:|u_N|^2:dx|\le K\}} \exp\Big(\frac{1}{4}\big\|\pi_{\ne 0}|u_N|^2\big\|_{L^2}^2 - CK^2\Big) d\mu_1(u) \\ &\ge \liminf_{N\to\infty} \big(c_K - \exp\big(\eta(M - c\log N)\big)\big) \exp\Big(\frac{1}{4}M^2 - CK^2\Big) \\ &= c_K \exp\Big(\frac{1}{4}M^2 - CK^2\Big) \longrightarrow \infty \end{aligned}$$

by taking $M \to \infty$. This shows the nonnormalizability of the Gibbs measure for the Zakharov system on $\mathbb{T}^2$ even if we apply the Wick renormalization on the potential energy $Q(u, W)$ in (A.6) and endow the measure with a Wick-ordered L^2-cutoff on the Schrödinger component.

Another way would be to apply a change of variables as in the one-dimensional case due to Bourgain [9]. Namely, rewrite the Hamiltonian in (A.4) as in the one-dimensional case by Bourgain [9]:

$$H(u, W, \vec{V}) = \frac{1}{2}\int_{\mathbb{T}^2} |\nabla u|^2 dx - \frac{1}{4}\int_{\mathbb{T}^2} |u|^4 dx + \frac{1}{2}\int_{\mathbb{T}^2} (W + \sqrt{2}|u|^2)^2 dx + \frac{1}{2}\int_{\mathbb{T}^2} |\vec{V}|^2 dx.$$

By introducing a new variable $\widetilde{W} = W + \sqrt{2}|u|^2$, we arrive at

$$\widetilde{H}(u, \widetilde{W}, \vec{V}) = \frac{1}{2}\int_{\mathbb{T}^2} |\nabla u|^2 dx - \frac{1}{4}\int_{\mathbb{T}^2} |u|^4 dx + \frac{1}{2}\int_{\mathbb{T}^2} \widetilde{W}^2 dx + \frac{1}{2}\int_{\mathbb{T}^2} |\vec{V}|^2 dx.$$

Then, we apply the Wick renormalization to the potential energy.

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In this formulation, we consider the renormalized truncated Gibbs measure $\widetilde{\rho}_N$ defined by

$$d\widetilde{\rho}_N = Z_N^{-1}\mathbf{1}_{\{|\int_{\mathbb{T}^2}:|u_N|^2:dx|\le K\}}e^{R_N(u)}d\mu_1(u)d\mu_0(\widetilde{W})\prod_{j=1}^{2}d\mu_0(V_j),$$

where the renormalized truncated potential energy R_N is defined by

$$R_N(u) = \frac{1}{4}\int_{\mathbb{T}^2}:|u_N|^4:\,dx.$$

Note that, in the complex-valued setting, the Wick-ordered fourth power is given by

$$:|u_N|^4: = |u_N|^4 - 4\sigma_N|u_N|^2 + 2\sigma_N^2.$$

See [47]. Then, by integrating in $\widetilde{W}$ and $\vec{V}$ and then by applying Theorem 1.4 (in the complex-valued setting), we have

$$\begin{aligned}\sup_{N\in\mathbb{N}}&\iiint \mathbf{1}_{\{|\int:|u_N|^2:dx|\le K\}}e^{R_N(u)}d\mu_1(u)d\mu_0(\widetilde{W})\prod_{j=1}^{2}d\mu_0(V_j)\\&= \sup_{N\in\mathbb{N}}\int \mathbf{1}_{\{|\int:|u_N|^2:dx|\le K\}}e^{R_N(u)}d\mu_1(u) = \infty\end{aligned}$$

for any $K>0$. This shows the nonnormalizability of the limiting Gibbs measure in this formulation.

Remark A.1. In the renormalization (A.7), we added the term $\frac{\sigma_N}{\sqrt{2}}\int_{\mathbb{T}^2}Wdx = \frac{\sigma_N}{2}\int_{\mathbb{T}^2}wdx$. Note that the spatial mean of w is conserved under the flow of the system (A.2). Thus, by imposing the spatial mean-zero condition on w, we can write $Q_N(u,W)$ in (A.7) as

$$Q_N(u,W) = -\frac{1}{\sqrt{2}}\int_{\mathbb{T}^2}:|u_N|^2:\,Wdx = -\frac{1}{\sqrt{2}}\int_{\mathbb{T}^2}|u_N|^2Wdx,$$

showing that this term is self-renormalizing, and thus the renormalization (A.7) does not affect the system (A.2).

B. Focusing quartic Gibbs measures with smoother Gaussian fields

In this appendix, we briefly discuss the construction of the focusing Gibbs measure ρ_α in (1.37) with a smoother base Gaussian measure μ_α in (1.36). We only discuss the uniform exponential integrability bound (1.35). Since the precise value of $\lambda\in\mathbb{R}\setminus\{0\}$ does not play any role, we set $\lambda=4$ in the following. As before, we also assume $p=1$ for simplicity.

Fix $\alpha>\frac{d}{2}$. The Gaussian measure μ_α in (1.37) is the induced probability measure under the map:

$$\omega\in\Omega\longmapsto u(\omega) = \sum_{n\in\mathbb{Z}^d}\frac{g_n(\omega)}{\langle n\rangle^\alpha}e_n,$$

where $\{g_n\}_{n\in\mathbb{Z}^d}$ is as in (1.3). In particular, a typical function u in the support of μ belongs to $L^\infty(\mathbb{T}^d)$.

We define Y^α by

$$Y^\alpha(t) = \langle\nabla\rangle^{-\alpha}W(t),$$

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where W is as in (3.1). Then, in view of the Boué–Dupuis formula (Lemma 3.1), it suffices to establish a lower bound on

$$\mathcal{W}_N^\alpha(\theta) = \mathbb{E}\bigg[-R_N^{\diamond,\gamma}(Y^\alpha(1) + I^\alpha(\theta)(1)) + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L_x^2}^2 dt \bigg], \tag{B.1}$$

uniformly in $N \in \mathbb{N}$ and $\theta \in \mathbb{H}_a$, where $R_N^{\diamond,\gamma}(u)$ and $I^\alpha(\theta)$ are defined by

$$R_N^{\diamond,\gamma}(u) = \int_{\mathbb{T}^d} u_N^4 dx - A\bigg(\int_{\mathbb{T}^d} u_N^2 dx\bigg)^\gamma \tag{B.2}$$

for some $\gamma > 0$ (to be chosen later) and

$$I^\alpha(\theta)(t) = \int_0^t \langle\nabla\rangle^{-\alpha}\theta(t')dt'.$$

For simplicity of notation, we set $Y_N^\alpha = \pi_N Y^\alpha = \pi_N Y^\alpha(1)$ and $\Theta_N^\alpha = \pi_N \Theta^\alpha = \pi_N I^\alpha(\theta)(1)$.

From (B.1) and (B.2), we have

$$\begin{aligned}
\mathcal{W}_N^\alpha(\theta) = \mathbb{E}\bigg[&-\int_{\mathbb{T}^d} (Y_N^\alpha)^4 dx - 4\int_{\mathbb{T}^d} (Y_N^\alpha)^3\Theta_N^\alpha dx - 6\int_{\mathbb{T}^d} (Y_N^\alpha)^2(\Theta_N^\alpha)^2 dx \\
&- 4\int_{\mathbb{T}^d} Y_N^\alpha(\Theta_N^\alpha)^3 dx - \int_{\mathbb{T}^d} (\Theta_N^\alpha)^4 dx + A\bigg\{\int_{\mathbb{T}^d} \big(Y_N^\alpha + \Theta_N^\alpha\big)^2 dx\bigg\}^2 \\
&+ \frac{1}{2}\int_0^1 \|\theta(t)\|_{L_x^2}^2 dt\bigg].
\end{aligned}$$

Let us first state a lemma, analogous to Lemma 4.1.

Lemma B.1. (i) *Let $\alpha > \frac{d}{2}$. Then, there exists $c > 0$ such that*

$$\bigg|\int_{\mathbb{T}^d} (Y_N^\alpha)^3\Theta_N^\alpha dx\bigg| \le c\|Y_N^\alpha\|_{L^\infty}^6 + \frac{1}{100}\|\Theta_N^\alpha\|_{L^2}^2, \tag{B.3}$$

$$\bigg|\int_{\mathbb{T}^d} (Y_N^\alpha)^2(\Theta_N^\alpha)^2 dx\bigg| \le c\|Y_N^\alpha\|_{L^\infty}^4 + \frac{1}{100}\|\Theta_N^\alpha\|_{L^2}^4, \tag{B.4}$$

$$\bigg|\int_{\mathbb{T}^d} Y_N^\alpha(\Theta_N^\alpha)^3 dx\bigg| \le c\|Y_N^\alpha\|_{L^\infty}^4 + \frac{1}{100}\|\Theta_N^\alpha\|_{L^4}^4, \tag{B.5}$$

$$\bigg|\int_{\mathbb{T}^d} (\Theta_N^\alpha)^4 dx\bigg| \le \frac{1}{100}\|\Theta_N^\alpha\|_{H^\alpha}^2 + \frac{A}{100}\|\Theta_N^\alpha\|_{L^2}^{\frac{8\alpha-2d}{2\alpha-d}} \tag{B.6}$$

for any sufficiently large $A > 0$, uniformly in $N \in \mathbb{N}$.

(ii) *Let $A, \gamma > 0$. Then, there exists $c = c(A, \gamma) > 0$ such that*

$$A\bigg\{\int_{\mathbb{T}^d} \big(Y_N^\alpha + \Theta_N^\alpha\big)^2 dx\bigg\}^\gamma \ge \frac{A}{4}\|\Theta_N^\alpha\|_{L^2}^{2\gamma} - c\|Y_N^\alpha\|_{L^\infty}^{2\gamma}, \tag{B.7}$$

uniformly in $N \in \mathbb{N}$.

Set

$$\gamma = \frac{4\alpha - d}{2\alpha - d}. \tag{B.8}$$

Then, by arguing as in Section 4 with Lemma B.1,[25] the almost sure L^∞-regularity of Y^α and a variant of (3.6) for $\Theta^\alpha = I^\alpha(\theta)(1)$:

$$\|\Theta^\alpha\|_{H^\alpha}^2 \le \int_0^1 \|\theta(t)\|_{L^2}^2 dt,$$

we obtain the following uniform lower bound:

$$\inf_{N\in\mathbb{N}} \inf_{\theta\in\mathbb{H}_a} \mathcal{W}_N^\alpha(\theta) \ge -C_0 > -\infty. \tag{B.9}$$

Then, the uniform exponential integrability (1.35) follows from (B.9) and Lemma 3.1.

We now present the proof of Lemma B.1.

Proof of Lemma B.1. (i) The estimates (B.3), (B.4) and (B.5) follow from Hölder's and Young's inequalities. As for the fourth estimate (B.6), it follows from Sobolev's inequality, Lemma 2.1 (i) and Young's inequality that

$$\begin{aligned}\left|\int_{\mathbb{T}^d}(\Theta_N^\alpha)^4 dx\right| &\le C\|\Theta_N^\alpha\|_{H^{\frac{d}{4}}}^4 \le C\|\Theta_N^\alpha\|_{H^\alpha}^{\frac{d}{\alpha}}\|\Theta_N^\alpha\|_{L^2}^{4-\frac{d}{\alpha}}\\ &\le \frac{1}{100}\|\Theta_N^\alpha\|_{H^\alpha}^2 + \frac{A}{100}\|\Theta_N^\alpha\|_{L^2}^{\frac{8\alpha-2d}{2\alpha-d}}\end{aligned}$$

for sufficiently large $A > 0$.

(ii) Note that

$$|a+b+c|^\gamma \ge \frac{1}{2}|c|^\gamma - C_\gamma(|a|^\gamma + |b|^\gamma) \tag{B.10}$$

for any $a, b, c \in \mathbb{R}$. Then, the bound (B.7) follows from (B.10) and

$$\left|\int_{\mathbb{T}^d} Y_N^\alpha \Theta_N^\alpha dx\right|^\gamma \le c\|Y_N^\alpha\|_{L^\infty}^{2\gamma} + \frac{1}{100C_\gamma}\|\Theta_N^\alpha\|_{L^2}^{2\gamma}. \qquad \square$$

Remark B.2. Let γ be as in (B.8). Then, we have $\gamma > 2$. Moreover, we have $\gamma \to \infty$ as $\alpha \to \frac{d}{2}+$, indicating an issue at $\alpha = \frac{d}{2}$ even if we disregard a renormalization required for $\alpha = \frac{d}{2}$.

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[25] We bound the second term on the right-hand side of (B.5) by (B.6).

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