A NOTE ON UNIQUENESS FOR ANISOTROPIC FLUIDS

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1. Introduction. In 1960 Ericksen [**1**] introduced a simple theory of anisotropic fluids. This theory differs from the classical theory of fluids in that the deformation of the material is no longer solely described by the usual vector displacement field but requires in addition the specification of a further vector field d_i, termed the director. Moreover, corresponding to this increased kinematic flexibility new types of stress, body force and inertia are introduced. Leslie [**2**], adopting the conservation laws of [**1**], formulated constitutive equations similar to those considered by Ericksen and discussed the thermodynamical restrictions imposed by the Clausius–Duhem inequality. Here we shall consider the case in which at each point the director is constrained to remain a unit vector. Then the usual interpretation is to regard d_i as indicating a single preferred direction in the material (see for example [**3**]). It is thought that the physical applications of this theory are likely to lie in such areas as polymeric fluids and suspensions.

In the present note we shall consider the uniqueness of the solutions of the equations governing the isothermal motions of an incompressible anisotropic fluid with a director of constant (unit) magnitude. We assume that the fluid occupies a bounded region D of three-space with a boundary ∂D which is smooth enough to allow applications of the divergence theorem. If we introduce the director velocity w_i by

$$w_i = \dot{d}_i,$$

where a superposed dot denotes material time differentiation, then the fluid velocity v_i and the director velocity w_i satisfy the equations [**2**]

$$v_{i,i} = 0, \tag{1.1}$$

$$\rho\dot{v}_i = \rho F_i - p_{,i} + \tfrac{1}{2}\alpha_4 v_{i,jj} + \{\alpha_1 d_i d_j d_k d_p v_{(k,p)} + \alpha_2(d_j w_i + v_{[k,i]} d_j d_k) + \alpha_3(d_i w_j + v_{[k,j]} d_i d_k) + \alpha_5 d_j d_k v_{(k,i)} + \alpha_6 d_i d_k v_{(k,j)}\}_{,j} \tag{1.2}$$

$$\rho_1 \dot{w}_i = \rho_1 L_i - \gamma d_i - (\alpha_3 - \alpha_2)[w_i + v_{[k,i]} d_k] - (\alpha_6 - \alpha_5) d_j v_{(j,i)} \tag{1.3}$$

in the space-time cylinder $D \times [0, T]$, where T is a finite instant of time and

$$v_{(i,j)} = \tfrac{1}{2}(v_{i,j} + v_{j,i}), \qquad v_{[i,j]} = \tfrac{1}{2}(v_{i,j} - v_{j,i}).$$

Throughout this note we employ the usual convention of summing over repeated indices and a subscript k following a comma indicates partial differentiation with respect to the space variable x_k. In equations (1.2) and (1.3), ρ denotes the (constant) density, ρ_1 is a positive constant and F_i and L_i are the prescribed body forces. The unknown scalar functions p and γ are called the pressure and director tension, respectively, and they arise from the constraints of

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incompressibility and the director having a fixed magnitude. The coefficients α_1 $(i = 1, 2, \ldots, 6)$ are constants which, from thermodynamical considerations, are restricted to satisfy the conditions [2]

$$\alpha_4 \geqq 0, \quad 2\alpha_1 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 \geqq 0, \quad 2\alpha_4 + \alpha_5 + \alpha_6 \geqq 0,$$
$$(\alpha_3 - \alpha_2) \geqq 0, \quad 4(\alpha_3 - \alpha_2)(2\alpha_4 + \alpha_5 + \alpha_6) \geqq (\alpha_2 + \alpha_3 + \alpha_6 - \alpha_5)^2. \tag{1.4}$$

For the purposes of this note it is convenient to introduce a new variable Ω_i termed the rotational velocity and defined by

$$\Omega_i = \varepsilon_{ijk} d_j w_k, \quad w_i = \varepsilon_{ijk} \Omega_j d_k \tag{1.5}$$

where ε_{ijk} denotes the alternating tensor. Employing $(1.5)_2$, we can rewrite (1.2) as

$$\rho \dot{v}_i = \rho F_i - p_{,i} + \tfrac{1}{2}\alpha_4 v_{i,jj} + \{\alpha_1 d_i d_j d_k d_p v_{(k,p)} + \alpha_2(\varepsilon_{ipk}\Omega_p d_j d_k + v_{[k,i]} d_j d_k)$$
$$+ \alpha_3(\varepsilon_{jpk}\Omega_p d_i d_k + v_{[k,j]} d_i d_k) + \alpha_5 v_{(k,i)} d_j d_k + \alpha_6 v_{(k,j)} d_i d_k\}_{,j} \tag{1.6}$$

and moreover, by multiplying (1.3) by $\varepsilon_{kji} d_j$ we obtain an equation that does not involve the director tension γ, namely,

$$\rho_1 \dot{\Omega}_i = \rho_1 \varepsilon_{ijk} d_j L_k - (\alpha_3 - \alpha_2)[\Omega_i + \varepsilon_{ijk} v_{[p,k]} d_j d_p] - (\alpha_6 - \alpha_5)\varepsilon_{ijk} v_{(p,k)} d_j d_p. \tag{1.7}$$

We shall restrict our attention to classical solutions which are assumed to exist in $[0, T)$ subject to prescribed initial conditions

$$v_i(\mathbf{x}, 0) = f_i(\mathbf{x}), \quad d_i(\mathbf{x}, 0) = g_i(\mathbf{x}), \quad w_i(\mathbf{x}, 0) = h_i(\mathbf{x}) \quad \text{in} \quad D \times 0 \tag{1.8}$$

and, on $\partial D \times (0, T)$, prescribed boundary conditions of one of the following types:

$$v_i(\mathbf{x}, t) = F_i(\mathbf{x}, t), \quad F_i n_i = 0, \tag{1.9}$$

or, if $F_i n_i \neq 0$,

$$v_i(\mathbf{x}, t) = F_i(\mathbf{x}, t), \quad d_i(\mathbf{x}, t) = G_i(\mathbf{x}, t), \quad w_i(\mathbf{x}, t) = H_i(\mathbf{x}, t), \tag{1.10}$$

$$v_i(\mathbf{x}, t) = F_i(\mathbf{x}, t), \quad d_i(\mathbf{x}, t) = G_i(\mathbf{x}, t), \quad \partial d_i(\mathbf{x}, t) = J_i(\mathbf{x}, t), \tag{1.11}$$

where n_i are the Cartesian components of the unit normal to ∂D and ∂ is the normal gradient operator. In what follows we employ the familiar energy arguments to show that the fluid motion in D is uniquely determined by the initial velocity, director and director velocity together with one of the sets of boundary conditions (1.9), (1.10) or (1.11).

2. Uniqueness. We shall say that a continuously differentiable pair (v_i, d_i) is a solution to problem $\mathscr{A}$ if it satisfies the equations (1.1)–(1.3), the boundary conditions (1.9) and at $t = 0$ reduces to the conditions $(1.8)_{1,2}$. Furthermore, if (v_i, d_i) is a solution pair of problem $\mathscr{A}$ with an associated rotational velocity Ω_i and the pair (v_i^*, d_i^*) a solution corresponding to the same body forces F_i and L_i with rotational velocity Ω_i^*, then it is easily verified that the difference fields

$$u_i = v_i - v_i^*, \qquad D_i = d_i - d_i^*, \qquad \mu_i = \Omega_i - \Omega_i^* \tag{2.1}$$

satisfy the equations

$$\rho u_{i,t} = -\rho v_j u_{i,j} - \rho v^*_{i,j} u_j - (p_{,i} - p^*_{,i}) + \tfrac{1}{2}\alpha_4 u_{i,jj} + \alpha_1 A_{ij,j} + \alpha_2 B_{ij,j} + \alpha_3 B_{ji,j} + \alpha_5 C_{ij,j} + \alpha_6 C_{ji,j} \tag{2.2}$$

and

$$\rho_1 \mu_{i,t} = -\rho_1 v_j \mu_{i,j} - \rho_1 \Omega^*_{i,j} u_j - (\alpha_3 - \alpha_2)[\mu_i - \varepsilon_{ijk} u_{[p,j]} d_k d_p - \varepsilon_{ijk} v^*_{[p,j]}(D_k d_p + D_p d^*_k)] + (\alpha_6 - \alpha_5)[\varepsilon_{ijk} u_{(p,j)} d_k d_p + \varepsilon_{ijk} v^*_{(p,j)}(D_k d_p + D_p d^*_k)], \tag{2.3}$$

where

$$A_{ij} = u_{(k,p)} d_i d_j d_k d_p + v^*_{(k,p)}(D_i d_j d_k d_p + D_j d_k d_p d^*_i + D_k d_p d^*_i d^*_j + D_p d^*_i d^*_j d^*_k),$$

$$B_{ij} = \varepsilon_{ipk} \mu_p d_j d_k + \varepsilon_{ipk} \Omega^*_p (D_j d_k + D_k d^*_j) + u_{[k,i]} d_j d_k + v^*_{[k,i]}(D_j d_k + D_k d^*_j),$$

$$C_{ij} = u_{(k,i)} d_j d_k + v^*_{(k,i)}(D_j d_k + D_k d^*_j),$$

and a comma followed by a t indicates partial differentiation with respect to time, holding the spatial variable fixed. Moreover, from the definitions (1.5) and (2.1), we deduce

$$D_i D_{i,t} = \varepsilon_{ijk} \mu_j D_i d_k - D_i D_{i,j} v_j - D_i u_j d^*_{i,j}. \tag{2.4}$$

We are now in a position to establish the

THEOREM. *If the coefficients* α_i $(i = 1, 2, \ldots, 6)$ *satisfy* (1.4) *but with the strict inequality holding for* $(1.4)_{1,2,3}$, *then there is at most one solution of the initial value problem* $\mathscr{A}$.

We first introduce the function $F(t)$ defined by

$$F(t) = \int_{D_t} (\rho u_i u_i + \rho_1 \mu_i \mu_i + D_i D_i)\, dx,$$

where the symbol D_t indicates that the integral is to be taken over the region D at time t. Differentiation using the boundary conditions (1.9) yields

$$F'(t) = 2\int_{D_t} (\rho u_i u_{i,t} + \rho_1 \mu_i \mu_{i,t} + D_i D_{i,t})\, dx.$$

Consequently, on using the equations (2.2)–(2.4) and integrating by parts, we obtain

$$F'(t) = -2\int_{D_t} \{\rho u_i u_j v^*_{(i,j)} + \tfrac{1}{2}\alpha_4 u_{i,j} u_{i,j} + \alpha_1 u_{(i,j)} A_{ij} + \alpha_2 u_{i,j} B_{ij} + \alpha_3 u_{i,j} B_{ji} + \alpha_5 u_{i,j} C_{ij} + \alpha_6 u_{i,j} C_{ji} + \rho_1 \mu_i u_j \Omega^*_{i,j} + (\alpha_3 - \alpha_2)\mu_i[\mu_i - \varepsilon_{ijk} u_{[p,j]} d_k d_p - \varepsilon_{ijk} v^*_{[p,j]}(D_k d_p + D_p d^*_k)] - (\alpha_6 - \alpha_5)\mu_i \varepsilon_{ijk}[u_{(p,j)} d_k d_p + v^*_{(p,j)}(D_k d_p + D_p d^*_k)] + D_i u_j d^*_{i,j} + \varepsilon_{ijk} \mu_k D_i d_j\}\, dx.$$

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Rearranging the terms, we can write

$$F'(t) = -2\int_{D_t} \{\rho u_i u_j v^*_{(i,j)} + \rho_1 \mu_i u_j \Omega^*_{i,j} + u_j D_i d^*_{i,j} + \varepsilon_{ijk}\mu_k D_i d_j$$
$$+\alpha_1 u_{i,j} v^*_{(k,p)}(D_i d_j d_k d_p + D_j d_k d_p d_i^* + D_k d_p d_i^* d_j^* + D_p d_i^* d_j^* d_k^*)$$
$$+\alpha_2 u_{i,j}(\Omega^*_p \varepsilon_{ipk} + v^*_{[k,i]})(D_j d_k + D_k d_j^*) + \alpha_3 u_{i,j}(\Omega^*_p \varepsilon_{jpk} + v^*_{[k,i]})(D_i d_k + D_k d_i^*)$$
$$+\alpha_5 u_{i,j} v^*_{(k,i)}(D_j d_k + D_k d_j^*) + \alpha_6 u_{i,j} v^*_{(k,j)}(D_i d_k + D_k d_i^*)$$
$$+(\alpha_3-\alpha_2)\mu_p v^*_{[i,j]}\varepsilon_{jpk}(D_i d_k + D_k d_i^*) + (\alpha_6-\alpha_5)\mu_p v^*_{(i,j)}\varepsilon_{jpk}(D_i d_k + D_k d_i^*)$$
$$+(\alpha_3-\alpha_2)\mu_i\mu_j d_i d_j\}\, dx - 2Q, \tag{2.5}$$

where

$$Q = \int_{D_t} \{\tfrac{1}{2}\alpha_4 u_{i,j}u_{i,j} + \alpha_1 u_{(i,j)}u_{(k,p)} d_i d_j d_k d_p + (\alpha_5+\alpha_6)u_{(i,j)}u_{(k,j)} d_i d_k$$
$$+(\alpha_2+\alpha_3+\alpha_6-\alpha_5)u_{(i,j)} d_i N_j + (\alpha_3-\alpha_2)N_i N_i\}\, dx$$

and

$$N_i = \varepsilon_{ipk}\mu_p d_k + u_{[k,i]} d_k.$$

We then proceed to obtain estimates for the terms in the braces on the right hand side of (2.5). We let M be the generic notation for an upper bound and note that, although the number M will differ for every estimate, nevertheless it is always possible to determine its size. We employ weighted arithmetic-geometric mean and Schwarz's inequalities to obtain

$$F'(t) \leqq -2Q + \int_{D_t}\left\{M(u_i u_i + \mu_i \mu_i + D_i D_i) + \frac{2}{\lambda}u_{i,j}u_{i,j}\right\} dx.$$

Here λ is an arbitrary positive constant which is to be prescribed, while M depends on λ and the bounds of Ω_i^* and the spatial derivatives of v_i^*, d_i^*, Ω_i^* during $[0, T]$.

Now, since α_4 is assumed strictly positive, we can choose a value of λ such that

$$\alpha_4' = \alpha_4 - \frac{1}{\lambda}$$

is nonnegative. Moreover, if the inequalities (1.4) hold with α_4' replacing α_4, then we can show that (see [2], (5.8))

$$Q - \frac{1}{\lambda}\int_{D_t} u_{i,j}u_{i,j}\, dx$$

is nonnegative. Consequently, if we restrict ourselves to those materials for which the inequalities $(1.4)_{1,2,3}$ hold strictly, we can deduce that

$$F'(t) \leqq MF(t).$$

Thus, by integrating from $t = 0$ to $t = T$ using the initial conditions, we conclude that

$$F(T)\exp(-MT) \leqq 0.$$

It follows that $F(t)$ is zero for all $t \in [0, T]$ and thus $u_i = \mu_i = D_i = 0$, so that the two flows are identical. Moreover, since T is an arbitrary instant, we can conclude that the flows are identical as long as they exist.

We remark that, in the preceding proof, the boundary condition $(1.9)_2$ enabled us to show that the surface integrals

$$\int_{\partial D_t} n_i v_i \mu_k \mu_k \, dx, \qquad \int_{\partial D_t} n_i v_i D_k D_k \, dx \tag{2.6}$$

are zero. Clearly these integrals will also vanish with boundary data (1.10). Moreover, given any surface, we can express the director velocity w_i over the surface in terms of the surface gradient operator ∂_j and the normal gradient operator ∂, thus:

$$w_i = d_{i,t} + v_j \partial_j d_i + v_j n_j \partial d_i,$$

where n_i is the unit normal to the surface, so that under data (1.11) the integrals (2.6) will again vanish. Consequently the corresponding uniqueness theorems for data (1.10) and (1.11) replacing (1.9) can be established.

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