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On the vanishing of local cohomology in characteristic $p > 0$

Gennady Lyubeznik

Abstract

Let R be a d-dimensional regular local ring of characteristic $p > 0$ with maximal ideal $\mathfrak{m}$, let I be an ideal of R and let $A = R/I$. We describe some properties of the local cohomology module $H^i_I(R)$, in particular its vanishing, in terms of the Frobenius action on the local cohomology module $H^{d-i}_{\mathfrak{m}}(A)$.

1. Introduction

A. Grothendieck stated a problem [Gro67, p. 79] that in the language of local algebra says the following. *Let R be a commutative Noetherian local ring and let $I \subset R$ be an ideal. If n is an integer, find conditions under which $H^i_I(M) = 0$ for all $i > n$ and all R-modules M.* Here $H^i_I(M)$ is the ith local cohomology module of M with support in I.

This problem has stimulated a very fruitful line of research; important conditions for the vanishing of $H^i_I(M)$ for all $i > n$ and all M have been found by Faltings [Fal80], Grothendieck [Gro67], Hartshorne [Har68], Hartshorne and Speiser [HS77], Huneke and Lyubeznik [HL90], Ogus [Ogu73] and Peskine and Szpiro [PS73].

It is known that $H^i_I(M) = 0$ for all $i > n$ and all R-modules M if and only if $H^i_I(R) = 0$ for all $i > n$ (see [Har68, p. 413]). Accordingly, it would be enough to find conditions for the vanishing of $H^i_I(R)$.

In the general case no reasonable necessary and sufficient conditions are known. However, if R is regular and contains a field, necessary and sufficient conditions have been found by Ogus [Ogu73, 2.8] in characteristic zero and by Hartshorne and Speiser [HS77, 2.5b] in characteristic $p > 0$. One of the main goals of this paper is to produce a considerably simpler necessary and sufficient condition in characteristic $p > 0$.

Standard assumption. For the rest of the paper all rings are assumed to contain a field of characteristic $p > 0$.

We now assume for the rest of this paper that R is a commutative Noetherian ring. Let $I \subset R$ be an ideal. We denote R/I by A. The natural ring homomorphism $A \xrightarrow{a \mapsto a^p} A$ induces a map $f : H^j_J(A) \to H^j_J(A)$ on local cohomology groups of A with support in any ideal J of A. This map f is called *the action of the Frobenius* on $H^j_J(A)$; it is a homomorphism of abelian groups such that $f(ax) = a^p f(x)$ for all $a \in A$ and all $x \in H^j_J(A)$.

One of our main results is Corollary 3.2, which states the following.

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THEOREM 1.1. *Let R be a regular local ring of dimension n with maximal ideal $\mathfrak{m}$. Let $I \subset R$ be an ideal and let $A = R/I$. Then $H_I^{n-i}(R) = 0$ if and only if there is s such that $f^s : H_{\mathfrak{m}}^i(A) \to H_{\mathfrak{m}}^i(A)$ is the zero map (where $f^1 = f$ and $f^s = f^{s-1} \circ f$ for $s > 1$).*

In fact, even if $H_I^{n-i}(R)$ does not vanish, a lot of information about its structure is encoded in the Frobenius action on $H_{\mathfrak{m}}^i(A)$ and the aim of this paper is to decode some of this information. Our Theorem 1.1 is a special case of a more general result (Theorem 3.1) that describes the support of $H_I^{n-i}(R)$ in terms of the Frobenius action on $H_{\mathfrak{m}}^i(A)$. Furthermore, we give a necessary and sufficient criterion for $H_I^{n-i}(R)$ to be supported at $\mathfrak{m}$ (Corollary 3.3) and describe the structure of $H_I^{n-i}(R)$ in this case (Corollary 3.4), all in terms of the Frobenius action on $H_{\mathfrak{m}}^i(A)$.

Our Theorem 1.1 leads to a solution of Grothendieck's problem in terms of the F-depth of A (Definition 4.1) which we define as the smallest integer i such that f^s does not send $H_{\mathfrak{m}}^i(A)$ to zero for any s (we call it the F-depth by analogy with the usual depth of A which equals the smallest i such that $H_{\mathfrak{m}}^i(A) \neq 0$). Namely, we prove that $H_I^i(M) = 0$ for all $i > r$ and all R-modules M if and only if F-depth $A \geqslant n - r$ (Theorem 4.3). This is a striking sharpening of the previously known fact that if depth $A \geqslant n - r$, then $H_I^i(M) = 0$ for all $i > r$ and all R-modules M [PS73, Remarque, p. 110].

In § 5 we apply our main results to closed subschemes Y of projective space $\mathbb{P}_k^n$. In particular, we express the cohomological dimension of $\mathbb{P}_k^n \setminus Y$ in terms of the F-depths of the local rings $\mathcal{O}_{Y,y}$ at the closed points $y \in Y$ and the Frobenius action on the cohomology groups $H^i(Y, \mathcal{O}_Y)$ (Corollary 5.4).

Our solution to Grothendieck's problem is in terms of our notion of the F-depth of a local ring A whereas Hartshorne and Speiser's solution [HS77] is in terms of their notion of the F-depth of the scheme Spec A (see Definition 6.1). Our definition of F-depth A is considerably simpler because it is in terms of the Frobenius action on the (finitely many) local cohomology modules $H_{\mathfrak{m}}^i(A)$ whereas Hartshorne and Speiser's definition of the F-depth of Spec A is in terms of the Frobenius action on the (infinitely many) local cohomology modules $H_{\mathfrak{p}}^i(A_{\mathfrak{p}})$ as $\mathfrak{p}$ runs through all the prime ideals of A.

Our techniques are quite elementary. While our proofs came out of the circle of ideas around our theory of F-modules [Lyu97], we do not use F-modules in this paper. Our proof of Theorem 1.1 and our solution to Grothendieck's problem use nothing beyond some fairly standard commutative algebra and local cohomology.

In § 6, the last section of this paper, we discuss connections between our notion of the F-depth of A and Hartshorne and Speiser's notion of the F-depth of Spec A. We prove that they coincide for local rings A that admit a surjection from a regular local ring (Corollary 6.3). However, the only known (to us) way to prove this fact is via ours and Hartshorne and Speiser's solutions to Grothendieck's problem. The two definitions are quite different from each other and we are unaware of any other way to connect them. Finally, we use our results to settle a question left open in [HS77, p. 61] (Theorem 6.4).

2. Preliminaries

In this section we have collected some material that is well known to the experts but it is difficult to find it in the literature in the form in which we need it. Throughout this section R is a commutative Noetherian regular ring containing a field of characteristic $p > 0$.

Let $R^{(e)}$ be the additive group of R regarded as an R-bimodule with the usual left R-action and with the right R-action defined by $r'r = r^{p^e}r'$ for all $r \in R, r' \in R^{(e)}$. The Frobenius functor

$$F : R\text{-mod} \to R\text{-mod}$$

of Peskine and Szpiro [PS73, I.1.2] is defined by

$$F(M) = R^{(1)} \otimes_R M$$

$$F(M \xrightarrow{\phi} N) = (R^{(1)} \otimes_R M \xrightarrow{\mathrm{id} \otimes_R \phi} R^{(1)} \otimes_R N)$$

for all R-modules M and all R-module homomorphisms ϕ, where $F(M)$ acquires its R-module structure via the left R-module structure on $R^{(1)}$. It is not hard to see that the map $R^{(1)} \otimes_R R^{(e-1)} \xrightarrow{r_1 \otimes r_2 \mapsto r_1 r_2^p \cdot 1} R^{(e)}$ is an isomorphism of bimodules. Hence F^e, the eth iteration of the Frobenius morphism, is given by

$$F(M) = R^{(e)} \otimes_R M$$

$$F(M \xrightarrow{\phi} N) = (R^{(e)} \otimes_R M \xrightarrow{\mathrm{id} \otimes_R \phi} R^{(e)} \otimes_R N).$$

Since R is regular, a theorem of Kunz [Kun69, 2.1] implies that F is exact.

If M is a free R-module, $F^t(M) \cong M$ for every t. An explicit isomorphism is given by $\Sigma_i r_i \otimes a_i e_i \mapsto \Sigma_i r_i a_i^{p^t} e_i$ where the set $\{e_1, e_2, \dots\}$ is an R-basis of M. In particular we have an isomorphism $F^t(R) \to R$ given by $r \otimes a \mapsto r a^{p^t}$ for $r \in R^{(t)}$, $a \in R$. If $I \subset R$ is an ideal, this isomorphism sends $F^t(I) \subset F^t(R)$ onto $I^{[p^t]}$, the ideal of R generated by the p^tth powers of the elements of I, and thereby induces an isomorphism

$$\phi_t : F^t(R/I) = R^{(t)} \otimes_R R/I \xrightarrow{r \otimes a \mapsto r \tilde{a}^{p^t}} R/I^{[p^t]}, \tag{1}$$

where $r \in R^{(t)}, a \in R/I$ and $\tilde{a} \in R/I^{[p^t]}$ is any lifting of a. Let

$$\alpha : F(R/I) = R^{(1)} \otimes_R R/I \to R/I \tag{2}$$

be the R-module homomorphism defined by $r \otimes a \mapsto r a^p$ for $r \in R^{(1)}$ and $a \in R/I$. Then $F^t(\alpha)$: $F^{t+1}(R/I) \to F^t(R/I)$ is defined by $r \otimes a \mapsto r \otimes a^p$ and we have the following commutative diagram, where the map in the top row is the natural surjection and the vertical maps are isomorphisms.

$$\begin{array}{ccc} R/I^{[p^{t+1}]} & \longrightarrow & R/I^{[p^t]} \\ \uparrow{\scriptstyle \phi_{t+1}} & & \uparrow{\scriptstyle \phi_t} \\ F^{t+1}(R/I) & \xrightarrow{F^t(\alpha)} & F^t(R/I) \end{array} \tag{3}$$

Applying the functor $\mathrm{Ext}^j_R(-, R)$ to (3) we get the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^j_R(R/I^{[p^t]}, R) & \xrightarrow{\xi_t} & \mathrm{Ext}^j_R(R/I^{[p^{t+1}]}, R) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^j_R(F^t(R/I), R) & \longrightarrow & \mathrm{Ext}^j_R(F^{t+1}(R/I), R) \end{array} \tag{4}$$

in which the vertical maps are isomorphisms.

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Let $\eta : M' \to M$ be an R-module homomorphism between finitely generated R-modules and let $\tilde{\eta} : \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_R(M', R)$ be the induced map. We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(R^{(t)} \otimes_R M, R) & \longrightarrow & \mathrm{Hom}_R(R^{(t)} \otimes_R M', R) \\ \uparrow & & \uparrow \\ R^{(t)} \otimes_R \mathrm{Hom}_R(M, R) & \xrightarrow{F^t(\tilde{\eta})} & R^{(t)} \otimes_R \mathrm{Hom}_R(M', R) \end{array} \tag{5}$$

in which the top map is induced by $F^t(\eta) : F^t(M') \to F^t(M)$ and the vertical maps are R-module isomorphisms which send $1 \otimes \varphi$ to the map sending $1 \otimes x$ to $\varphi(x)^{p^t} \in R$, where $\varphi \in \mathrm{Hom}_R(M, R)$ and $x \in M$ (respectively, $\varphi \in \mathrm{Hom}_R(M', R)$ and $x \in M'$).

To see that the vertical maps in (5) are indeed isomorphisms note that it is enough to show this for the left map, as the right map is completely analogous. It is straightforward that the left map is an isomorphism if $M \cong R$, hence also if M is a finite free R-module. For a general finitely generated M there is an exact sequence $L_1 \to L_0 \to M \to 0$ where L_0 and L_1 are finite free R-modules. This induces the following commutative diagram in which $H(-)$ denotes $\mathrm{Hom}_R(-, R)$ for brevity.

$$\begin{array}{ccccccc} 0 & \longrightarrow & H(R^{(t)} \otimes_R M) & \longrightarrow & H(R^{(t)} \otimes_R L_0) & \longrightarrow & H(R^{(t)} \otimes_R L_1) \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & R^{(t)} \otimes_R H(M) & \longrightarrow & R^{(t)} \otimes_R H(L_0) & \longrightarrow & R^{(t)} \otimes_R H(L_1) \end{array}$$

Since the vertical maps in the middle and on the right are isomorphisms and the rows are exact, the vertical map on the left is also an isomorphism. However, this is the same map as the left vertical map in the diagram (5).

The commutative diagram (5) establishes an isomorphism of functors

$$F^t(\mathrm{Hom}_R((-), R)) \to \mathrm{Hom}_R(F^t(-), R) \tag{6}$$

on the category of finitely generated R-modules. Since F^t is exact and takes finite free resolutions to finite free resolutions, we have an induced isomorphism of functors

$$\psi_t : F^t(\mathrm{Ext}^j_R((-), R)) \to \mathrm{Ext}^j_R(F^t(-), R) \tag{7}$$

on the category of finitely generated modules. Thus we get the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^j_R(F^t(R/I), R) & \longrightarrow & \mathrm{Ext}^j_R(F^{t+1}(R/I), R) \\ \uparrow{\scriptstyle \psi_t} & & \uparrow{\scriptstyle \psi_{t+1}} \\ F^t(\mathrm{Ext}^j_R(R/I, R)) & \xrightarrow{F^t(\beta)} & F^{t+1}(\mathrm{Ext}^j_R(R/I, R)) \end{array} \tag{8}$$

in which the top row is the same as the bottom row of (4) and

$$\beta : \mathrm{Ext}^j_R(R/I, R) \to F(\mathrm{Ext}^j_R(R/I, R))$$

is the composition $\mathrm{Ext}^j_R(R/I, R) \xrightarrow{\tilde{\alpha}} \mathrm{Ext}^j_R(F(R/I), R) \xrightarrow{\psi_1^{-1}} F(\mathrm{Ext}^j_R(R/I, R))$ where $\tilde{\alpha}$ is induced by the map $\alpha : F(R/I) \to R/I$ of (2).

LEMMA 2.1. *There is a commutative diagram*

$$\begin{array}{ccccccc}
\operatorname{Ext}^j_R(R/I,R) & \xrightarrow{\xi_0} & \operatorname{Ext}^j_R(R/I^{[p]},R) & \xrightarrow{\xi_1} & \operatorname{Ext}^j_R(R/I^{[p^2]},R) & \xrightarrow{\xi_2} & \cdots \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\
\operatorname{Ext}^j_R(R/I,R) & \xrightarrow{\beta} & F(\operatorname{Ext}^j_R(R/I,R)) & \xrightarrow{F(\beta)} & F^2(\operatorname{Ext}^j_R(R/I,R)) & \xrightarrow{F^2(\beta)} & \cdots
\end{array}$$

in which the vertical maps are isomorphisms. The module $H^j_I(R)$ is isomorphic to the limit of the inductive system in the bottom row of this diagram.

Proof. The vertical maps in the tth square of the diagram are obtained by composing the vertical maps in diagram (4) with the inverses of the vertical maps in diagram (8). This proves the existence of the diagram.

Since for every s there is t such that $I^s \supset I^{[p^t]}$ and for every t there is s such that $I^{[p^t]} \supset I^s$, it follows from [Gro67, 2.8] that $H^j_I(R)$ is the limit of the inductive system in the top row. However, all vertical maps are isomorphisms. □

We recall that *an action of the Frobenius* on an R-module M is an additive map $f : M \to M$ such that $f(rx) = r^p f(x)$ for all $r \in R$ and $x \in M$. We also recall that if $\mathfrak{m}$ is an ideal of R generated by $g_1, \dots, g_n \in R$, then $H^i_{\mathfrak{m}}(M)$ is the ith cohomology module of the Čech complex

$$C^\bullet(M) = 0 \to M \to \bigoplus M_{g_i} \to \bigoplus_{i<j} M_{g_ig_j} \to \bigoplus_{i<j<k} M_{g_ig_jg_k} \to \cdots \quad \text{(see [BS98, 5.1.5])}.$$

Finally, we let $A = R/I$ and recall that the natural action of the Frobenius $f : H^i_{\mathfrak{m}}(A) \to H^i_{\mathfrak{m}}(A)$ is induced by the map $\tilde{f} : C^\bullet(A) \to C^\bullet(A)$ in the category of complexes of abelian groups which sends $A_{g_{i_1}\cdots g_{i_j}}$ to itself via

$$\frac{a}{(g_{i_1}\cdots g_{i_j})^q} \mapsto \frac{a^p}{(g_{i_1}\cdots g_{i_j})^{pq}}.$$

We denote by $H^i_{\mathfrak{m}}(A)^{f^t}$ the A-submodule of $H^i_{\mathfrak{m}}(A)$ generated by the image of the map $f^t : H^i_{\mathfrak{m}}(A) \to H^i_{\mathfrak{m}}(A)$.

LEMMA 2.2. *We have that $H^i_{\mathfrak{m}}(A)^{f^t}$ is the image of the map $H^i_{\mathfrak{m}}(R/I^{[p^t]}) \to H^i_{\mathfrak{m}}(A)$ induced by the natural surjection $\sigma_t : R/I^{[p^t]} \to R/I = A$.*

Proof. The map of complexes of abelian groups $\tilde{f}^t$ defined just before the statement of Lemma 2.2 is the composition $C^\bullet(A) \xrightarrow{\theta} R^{(t)} \otimes_R C^\bullet(A) \xrightarrow{\psi} C^\bullet(A)$ where θ is the map of complexes of abelian groups defined by $x \mapsto 1 \otimes x$ and ψ is the map of complexes of R-modules defined by $r \otimes x \mapsto rx^{p^t}$. Since $R^{(t)}$ is a flat right R-module, the action of the Frobenius $H^i(\tilde{f}^t)$ on $H^i_{\mathfrak{m}}(A)$ is the composition $H^i_{\mathfrak{m}}(A) \xrightarrow{H^i(\theta)} R^{(t)} \otimes_R H^i_{\mathfrak{m}}(A) \xrightarrow{H^i(\psi)} H^i_{\mathfrak{m}}(A)$, hence the R-module generated by the image of $H^i(\tilde{f}^t)$ is the same as the image of $H^i(\psi)$.

For every R-module M there is a natural isomorphism of complexes of R-modules $\omega_M : M \otimes_R C^\bullet(R) \to C^\bullet(M)$ defined by $m \otimes x \mapsto mx$. We have a commutative diagram

$$\begin{array}{ccccc}
R^{(t)} \otimes_R A \otimes_R C^\bullet(R) & \xrightarrow{\phi_t \otimes_R C^\bullet(R)} & (R/I^{[p^t]}) \otimes_R C^\bullet(R) & \xrightarrow{\omega_{R/I^{[p^t]}}} & C^\bullet(R/I^{[p^t]}) \\
\downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle C^\bullet(\sigma_t)} \\
R^{(t)} \otimes_R A \otimes_R C^\bullet(R) & \xrightarrow{R^{(t)} \otimes_R \omega_A} & R^{(t)} \otimes_R C^\bullet(A) & \xrightarrow{\psi} & C^\bullet(A)
\end{array}$$

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where $C^\bullet(\sigma_t)$ is induced by the natural surjection $\sigma_t : R/I^{[p^t]} \to R/I$ and ϕ_t is defined in (1). Since all the maps in this commutative diagram, except ψ and $C^\bullet(\sigma_t)$, are isomorphisms and the map $H^i_{\mathfrak{m}}(R/I^{[p^t]}) \to H^i_{\mathfrak{m}}(A)$ induced by the natural surjection $\sigma_t : R/I^{[p^t]} \to R/I = A$ is nothing but $H^i(C^\bullet(\sigma_t))$, we are done. □

3. The main results

Let M be an R-module with an action of the Frobenius $f : M \to M$ and let M^{f^t} be the R-submodule of M generated by the set $f^t(M) \subset M$. We set $M^* = \bigcap_t M^{f^t}$ to be the intersection of all the M^{f^t}. Our main results in this paper are the following theorem and its corollaries.

Theorem 3.1. *Let R be a regular local ring of dimension n with maximal ideal $\mathfrak{m}$. Let $I \subset R$ be an ideal and let $A = R/I$. The support of the module $H^{n-i}_I(R)$ is the Zariski closed subset $V(J)$ of* $\operatorname{Spec} R$ *whose defining ideal J is the annihilator ideal of $H^i_{\mathfrak{m}}(A)^*$.*

Proof. Let $M = \operatorname{Ext}^{n-i}_R(R/I, R)$, let $\beta : M \to F(M)$ be as in Lemma 2.1 and let $\beta_t : M \to F^t(M)$, for any t, be the composition

$$M \xrightarrow{\beta} F(M) \xrightarrow{F(\beta)} \cdots \xrightarrow{F^{t-1}(\beta)} F^t(M).$$

Considering that $\operatorname{im}\beta_t \subset F^t(M)$ while $\operatorname{im} F(\beta_t) \subset F^{t+1}(M)$, we conclude that the map $F^t(\beta) : F^t(M) \to F^{t+1}(M)$ induces a map

$$\bar\beta : \operatorname{im}\beta_t \to \operatorname{im} F(\beta_t) \cong F(\operatorname{im}\beta_t)$$

where the last isomorphism follows from the exactness of the functor F. The resulting commutative diagram

$$\begin{array}{ccccccc}
M & \xrightarrow{\beta} & F(M) & \xrightarrow{F(\beta)} & F^2(M) & \xrightarrow{F^2(\beta)} & \cdots \\
\downarrow{\scriptstyle \beta_t} & & \downarrow{\scriptstyle F(\beta_t)} & & \downarrow{\scriptstyle F^2(\beta_t)} & & \\
\operatorname{im}\beta_t & \xrightarrow{\bar\beta} & F(\operatorname{im}\beta_t) & \xrightarrow{F(\bar\beta)} & F^2(\operatorname{im}\beta_t) & \xrightarrow{F^2(\bar\beta)} & \cdots
\end{array}$$

induces a map from the limit of the inductive system of the top row to the limit of the inductive system of the bottom row. This map is surjective because each vertical arrow is surjective. The map is injective because the kernel of every vertical arrow $F^r(\beta_t) : F^r(M) \to F^r(\operatorname{im}\beta_t)$ goes to zero after t steps in the top row (the composition of those t steps is nothing but $F^r(\beta_t) : F^r(M) \to F^{r+t}(M)$). Hence, this map is an isomorphism. By Lemma 2.1, $H^{n-i}_I(R)$ is isomorphic to the limit of the top row. Hence, $H^{n-i}_I(R)$ is isomorphic to the limit of the bottom row.

The ascending chain $\ker\beta_1 \subset \ker\beta_2 \subset \cdots$ of submodules of M eventually stabilizes because M is finitely generated, hence Noetherian. From now on we assume for the rest of the proof that s is such that $\ker\beta_s = \ker\beta_{s+1}$. Setting $t = s$ in the bottom row of the diagram we conclude that $H^{n-i}_I(R)$ is isomorphic to the limit of the inductive system

$$\operatorname{im}\beta_s \xrightarrow{\bar\beta} F(\operatorname{im}\beta_s) \xrightarrow{F(\bar\beta)} F^2(\operatorname{im}\beta_s) \xrightarrow{F^2(\bar\beta)} \cdots. \tag{9}$$

Equality $\ker\beta_s = \ker\beta_{s+1}$ implies that the natural map $\operatorname{im}\beta_s \to \operatorname{im}\beta_{s+1}$ is an isomorphism. Since $\bar\beta$ is the composition of the natural map $\operatorname{im}\beta_s \to \operatorname{im}\beta_{s+1}$ with the natural inclusion $\operatorname{im}\beta_{s+1} \to \operatorname{im} F(\beta_s)$ followed by the isomorphism $\operatorname{im} F(\beta_s) \cong F(\operatorname{im}\beta_s)$, we find that $\bar\beta$ is injective. Since F is exact, $F^r(\bar\beta)$ is injective for all r, so all the maps in (9) are injective.

Hence, the support of the limit of this inductive system is the union of the supports of the modules in it. However, the support of $F^r(\operatorname{im}\beta_s)$ coincides with the support of $\operatorname{im}\beta_s$, for if P is a prime of R, then $(F^r(\operatorname{im}\beta_s))_P = F^r((\operatorname{im}\beta_s)_P)$ and if $\mathcal{M}$ is an R-module, for example, $\mathcal{M} = (\operatorname{im}\beta_s)_P$, then $F^r(\mathcal{M}) = 0$ if and only if $\mathcal{M} = 0$. Hence, the support of $H^{n-i}_I(R)$ coincides with the support of $\operatorname{im}\beta_s$. Since $\operatorname{im}\beta_s$ is a finitely generated module, its support is the closed subset $V(J)$ of $\operatorname{Spec} R$ where J is the annihilator ideal of $\operatorname{im}\beta_s$.

It remains to show that the same ideal J is the annihilator ideal of $H^i_{\mathfrak{m}}(A)^*$. Let D be the Matlis duality functor in the category of R-modules. If $\mathcal{M}$ is a finitely generated R-module with annihilator ideal $J \subset R$, then $D(\mathcal{M})$ is an Artinian R-module with the same annihilator ideal. Thus it is enough to prove that $D(\operatorname{im}\beta_s) \cong H^i_{\mathfrak{m}}(A)^*$.

For any t let

$$\chi_t : \operatorname{Ext}^{n-i}_R(R/I, R) \to \operatorname{Ext}^{n-i}_R(R/I^{[p^t]}, R)$$

be the map induced by the natural surjection $R/I^{[p^t]} \to R/I$. Since the two inductive systems in the top and bottom rows of the diagram of Lemma 2.1 are isomorphic and $\chi_t = \xi_{t-1} \circ \cdots \circ \xi_1 \circ \xi_0$, the commutative diagram of Lemma 2.1 implies that $\ker\beta_t \cong \ker\chi_t$ and $\operatorname{im}\beta_t \cong \operatorname{im}\chi_t$ for every t. Hence, it is enough to prove that $D(\operatorname{im}\chi_s) \cong H^i_{\mathfrak{m}}(A)^*$.

Local duality [Gro67, 6.3] implies an isomorphism of functors

$$D(\operatorname{Ext}^{n-i}_R(-, R)) \cong H^i_{\mathfrak{m}}(-)$$

on the category of finitely generated R-modules, so applying Matlis duality to the map χ_t we get the map

$$D(\chi_t) : H^i_{\mathfrak{m}}(R/I^{[p^t]}) \to H^i_{\mathfrak{m}}(R/I).$$

Clearly $D(\chi_t)$ is the map induced on local cohomology by the natural surjection $R/I^{[p^t]} \to R/I$, because χ_t is induced by this natural surjection and local duality is functorial. Hence, $\operatorname{im} D(\chi_t) \cong H^i_{\mathfrak{m}}(A)^{f^t}$ by Lemma 2.2. Since $\ker\beta_t \cong \ker\chi_t$ for every t and $\ker\beta_s = \ker\beta_{s+1}$, we conclude that $\ker\chi_s = \ker\chi_{s+1}$. Exactness and contravariance of D imply that we have $\operatorname{im} D(\chi_s) \cong \operatorname{im} D(\chi_{s+1})$, and therefore $H^i_{\mathfrak{m}}(A)^{f^s} = H^i_{\mathfrak{m}}(A)^{f^{s+1}}$. This implies $H^i_{\mathfrak{m}}(A)^* = H^i_{\mathfrak{m}}(A)^{f^s}$, hence $H^i_{\mathfrak{m}}(A)^* \cong \operatorname{im} D(\chi_s)$. □

It is worth pointing out that one could have replaced chunks of the above proof of Theorem 3.1 with references to our old results on F-modules [Lyu97, 4.2, 4.8]. This would have shortened the proof but only at the cost of making it considerably less transparent.

COROLLARY 3.2. *Let R, I and A be as in Theorem 3.1. Then $H^{n-i}_I(R) = 0$ if and only if there is s such that $f^s : H^i_{\mathfrak{m}}(A) \to H^i_{\mathfrak{m}}(A)$ is the zero map.*

Proof. Clearly, $H^{n-i}_I(R) = 0$ if and only if the unit ideal is a defining ideal of the support of $H^{n-i}_I(R)$. According to Theorem 3.1, this happens if and only if the unit ideal is the annihilator ideal of $H^i_{\mathfrak{m}}(A)^*$. However, the unit ideal annihilates a module if and only if the module is zero. Hence, $H^{n-i}_I(R) = 0$ if and only if $H^i_{\mathfrak{m}}(A)^* = 0$.

The descending chain $H^i_{\mathfrak{m}}(A) \supset H^i_{\mathfrak{m}}(A)^f \supset H^i_{\mathfrak{m}}(A)^{f^2} \supset \cdots$ eventually stabilizes since $H^i_{\mathfrak{m}}(A)$ is Artinian. Thus, $H^i_{\mathfrak{m}}(A)^* = H^i_{\mathfrak{m}}(A)^{f^s}$ for some s. Hence $H^{n-i}_I(R) = 0$ if and only if $H^i_{\mathfrak{m}}(A)^{f^s} = 0$ for some s, i.e. if and only if $f^s : H^i_{\mathfrak{m}}(A) \to H^i_{\mathfrak{m}}(A)$ is the zero map. □

We note that if M is an R-module and $\operatorname{Supp} M \subset \operatorname{Spec} R$ is the support of M, then $\operatorname{Supp} M \subset \{\mathfrak{m}\}$ if and only if either $M = 0$ (i.e. $\operatorname{Supp} M$ is the empty set) or $M \neq 0$ and $\operatorname{Supp} M = \{\mathfrak{m}\}$.

COROLLARY 3.3. *Let R, I and A be as in 3.1. Then $\operatorname{Supp} H^{n-i}_I(R) \subset \{\mathfrak{m}\}$ if and only if $H^i_{\mathfrak{m}}(A)^*$ has finite length.*

Proof. We have that $H^i_{\mathfrak{m}}(A)^*$ is a submodule of the Artinian module $H^i_{\mathfrak{m}}(A)$, hence Artinian itself. By Theorem 3.1, $H^{n-i}_I(R)$ is supported at $\mathfrak{m}$ if and only if the annihilator ideal of $H^i_{\mathfrak{m}}(A)^*$

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is $\mathfrak{m}$-primary. However, an Artinian module is annihilated by an $\mathfrak{m}$-primary ideal if and only if the module has finite length. □

Corollary 3.4. *Let R, I and A be as in Theorem 3.1. If* $\operatorname{Supp} H_I^{n-i}(R) \subset \{\mathfrak{m}\}$, *then* $H_I^{n-i}(R) \cong E^v$, *where E is the injective hull of $R/\mathfrak{m}$ in the category of R-modules and* $v = \dim_{R/\mathfrak{m}} H_{\mathfrak{m}}^i(A)^*/\mathfrak{m}H_{\mathfrak{m}}^i(A)^*$ *is the minimum number of generators of* $H_{\mathfrak{m}}^i(A)^*$.

Proof. It has been shown in the course of proof of Theorem 3.1 that $H_I^{n-i}(R)$ is the limit of inductive system (9) in which all the maps are injective. Since $D(\operatorname{im}\beta_s) \cong H_{\mathfrak{m}}^i(A)^*$, exactness and contravariance of Matlis duality imply that the minimum number of generators of $H_{\mathfrak{m}}^i(A)^*$ equals the dimension of the socle of $\operatorname{im}\beta_s$ as $(R/\mathfrak{m})$-vector space.

We claim that for any R-module M the dimensions of the socles of M and $F^r(M)$ coincide. Indeed, we have an exact sequence

$$0 \to \operatorname{Socle}(M) \to M \xrightarrow{m \mapsto \oplus x_i m} M^n,$$

where $m \in M$ and $x_1, \dots, x_n \in \mathfrak{m}$ generate the maximal ideal $\mathfrak{m}$ of R. Applying the exact functor F^r we get an exact sequence

$$0 \to F^r(\operatorname{Socle}(M)) \to F^r(M) \xrightarrow{m \mapsto \oplus x_i^{p^r} m} F^r(M)^n,$$

where $m \in F^r(M)$. Thus, the annihilator of the ideal $(x_1^{p^r}, \dots, x_n^{p^r})$ in $F^r(M)$ is $F^r(\operatorname{Socle}(M))$ and, therefore, the socle of $F^r(M)$ coincides with the socle of $F^r(\operatorname{Socle}(M))$. However, $\operatorname{Socle}(M) \cong (R/\mathfrak{m})^q$ where $q = \dim_{R/\mathfrak{m}} \operatorname{Socle}(M)$, hence $F^r(\operatorname{Socle}(M)) \cong F^r(R/\mathfrak{m})^q$. However, $F^r(R/\mathfrak{m}) \cong R/(x_1^{p^r}, \dots, x_n^{p^r})$ and the socle of this module is isomorphic to $R/\mathfrak{m}$ and generated by $(x_1 \cdots x_n)^{p^r-1}$. Hence, the socle of $F^r(M)$ is isomorphic to a direct sum of q copies of the socle of $F^r(R/\mathfrak{m})$, i.e. has dimension q over $R/\mathfrak{m}$. This proves the claim.

The claim implies that the dimensions of the socles of all the modules in (9) are the same. The injectivity of the maps in (9) implies that the induced maps on the socles are all isomorphisms, hence the socle of $\operatorname{im}\beta_s$ maps isomorphically onto the socle of $H_I^{n-i}(R)$. However, $H_I^{n-i}(R)$ is a direct sum of copies of E (this fact has been originally proven in [HS77, Corollary 2.4] in characteristic $p > 0$; a characteristic-free proof is given in [Lyu00, Corollary 2]). Since E has one-dimensional socle, the dimension of the socle of $H_I^{n-i}(R)$ equals the number of copies of E in this direct sum. □

4. The *F*-depth of a local ring: an application to Grothendiek's problem

In this section we define the F-depth of a local ring, use it to give a solution to Grothendieck's problem and discuss some of its elementary properties.

Definition 4.1. Let A be a local ring with maximal ideal $\mathfrak{m}$. The F-depth of A is the smallest i such that f^s does not send $H_{\mathfrak{m}}^i(A)$ to zero for any s.

The next lemma shows that this is well defined because the set of integers i such that f^s does not send $H_{\mathfrak{m}}^i(A)$ to zero for any s, is non-empty.

Lemma 4.2. *Let A and $\mathfrak{m}$ be as in Definition 4.1. Then f^s does not send $H_{\mathfrak{m}}^{\dim A}(A)$ to zero for any s. In particular,* $0 \leqslant F\text{-depth}\, A \leqslant \dim A$.

Proof. Let $\dim A = d$ and let $x_1, \dots x_d \in A$ be a system of parameters. The module $H_{\mathfrak{m}}^d(A)$ is generated by the elements $\overline{(x_1 \cdots x_d)^{-j}}$ as j runs through all positive integers, where $\overline{(x_1 \cdots x_d)^{-j}}$ is the image of the element $(x_1 \cdots x_d)^{-j} \in A_{x_1 \cdots x_d}$ under the natural surjection $A_{x_1 \cdots x_d} \to H_{\mathfrak{m}}^d(A)$. However, $f^s(\overline{(x_1 \cdots x_d)^{-j}}) = \overline{(x_1 \cdots x_d)^{-p^s j}}$ and these elements also generate $H_{\mathfrak{m}}^d(A)$ as j runs through all positive integers. Since the set $f^s(H^d_{\mathfrak{m}}(A))$ generates the A-module $H^d_{\mathfrak{m}}(A)$, this set cannot be zero. □

Our solution to Grothendieck's problem is as follows (for Hartshorne and Speiser's solution, see Theorem 6.2 below).

THEOREM 4.3. *Let R be a regular local ring, let I be an ideal of R, let $A = R/I$, let $n = \dim R$ and let r be an integer. Then $H^i_I(M) = 0$ for all $i > r$ and all R-modules M if and only if $F\text{-depth}\, A \geqslant n - r$.*

Proof. We have $H^i_I(M) = 0$ for all $i > r$ and all R-modules M if and only if $H^i_I(R) = 0$ for all $i > r$ (see [Har68, p. 413]). Now we are done by Corollary 3.2 considering that $i > r$ if and only if $n - i < n - r$. □

The F-depth is an interesting characteristic of a local ring A and the rest of this section is devoted to a discussion of some of its elementary properties. We note the close similarity between the F-depth and the usual depth of A which equals the minimum integer i such that $H^i_{\mathfrak{m}}(A) \neq 0$. Another similarity is given by the following proposition which shows that the F-depth, like the usual depth, is preserved under completion.

PROPOSITION 4.4. *Let A be a local ring. Then $F\text{-depth}\, A = F\text{-depth}\, \hat{A}$ where $\hat{A}$ is the completion of A with respect to the maximal ideal $\mathfrak{m}$.*

Proof. We have $H^i_{\mathfrak{m}}(A) \cong H^i_{\hat{\mathfrak{m}}}(\hat{A})$ and the action of the Frobenius is the same in both cases. □

However, the following proposition indicates a striking difference because the usual depth of A may be very different from the depth of $A_{\mathrm{red}} = A/\sqrt{(0)}$.

PROPOSITION 4.5. *Let A be a local ring. Then $F\text{-depth}\, A = F\text{-depth}\, A_{\mathrm{red}}$.*

Proof. We have $F\text{-depth}(A) = F\text{-depth}(\hat{A})$ and $F\text{-depth}(A_{\mathrm{red}}) = F\text{-depth}(\widehat{A_{\mathrm{red}}})$ by the previous proposition. Since $\hat{A}$ is complete and contains a field of characteristic $p > 0$, there is a surjection $R \to \hat{A}$ with kernel ideal $I \subset R$, where R is a complete regular local ring containing a field of characteristic $p > 0$. Composing this surjection with the surjection $\phi : \hat{A} \to \widehat{A_{\mathrm{red}}}$ which is the completion of the natural surjection $A \to A_{\mathrm{red}}$ we get a surjection $R \to \widehat{A_{\mathrm{red}}}$ with kernel ideal $J \subset R$. Since ϕ is a surjection, $I \subset J$. Since the kernel of ϕ is a nilpotent ideal of $\hat{A}$, we conclude that $J \subset \sqrt{I}$. This implies that $H^j_I(R) \cong H^j_J(R)$ for all j. Now we are done by Theorem 4.3 considering that $\hat{A} \cong R/I$ and $\widehat{A_{\mathrm{red}}} \cong R/J$. □

Next we give necessary and sufficient conditions for small values of the F-depth of A (recall that $F\text{-depth}(A) \geqslant 0$ by Lemma 4.2).

COROLLARY 4.6. *Let A be a local ring. Then:*

(a) *$F\text{-depth}\, A > 0$ if and only if $\dim A > 0$;*
(b) *$F\text{-depth}\, A > 1$ if and only if $\dim A \geqslant 2$ and the punctured spectrum of A is formally geometrically connected.*

'The punctured spectrum of A is formally geometrically connected' means that the punctured spectrum of $\tilde{A} = \widehat{\hat{A}^{\mathrm{sh}}}$, the completion of the strict Henselization of the completion of A, is connected. In particular, if A is complete with separably closed residue field, this simply means that the punctured spectrum of A is connected.

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Proof. (a) If $\dim A = 0$, then $\mathfrak{m}$ is a nilpotent ideal, so $H^0_{\mathfrak{m}}(A) = A$. However, f^s never vanishes on A because $f^s(1) = 1$, hence F-depth $A = 0$. If $\dim A > 0$, then $H^0_{\mathfrak{m}}(A)$ is a nilpotent ideal of A, hence annihilated by f^s for some s.

(b) We have $\dim A = \dim \hat{A}$, so by Proposition 4.4 we may replace A by $\hat{A}$, i.e. we may assume that A is complete. Hence, there is a surjection $R \to A$ with kernel ideal $I \subset R$ where R is a complete regular local ring containing a field. In this case $H_I^{\dim R - i}(R) = 0$ for $i < 2$ if and only if $\dim A \geqslant 2$ and the punctured spectrum of $\tilde{R}/I\tilde{R} = \tilde{A}$ is connected where $\tilde{R} = \widehat{\hat{R}^{\mathrm{sh}}}$; this result was originally proven in characteristic 0 by Ogus [Ogu73, Corollary 2.11] and in characteristic $p > 0$ by Peskine and Szpiro [PS73, III, 5.5] and later a completely characteristic-free proof was given in [HL90, 2.9]. $\square$

A similar necessary and sufficient topological criterion for F-depth $A > 2$ is unlikely to exist. Indeed, at the end of the next section we give an example of a local ring A such that F-depth $A > 2$ if and only if $3|(p-2)$ while the topology of $\operatorname{Spec} A$ seems to be 'the same' in all characteristics since A is the local ring at the vertex of the affine cone over 'the same' variety Y.

It should be pointed out that results such as [Fal80, Korollar 1–4] and [HL90, 5.1] on the vanishing of $H^j_I(R)$ for high j lead to corresponding lower bounds on F-depth A via Theorem 4.3. However, of course, it would be very interesting to do the opposite, i.e. prove that F-depth $A > r$ and then via Theorem 4.3 obtain a (previously unknown) vanishing of $H^j_I(R)$ for $j > \dim R - r$. With a view toward obtaining such results, the F-depth of a local ring emerges as a promising object of study.

5. The projective case

Let k be a field of characteristic $p > 0$, let $Y \subset \mathbb{P}^n_k$ be a closed subscheme of projective n-space over k, let $R = k[X_0, \ldots, X_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n_k$, let $I \subset R$ be the homogeneous defining ideal of Y and let $\mathfrak{m} = (X_0, \ldots, X_n)$ be the irrelevant ideal.

In this section we prove projective analogs of Corollaries 3.2, 3.3 and 3.4 (Theorems 5.2, 5.1 and 5.3, respectively) and apply Theorems 5.1 and 5.2 to express the cohomological dimension of $\mathbb{P}^n_k \setminus Y$ in terms of the F-depths of the local rings at the closed points of Y and the Frobenius actions on the cohomology groups of $\mathcal{O}_Y$ (Corollaries 5.4 and 5.5).

Theorem 5.1. *Let $R, \mathfrak{m}, Y$ and I be as above. Then $\operatorname{Supp} H_I^{n-j}(R) \subset \{\mathfrak{m}\}$ if and only if for every closed point $y \in Y$ there exists s such that f^s acts as the zero map on $H^j_{\mathfrak{m}_y}(\mathcal{O}_{Y,y})$, where $\mathcal{O}_{Y,y}$ is the local ring of Y at y and $\mathfrak{m}_y$ is the maximal ideal of $\mathcal{O}_{Y,y}$.*

Proof. $H_I^{n-j}(R)$ is a graded module, so $\operatorname{Supp} H_I^{n-j}(R) \subset \{\mathfrak{m}\}$ if and only if its localization at every one-dimensional homogeneous prime containing I is zero. Let $\mathcal{P} \subset R$ be a one-dimensional homogeneous prime containing I and let $y \in Y$ be the corresponding closed point. Then $H_I^{n-j}(R)_{\mathcal{P}} = 0$ if and only if $H_{\mathcal{I}}^{n-j}(\mathcal{O}_{\mathbb{P}^n_k,y}) = 0$, where $\mathcal{I}$ is the stalk of the defining ideal I of Y in $\mathcal{O}_{\mathbb{P}^n_k,y}$. There is a natural surjection $\mathcal{O}_{\mathbb{P}^n_k,y} \to \mathcal{O}_{Y,y}$ with kernel $\mathcal{I}$, so we are done by Corollary 3.2. $\square$

It has been known that:

(i) $H^s_I(R) = 0$ if $s > n+1$ (see [Gro67, 1.12]);

(ii) $H^{n+1}_I(R) = 0$ if and only if Y is non-empty [Har68, 3.1];

(iii) $H^n_I(R) = 0$ if and only if $\dim Y > 0$ and Y is geometrically connected (i.e. $Y \otimes_k \bar{k}$ is connected, where $\bar{k}$ is the algebraic closure of k) [Har68, 7.5].

These facts provide characteristic-free necessary and sufficient criteria for the vanishing of $H_I^{n-j}(R)$ for $j < 1$. While characteristic-free criteria for $j \geqslant 1$ are not known, Theorem 5.2 gives a characteristic $p > 0$ necessary and sufficient criterion in the case that $j \geqslant 1$.

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Let $A = R/I$ be the homogeneous coordinate ring of Y. The homomorphism of rings $A \xrightarrow{a \mapsto a^p} A$ induces a morphism $\mathcal{O}_Y \to \mathcal{O}_Y$ in the category of sheaves of abelian groups on Y, which, in turn, induces a Frobenius map $f : H^i(Y, \mathcal{O}_Y) \to H^i(Y, \mathcal{O}_Y)$.

THEOREM 5.2. *Let Y, R, I and $\mathfrak{m}$ be as above and assume $j \geqslant 1$. Then $H^{n-j}_I(R) = 0$ if and only if the following two conditions hold.*

(a) *For every closed point $y \in Y$ there exists s such that f^s acts as the zero map on $H^j_{\mathfrak{m}_y}(\mathcal{O}_{Y,y})$, where $\mathcal{O}_{Y,y}$ is the local ring of Y at y and $\mathfrak{m}_y$ is the maximal ideal of $\mathcal{O}_{Y,y}$.*

(b) *There is s such that $f^s : H^j(Y, \mathcal{O}_Y) \to H^j(Y, \mathcal{O}_Y)$ is the zero map.*

Proof. Since I is a homogeneous ideal, the vanishing of $H^{n-j}_I(R)$ is equivalent to the vanishing of $H^{n-j}_I(R)_{\mathfrak{m}} = H^{n-j}_{I_{\mathfrak{m}}}(R_{\mathfrak{m}})$, which by Corollary 3.2 is equivalent to the vanishing of $H^{j+1}_{\mathfrak{m}}(A_{\mathfrak{m}})^*$ (note that $\dim R_{\mathfrak{m}} = n+1$, hence $\dim R_{\mathfrak{m}} - (n-j) = j+1$).

By Theorem 5.1, condition (a) is equivalent to $\operatorname{Supp} H^{n-j}_I(R) \subset \{\mathfrak{m}\}$. Thus, it is enough to prove that if $\operatorname{Supp} H^{n-j}_I(R) \subset \{\mathfrak{m}\}$, then the vanishing of $H^{j+1}_{\mathfrak{m}}(A_{\mathfrak{m}})^*$ is equivalent to condition (b). Accordingly, we assume that $\operatorname{Supp} H^{n-j}_I(R) \subset \{\mathfrak{m}\}$.

Since $H^{n-j}_{I_{\mathfrak{m}}}(R_{\mathfrak{m}}) = H^{n-j}_I(R)_{\mathfrak{m}}$, we conclude that $\operatorname{Supp} H^{n-j}_{I_{\mathfrak{m}}}(R_{\mathfrak{m}}) \subset \{\mathfrak{m}\}$. Since $R_{\mathfrak{m}}/I_{\mathfrak{m}} = A_{\mathfrak{m}}$, Corollary 3.3 implies that $H^{j+1}_{\mathfrak{m}}(A_{\mathfrak{m}})^*$ has finite length. However, $\operatorname{Supp} H^{j+1}_{\mathfrak{m}}(A) \subset \{\mathfrak{m}\}$, hence $H^{j+1}_{\mathfrak{m}}(A) = H^{j+1}_{\mathfrak{m}}(A)_{\mathfrak{m}} = H^{j+1}_{\mathfrak{m}}(A_{\mathfrak{m}})$ and, therefore, $H^{j+1}_{\mathfrak{m}}(A_{\mathfrak{m}})^* \cong H^{j+1}_{\mathfrak{m}}(A)^*$. Hence, $H^{j+1}_{\mathfrak{m}}(A)^*$ has finite length.

Since $H^{j+1}_{\mathfrak{m}}(A)^*$ is a graded R-module, there is a positive integer L such that a homogeneous element of $H^{j+1}_{\mathfrak{m}}(A)^*$ of degree d is non-zero only if $-L < d < L$. Let i be such that $p^i \geqslant L$. If $x \in H^{j+1}_{\mathfrak{m}}(A)^*$ is a homogeneous element of degree $d \neq 0$, then $f^i(x)$ is a homogeneous element of degree $p^i d$ and either $p^i d \geqslant L$ or $p^i d \leqslant -L$ in which case $f^i(x) = 0$, so $f^i(H^{j+1}_{\mathfrak{m}}(A)^*) = f^i(H^{j+1}_{\mathfrak{m}}(A)^*_0)$, where $H^{j+1}_{\mathfrak{m}}(A)^*_0$ is the degree zero piece of $H^{j+1}_{\mathfrak{m}}(A)^*$. Since $f^i(H^{j+1}_{\mathfrak{m}}(A)^*) = f^i(H^{j+1}_{\mathfrak{m}}(A)^*_0) \subset H^{j+1}_{\mathfrak{m}}(A)^*_0$ generates $H^{j+1}_{\mathfrak{m}}(A)^*$ as an A-module, $H^{j+1}_{\mathfrak{m}}(A)^*$ is generated as an A-module by $H^{j+1}_{\mathfrak{m}}(A)^*_0$. Hence, $H^{j+1}_{\mathfrak{m}}(A)^* = 0$ if and only if $H^{j+1}_{\mathfrak{m}}(A)^*_0 = 0$, i.e. if and only if $f^s : H^{j+1}_{\mathfrak{m}}(A)^*_0 \to H^{j+1}_{\mathfrak{m}}(A)^*_0$ is the zero map for some s.

Set $U = \operatorname{Spec} A \setminus \{\mathfrak{m}\}$. There is an isomorphism $H^{j+1}_{\mathfrak{m}}(A) \cong H^j(U, \mathcal{O}_U)$ for all $j \geqslant 1$ (see [Gro67, 2.2]). The standard map $\pi : U \to Y$ is an affine morphism. Hence, we get an isomorphism $H^j(U, \mathcal{O}_U) \cong H^j(Y, \pi_* \mathcal{O}_U)$. However, $\pi_* \mathcal{O}_U \cong \bigoplus_{\nu \in \mathbb{Z}} \mathcal{O}_Y(\nu)$, hence $H^{j+1}_{\mathfrak{m}}(A) \cong \bigoplus_{\nu \in \mathbb{Z}} H^j(Y, \mathcal{O}_Y(\nu))$. This is a degree-preserving isomorphism of graded A-modules; the degree ν piece of the A-module $\bigoplus_{\nu \in \mathbb{Z}} H^j(Y, \mathcal{O}_Y(\nu))$ is $H^j(Y, \mathcal{O}_Y(\nu))$.

Hence, we have an isomorphism $H^{j+1}_{\mathfrak{m}}(A)_0 \cong H^j(Y, \mathcal{O}_Y)$ which, as is not hard to see, respects the action of the Frobenius on both sides. □

Let E be the injective hull of $R/\mathfrak{m}$ in the category of R-modules. If $\operatorname{Supp} H^n_I(R) \subset \{\mathfrak{m}\}$, then $H^n_I(R) \cong E^{c-1}$ where c is the number of connected components of Y (this is a consequence, by induction on c using the Mayer–Vietoris sequence, of item (iii) following the proof of Theorem 5.1). The following theorem describes the structure of $H^{n-j}_I(R)$ in the case where $j \geqslant 1$ and $\operatorname{Supp} H^{n-j}_I(R) \subset \{\mathfrak{m}\}$. This may be regarded as a generalization of Theorem 5.2 to the case $v = \dim_k H^j(Y, \mathcal{O}_Y)^* > 0$.

The k-linear spans of $f^t(H^j(Y, \mathcal{O}_Y))$, as t runs through all positive integers, form a descending chain of k-vector subspaces of $H^j(Y, \mathcal{O}_Y)$ since $f^{t+1}(H^j(Y, \mathcal{O}_Y)) \subset f^t(H^j(Y, \mathcal{O}_Y))$. Since $H^j(Y, \mathcal{O}_Y)$ is finite-dimensional, this chain eventually stabilizes. Let $H^j(Y, \mathcal{O}_Y)^*$ be the k-linear span of $f^t(H^j(Y, \mathcal{O}_Y))$ for all sufficiently large t.

THEOREM 5.3. *Let $k, Y, R, I, \mathfrak{m}$ and E be as above and assume $j \geqslant 1$. If $\operatorname{Supp} H^{n-j}_I(R) \subset \{\mathfrak{m}\}$, then $H^{n-j}_I(R) \cong E^v$, where $v = \dim_k H^j(Y, \mathcal{O}_Y)^*$.*

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Proof. Since $\operatorname{Supp} H_I^{n-j}(R) \subset \{\mathfrak{m}\}$ we conclude as in the proof of the preceding theorem that $H_{\mathfrak{m}}^{j+1}(A)^*$ is generated as an A-module by $H_{\mathfrak{m}}^{j+1}(A)_0^*$. Hence, $H_{\mathfrak{m}}^{j+1}(A)^*/\mathfrak{m}H_{\mathfrak{m}}^{j+1}(A)^* \cong H_{\mathfrak{m}}^{j+1}(A)_0^* \cong H^j(Y, \mathcal{O}_Y)^*$ and we are done by Corollary 3.4. □

Theorems 5.1 and 5.3 provide some supporting evidence for a positive answer to the open question whether the qth Bass number of the module $H_I^{n-j}(R)$ with respect to $\mathfrak{m}$ depends only on the integers q, j and the variety Y but is independent of the embedding $Y \hookrightarrow \mathbb{P}_k^n$ (see [Lyu02, p. 133]). Indeed, the action of the Frobenius on $H_{\mathfrak{m}_y}^j(\mathcal{O}_{Y,y})$ is independent of the embedding, as is $v = \dim_k H^j(Y, \mathcal{O}_Y)^*$ while the qth Bass number of E^v with respect to $\mathfrak{m}$ depends only on q (it is 0 if $q > 0$ and v if $q = 0$).

The rest of this section deals with applications to the cohomological dimension of $\mathbb{P}_k^n \setminus Y$. The cohomological dimension of a scheme U, denoted $\operatorname{cd} U$, is the largest integer i such that there exists a quasi-coherent sheaf $\mathcal{F}$ on U with $H^i(U, \mathcal{F}) \neq 0$. It follows from [Har68, pp. 412–413] that $\operatorname{cd}(\mathbb{P}_k^n \setminus Y) < t$ if and only if $H_I^i(R) = 0$ for all $i > t$. In particular, items (i)–(iii) following the proof of Theorem 5.1 imply that:

(i) $\operatorname{cd}(\mathbb{P}_k^n \setminus Y) < n + 1$ for all Y;

(ii) $\operatorname{cd}(\mathbb{P}_k^n \setminus Y) < n$ if and only if Y is non-empty;

(iii) $\operatorname{cd}(\mathbb{P}_k^n \setminus Y) < n - 1$ if and only if Y is geometrically connected and $\dim Y > 0$.

These facts provide characteristic-free necessary and sufficient conditions for $\operatorname{cd}(\mathbb{P}_k^n \setminus Y) < n - r$ where $r < 2$. While characteristic-free conditions for $r \geqslant 2$ are not known, we can now give a characteristic $p > 0$ necessary and sufficient condition for $\operatorname{cd}(\mathbb{P}_k^n \setminus Y) < n - r$ where $r \geqslant 2$ (for a corresponding result of Hartshorne and Speiser, cf. [HS77, 4.2]).

Corollary 5.4. *If $r \geqslant 2$, then $\operatorname{cd}(\mathbb{P}_k^n \setminus Y) < n - r$ if and only if the following three conditions hold:*

(a) *Y is geometrically connected and $\dim Y > 0$;*

(b) *F-depth $\mathcal{O}_{Y,y} \geqslant r$ for every closed point $y \in Y$;*

(c) *for every j such that $0 < j < r$ there is some s such that the map $f^s : H^j(Y, \mathcal{O}_Y) \to H^j(Y, \mathcal{O}_Y)$ is zero.*

Proof. $\operatorname{cd}(\mathbb{P}_k^n \setminus Y) < n - r$ if and only if $H_I^{n-j}(R) = 0$ provided $n - j > n - r$, i.e. $j < r$. According to item (iii) following the proof of Theorem 5.1, the vanishing of $H_I^n(R)$ is equivalent to condition (a). The vanishing of $H_I^{n-j}(R)$ for a fixed j such that $0 < j < r$ is equivalent to the two conditions of Theorem 5.2. Hence, the vanishing of $H_I^{n-j}(R)$ for all j such that $0 < j < r$ is equivalent to conditions (b) and (c). □

The case $r = 2$ deserves to be stated separately (cf. [HS77, p. 75, 2c]).

Corollary 5.5. $\operatorname{cd}(\mathbb{P}_k^n \setminus Y) < n - 2$ *if and only if the following conditions hold.*

(a) *Y is geometrically connected and every irreducible component has dimension $\geqslant 2$.*

(b) *For each closed point $y \in Y$ the punctured spectrum of $\mathcal{O}_{Y,y}$ is formally geometrically connected.*

(c) *There is s such that $f^s : H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y)$ is the zero map.*

Proof. This follows from Corollaries 5.4 and 4.6(b). □

It is worth pointing out that if k is separably closed, conditions (a) and (b) become somewhat simpler. Namely, in this case 'geometrically connected' in condition (a) is equivalent to 'connected' and 'the punctured spectrum of $\mathcal{O}_{Y,y}$ is formally geometrically connected' in condition (b) is equivalent to 'the punctured spectrum of the completion of $\mathcal{O}_{Y,y}$ is connected'.

We conclude this section with an example (cf. [HS77, p. 75]). Assume $p \neq 3$ and let $\mathcal{C} \subset \mathbb{P}^2_k$ be an elliptic curve. Then $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \cong k$. Therefore, either $f : H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ is the zero map, i.e. the Hasse invariant of $\mathcal{C}$ is 0 (see [Har77, pp. 332–335]), or the Hasse invariant is 1, i.e. $f^s : H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ is injective for all s. Set $Y = \mathcal{C} \times \mathbb{P}^1_k$. The Leray spectral sequence associated with the projection morphism $Y \to \mathcal{C}$ degenerates at E_2 and induces an isomorphism $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \cong H^1(Y, \mathcal{O}_Y)$. Hence, $f : H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y)$ is the zero map if the Hasse invariant of $\mathcal{C}$ is 0 and if the Hasse invariant of $\mathcal{C}$ is 1, then $f^s : H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y)$ is injective for all s.

Now embed Y in $\mathbb{P}^5_k$. Corollary 5.5 implies that $\operatorname{cd}(\mathbb{P}^5_k \setminus Y) < 3$ if the Hasse invariant of $\mathcal{C}$ is zero and $\operatorname{cd}(\mathbb{P}^5_k \setminus Y) = 3$ if the Hasse invariant is one.

For instance, if $\mathcal{C} \subset \mathbb{P}^2_k$ is the Fermat curve defined by $x^3 + y^3 + z^3 = 0$, it follows from [Har77, IV, 4.21] that the Hasse invariant is zero if and only if $3|(p-2)$. Hence, in this case $\operatorname{cd}(\mathbb{P}^5_k \setminus Y) < 3$ if and only if $3|(p-2)$.

Algebraically, let $I \subset R = k[X_0, \dots, X_5]$ be the defining ideal of Y. Then $H^4_I(R) = 0$ if and only if $3|(p-2)$. Let $A = (R/I)_{\mathfrak{m}}$. It follows from Corollary 3.2 that the F-depth $A > 2$ if and only if $3|(p-2)$.

6. The F-depth of a local ring versus the F-depth of a scheme

In this section we recall the notion of the F-depth of a scheme introduced by Hartshorne and Speiser (Definition 6.1 and Theorem 6.2), compare it to our notion of the F-depth of a local ring (Corollary 6.3 and the discussion section in the following) and settle a question left open in [HS77] (Theorem 6.4).

Definition 6.1 [HS77, p. 60]. Let Y be a Noetherian scheme of finite dimension, whose local rings are all of characteristic $p > 0$. Let $y \in Y$ be a (not necessarily closed) point. Let $d(y)$ be the dimension of the closure $\{y\}^-$ of the point y. Let $\mathcal{O}_y$ be the local ring of y, let k_0 be its residue field, let k be a perfect closure of k_0 and let $\hat{\mathcal{O}}_y$ be the completion of $\mathcal{O}_y$. Choose a field of representatives for k_0 in $\hat{\mathcal{O}}_y$, let R_y be the local ring $\hat{\mathcal{O}}_y \otimes_{k_0} k$ obtained by base extension to k, and let $\mathfrak{m}_y$ be the maximal ideal of R_y. The F-depth of Y is the largest integer r (or $+\infty$) such that for all points $y \in Y$, we have $H^i_{\mathfrak{m}_y}(R_y)_s = 0$ for all $i < r - d(y)$.

The object $H^i_{\mathfrak{m}_y}(R_y)_s$ requires a definition of its own. Let $M = H^i_{\mathfrak{m}_y}(R_y)$ and let $f : M \to M$ be the natural action of the Frobenius. The image $f^i(M)$ is not, in general, a submodule of M, but it is a k-vector subspace. Hartshorne and Speiser set M_s to be the intersection $\bigcap_i f^i(M)$ and call it the *stable part* of M (see [HS77, p. 46]).

Hartshorne and Speiser proved the following theorem (cf. [HS77, 2.5b]).

Theorem 6.2. *Let R be a regular local ring and let I be an ideal of R. Let $Y = \operatorname{Spec}(R/I) = V(I) \subset \operatorname{Spec} R$, let $n = \dim R$ and let r be an integer. Then $H^i_I(R) = 0$ for all $i > r$ if and only if F-depth $Y \geqslant n - r$.*

Theorems 6.2 and 4.3 provide two different necessary and sufficient criteria for the vanishing of $H^i_I(R)$ for all $i > r$. They clearly imply the following.

Corollary 6.3. *Let R be a regular local ring and let I be an ideal of R. Let $A = R/I$ and $Y = \operatorname{Spec} A = V(I) \subset \operatorname{Spec} R$. Then F-depth $Y = F$-depth A.*

Discussion

If A does not admit a surjection $R \to A$ from a regular local ring R of characteristic $p > 0$, we do not know how the F-depth of Y and the F-depth of A are related. However, if A admits such

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a surjection, Corollary 6.3 shows that Hartshorne and Speiser's Definition 6.1 defines the same integer for the scheme $Y = \operatorname{Spec} A$ as does our Definition 4.1 for the ring A, but our definition involves local cohomology modules only at the closed point of Y, whereas Definition 6.1 involves local cohomology modules at all the points of Y.

This difference is striking. It is known that $H^i_{\mathfrak{m}_y}(R_y)_s = 0$ for all but finitely many points $y \in Y$ (see [Lyu97, 4.14]; for example, if A is regular, then the only point $y \in Y$ such that $H^i_{\mathfrak{m}_y}(R_y)_s \neq 0$ for some i is the generic point of Y), so Definition 6.1 measures the F-depth of Y in terms of local cohomology modules at those finitely many points. However, to identify those points may be a highly non-trivial task and the closed point need not be one of them. That is, local cohomology modules at the closed point need not make any contribution at all to the computation of the F-depth of Y according to Definition 6.1 because $H^i_{\mathfrak{m}}(A)_s$ may be zero for all i. However, according to Corollary 6.3 our Definition 4.1 still expresses the F-depth of Y in terms of local cohomology modules at the closed point only!

In fact, we can express the F-depth of Y in terms of closed points for quite a large class of schemes Y, as the following theorem shows.

Theorem 6.4. *Let Y be a scheme isomorphic to a closed subscheme of a Noetherian regular scheme X of finite dimension such that the local rings of X are all of characteristic $p > 0$ and the dimension of the local rings of the closed points of X are all equal to the dimension of X. Then the F-depth of Y equals the minimum of the F-depths of the local rings $\mathcal{O}_{Y,y}$ at all the closed points $y \in Y$.*

Proof. Let $y \in Y$ be a point, let $\mathcal{O}_{Y,y}$ be the local ring of Y at y and let $Y_y = \operatorname{Spec} \mathcal{O}_{Y,y}$. Definition 6.1 implies that the F-depth of Y equals the minimum of the F-depths of Y_y as y runs through all the closed points of Y. However, Corollary 6.3 (with R being the local ring of X at a closed point y and $A = \mathcal{O}_{Y,y}$) implies that the F-depth of the scheme Y_y equals the F-depth of the local ring $\mathcal{O}_{Y,y}$. □

Hartshorne and Speiser point out that they 'do not know if the F-depth can be measured in terms of closed points only' [HS77, p. 61]. However, our Theorem 6.4 does just that (for reasonable schemes). It shows, for example, that if Y is a scheme of finite type over a field of characteristic $p > 0$, then the F-depth of Y can be measured in terms of closed points only.

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Gennady Lyubeznik gennady@math.umn.edu

Department of Mathematics, 206 Church Street SE, University of Minnesota, Minneapolis, MN 55455, USA

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