

MAXIMA OF SUMS OF HEAVY-TAILED RANDOM VARIABLES

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ABSTRACT

In this paper, we investigate asymptotic properties of the tail probabilities of the maxima of partial sums of independent random variables. For some large classes of heavy-tailed distributions, we show that the tail probabilities of the maxima of the partial sums asymptotically equal to the sum of the tail probabilities of the individual random variables. Then we partially extend the result to the case of random sums. Applications to some commonly used risk processes are proposed. All heavy-tailed distributions involved in this paper are supposed on the whole real line.

KEYWORDS

Asymptotics, Heavy tails, Insurance risk model, Partial sums, Random sums, Subexponential distribution, Tail probabilities.

1. INTRODUCTION

Since the classical work of Lévy's formula

$$\mathbb{P}\left(\sup_{0 \leq t \leq 1} B(t) > x\right) = 2\mathbb{P}(B(1) > x),$$

where $B(t)$ is the Brownian motion satisfying $B(0) = 0$, the question as how to establish relations of the type

$$\mathbb{P}\left(\sup_{0 \leq t \leq 1} W(t) > x\right) \sim c\mathbb{P}(W(1) > x), \quad (1.1)$$

for certain stochastic process $W(t)$ has been in the focus of many researchers, where $c > 0$ is a constant. In a special case where $W(t)$, with $W(0) = 0$, is a

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continuous time process with independent and stationary increments, relations like (1.1) have been studied by many authors. Berman (1986) proved (1.1) for $c = 1$ under the assumptions that $W(t)$ has symmetric increments and that the tail of its Lévy’s spectral measure is of regular variation. Some related papers are Willekens (1987), Braverman & Samorodnitsky (1995). Most recently, Braverman (2000) further considered the tail asymptotics for the suprema of Lévy process $W(t)$ with light-tailed spectral Lévy measure. After introducing the notation of the right *light-tailed* distributions, he obtained (1.1) for the compound Poisson process

$$W(t) = \sum_{k=1}^{N(t)} X_k - bt, \quad t \geq 0, \tag{1.2}$$

where, $b > 0$ is a constant, and $\{X_k, k \geq 1\}$ is a sequence of i.i.d. and light-tailed random variables, independent of the homogeneous Poisson process $N(t)$. See also Braverman (1999). All the works above assume that the process is infinitely divisible.

Now one question naturally arises: does (1.1) holds if $W(t)$ is not an infinitely divisible process? The works cited above give no clue to this question. In this paper we shall establish a result (in Theorem 2.3 below) that is more general than giving a positive answer. From Theorem 2.3 with τ replaced by a counting process, we can easily obtain asymptotics like (1.1) for some commonly used processes in insurance risk models; see also Example 3.2 below. The methodology we used in this paper is different from those in the references above.

2. MAIN RESULTS

Throughout this paper, $\{X_k, k \geq 1\}$ denotes a sequence of independent random variables (r.v.’s), each X_k has a distribution function (d.f.), $F_k(x) = 1 - \overline{F}_k(x) = \mathbb{P}(X_k \leq x)$, $k \geq 1$. We denote by S_n the n th partial sum of the sequence $\{X_k, k \geq 1\}$. In the sequel, each F , or F_k , is always assumed to satisfy that $\overline{F}(x) > 0$ for all x . All limit relationships, unless otherwise stated, are for $x \rightarrow \infty$. For two positive infinitesimals $A(x)$ and $B(x)$, we write $A(x) \leq B(x)$ if $\limsup A(x)/B(x) \leq 1$, and define the reverse relation $A(x) \geq B(x)$ in the natural way. We write $A(x) \asymp B(x)$ if

$$0 < \liminf A(x)/B(x) \leq \limsup A(x)/B(x) < \infty.$$

We further write $A(x) \sim B(x)$, as usual, if both the limits above equal to 1.

Like many recent researchers in the fields of applied probability and risk theory, we restrict our interest to the case of heavy-tailed risks. A r.v. X (or its d.f. F) is said to be *heavy-tailed* on right-hand if $\mathbb{E}e^{rX} = \infty$ for any $r > 0$. The most important heavy-tailed subclass is the *subexponential class* (denoted as \mathcal{S}). By definition, a d.f. F supported on $[0, \infty)$ belongs to \mathcal{S} iff

$$\lim_{x \rightarrow \infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n \tag{2.1}$$

for all $n \geq 2$ (or equivalently for some $n \geq 2$), where F^{*n} denotes the n -fold convolution of F . That is, for a sequence of i.i.d. r.v.'s $\{X_k, k \geq 1\}$ with a common d.f. $F \in \mathcal{S}$, it holds that, for each $n \geq 2$,

$$\mathbb{P}(S_n > x) \sim \mathbb{P}\left(\max_{1 \leq k \leq n} X_k > x\right). \tag{2.2}$$

More generally, a r.v. X with a d.f. supported on $(-\infty, \infty)$ belongs to \mathcal{S} if $X^+ = \max\{0, X\}$ belongs to \mathcal{S} ; see Willekens (1986). There are two other heavy-tailed subclasses, class \mathcal{L} of long-tailed d.f.'s and class \mathcal{D} of d.f.'s with dominatedly varying tails, which are closely related to class \mathcal{S} . A d.f. F supported on $(-\infty, \infty)$ belongs to class \mathcal{L} iff

$$\lim_{x \rightarrow \infty} \bar{F}(x+y) / \bar{F}(x) = 1 \tag{2.3}$$

for any $y > 0$ (or equivalently for some $y > 0$); F belongs to class \mathcal{D} iff

$$\limsup_{x \rightarrow \infty} \bar{F}(xy) / \bar{F}(x) < \infty \tag{2.4}$$

for any $0 < y < 1$ (or equivalently for some $0 < y < 1$). It is well-known that

$$\mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}.$$

For details of heavy-tailed subclasses and their applications to insurance and finance, the readers are referred to Embrechts et al. (1997) and Goldie & Klüppelberg (1998).

In this paper we address the asymptotic properties of the tail probabilities of the maxima of the first n partial sums, $\max_{1 \leq k \leq n} S_k$, where n is either a constant or random. Now we state the main results of this paper as follows.

Theorem 2.1. *Suppose that the d.f. $F_k \in \mathcal{L}$ for $k \geq 1$. Then we have that, for each $n \in \mathcal{N}$.*

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k > x\right) \sim \mathbb{P}(S_n > x). \tag{2.5}$$

If we restrict ourselves to the subexponential case, from Theorem 2.1 we can obtain

Theorem 2.2. *Suppose that the d.f. $\bar{F}_k(x) \sim c_k \bar{F}(x)$ for $k \geq 1$, where \bar{F} is the tail of some subexponential d.f. F , and $c_k, k \geq 1$, are some non-negative constants such that $C(n) = \sum_{k=1}^n c_k > 0$. Then we have that, for each $n \in \mathcal{N}$,*

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k > x\right) \sim \mathbb{P}(S_n > x) \sim \mathbb{P}\left(\max_{1 \leq k \leq n} X_k > x\right) \sim C(n) \bar{F}(x). \tag{2.6}$$

We remark that, the conclusion $\mathbb{P}(\max_{1 \leq k \leq n} S_k > x) \sim C(n) \bar{F}(x)$ in (2.6) partially coincides with Theorem 1 in Sgibnev (1996). In the latter the author used

sophisticated analytical arguments of the Pollaczek-Spitzer identity to derive an asymptotic formula similar to (2.6) but only for the i.i.d. case. In this paper we shall provide an elementary proof of (2.6). We suggest the readers to compare our (2.6) with the definition of (2.2).

We continue to state the main results of the paper. In the following theorem we partially extend the asymptotic formula (2.6) to the case of random sums.

Theorem 2.3. *Suppose that $\{X_k, k \geq 1\}$ is a sequence of i.i.d. r.v.'s with a common d.f. $F \in \mathcal{L} \cap \mathcal{D}$ and a finite expectation. Let τ be a non-negative and integer-valued r.v., independent of the sequence $\{X_k, k \geq 1\}$ and satisfying that $\mathbb{P}(\tau > x) = o(\bar{F}(x))$. Then,*

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k > x\right) \sim \mathbb{P}(S_\tau > x) \sim \mathbb{E}\tau \bar{F}(x). \quad (2.7)$$

Note that the conditions in Theorem 2.3 imply the existence of $\mathbb{E}\tau$. It is well-known that the intersection $\mathcal{L} \cap \mathcal{D}$ is a very large subexponential subclass. It contains many useful subclasses of heavy-tailed distributions. See Bingham et al. (1987) and Embrechts et al. (1997) for details. We also remark that, the second asymptotic relationship in (2.7) describes the tail asymptotic behavior of the compound sum

$$\mathbb{P}(S_\tau \leq x) = \sum_{n=0}^{\infty} \mathbb{P}(\tau = n) F^{*n}(x).$$

For more general discussion about the tail asymptotics of the compound distributions, please refer to Chover et al. (1973), Embrechts et al. (1979), Embrechts & Goldie (1982) and Cline (1987).

3. APPLICATIONS TO RISK PROCESSES

In this section we show some applications of our results to some commonly used risk processes. All of the following examples are complementary to recent research on the tail behavior of the maxima/suprema of processes in discrete/continuous time.

Example 3.1. We now consider a discrete time insurance risk model. Upper bounds for the ultimate ruin probability under this model were discussed in Yang (1999). Suppose within the k th time period, the total claim size is Z_k and the total premium income is Y_k , and the claim is paid at the end of the time period while the premium is paid at the beginning. We assume $\{(Z_k, Y_k), k \geq 1\}$ constitutes a sequence of independent, but not necessarily identically distributed, random vectors. Let $r \geq 0$ be the compound interest rate and $U_0 = x$ be the initial surplus. Then the surplus at time n is

$$U_n = x(1+r)^n + \sum_{k=1}^n Y_k (1+r)^{n-k+1} - \sum_{k=1}^n Z_k (1+r)^{n-k}, \quad n \geq 1. \quad (3.1)$$

Here r can be dependent on the time k or even random such that $\{r_k, k \geq 1\}$ is an independent sequence and independent of the sequence $\{(Z_k, Y_k), k \geq 1\}$. But for notational simplicity, we assume that r is a constant. We can model the seasonal effects by assigning different distributions to (Z_k, Y_k) for different k .

Now we are interested in the asymptotic behavior of the probability

$$\psi_n(x) = \mathbb{P}\left(\inf_{1 \leq k \leq n} U_k < 0 \mid U_0 = x\right).$$

Let

$$\Delta_k = Z_k - Y_k(1+r), \quad X_k = \Delta_k(1+r)^{-k} \quad \text{and} \quad S_n = \sum_{k=1}^n X_k. \tag{3.2}$$

We can rewrite the process U_n as

$$U_n = (1+r)^n(x - S_n).$$

Hence,

$$\psi_n(x) = \mathbb{P}\left(\inf_{1 \leq k \leq n} \frac{U_k}{(1+r)^k} < 0 \mid U_0 = x\right) = \mathbb{P}\left(\max_{1 \leq k \leq n} S_k > x\right). \tag{3.3}$$

If we assume that each Δ_k , therefore X_k , is long-tailed, then from (3.3) and Theorem 2.1, we obtain

$$\psi_n(x) \sim \mathbb{P}(S_n > x). \tag{3.4}$$

Now we further assume that the sequence $\{(Z_k, Y_k), k \geq 1\}$ is i.i.d. and that Δ_1 follows a Pareto law with tail

$$\mathbb{P}(\Delta_1 > x) \sim x^{-\alpha} L(x)$$

for some $\alpha > 0$ and a function $L(x)$, which slowly varies at infinity. Under these conditions we have from (3.3) and Theorem 2.2 that

$$\begin{aligned} \psi_n(x) &\sim \sum_{k=1}^n \mathbb{P}(X_k > x) = \sum_{k=1}^n \mathbb{P}(\Delta_1(1+r)^{-k} > x) \\ &\sim \sum_{k=1}^n (1+r)^{k\alpha} \mathbb{P}(\Delta_1 > x) \\ &\sim \frac{(1+r)^{(n+1)\alpha} - (1+r)^\alpha}{(1+r)^\alpha - 1} x^{-\alpha} L(x), \end{aligned}$$

where the coefficient in the last step naturally takes value of n provided that $r = 0$.

Example 3.2. In the classical insurance risk models the collective risk process $W(t)$ is often in the form of (1.2). In a recent paper by Ng et al. (2001), we

proposed a customer-arrival based model for $W(t)$. Such a $W(t)$ may or may not be infinitely divisible. We assume that the customer arrival process $N(t)$ is a counting process. When the k th ($k \geq 1$) customer arrives, he/she buys an insurance contract. The insurance company will therefore bear an underlying risk from this policy holder within a fixed term, say T . Suppose that the total potential claims due to the k th customer within the term T is Z_k , and that $\{Z_k, k \geq 1\}$ forms a sequence of i.i.d. non-negative r.v.'s with a finite expectation μ and independent of the customer-arrival process $N(t)$. The price of each policy is $(1 + \delta)\mu$, where the constant $\delta > 0$ can be interpreted as the safety loading coefficient. The collective risk process of the company within the period $[0, t]$ in the customer-arrival based model is

$$W(t) = \sum_{k=1}^{N(t)} (Z_k - (1 + \delta)\mu), \quad t \geq 0. \quad (3.5)$$

We write $X_k = Z_k - (1 + \delta)\mu$, $k \geq 1$, and denote by F the distribution of X_1 . In view of Theorem 2.3, if we assume $F \in \mathcal{L} \cap \mathcal{D}$ and that

$$\mathbb{P}(N(1) > x) = o(\bar{F}(x)), \quad (3.6)$$

then for the collective risk process in (3.5) we have

$$\mathbb{P}\left(\sup_{0 \leq t \leq 1} W(t) > x\right) \sim \mathbb{P}(W(1) > x) \sim \mathbb{E}N(1)\bar{F}(x). \quad (3.7)$$

We note that (3.6) is a very weak condition. It can be satisfied by many commonly used counting processes such as the ordinary renewal process and the compound Poisson process. More generally, (3.6) can be satisfied by the *compound renewal process*, as demonstrated by the following

Lemma 3.3. *Consider the compound renewal process*

$$N(t) = \sum_{k=1}^{\infty} Y_k \mathbb{1}_{(\sigma_k \leq t)}, \quad t \geq 0,$$

where, σ_k denotes the k th partial sum of an i.i.d. and non-negative sequence $\{\theta_k, k \geq 1\}$, which is independent of another i.i.d., non-negative and integer-valued sequence $\{Y_k, k \geq 1\}$. Let $F \in \mathcal{D}$ be a d.f. with a finite expectation (F represents the distribution of X_1 in Example 3.2). Then we have that (3.6) holds if $\mathbb{P}(\theta_1 > 0) > 0$ and $\mathbb{P}(Y_1 > x) = o(\bar{F}(x))$.

Proof Let $M(t)$ be an ordinary renewal process driven by the occurrence-times $\{\sigma_k, k \geq 0\}$ with $\sigma_0 = 0$, i.e.,

$$M(t) = \max\{k \geq 0 : \sigma_k \leq t\}, \quad t \geq 0.$$

From the assumption $\mathbb{P}(\theta_1 > 0) > 0$ we know that $\mathbb{P}(\sigma_{m_0} \leq 1) = p_0 < 1$ holds for some $m_0 \geq 1$. It follows that, for any $n \geq 1$,

$$\mathbb{P}(M(1) \geq n) = \mathbb{P}(\sigma_n \leq 1) \leq p_0^{n/m_0}.$$

Hence $M(1)$ has exponential moments. We remark that a short proof of the fact that $M(1)$ has exponential moments can also be found in Smith (1958). Now we start to prove (3.6). By Lemma 3.2 in Tang et al. (2001) we can obtain that, for all $u > 0$, $x > 0$ and $n \geq 1$,

$$\mathbb{P}\left(\sum_{k=1}^n Y_k > x\right) \leq n\mathbb{P}(Y_1 > x/u) + (edn/x)^u, \tag{3.8}$$

where d is the expectation of Y_1 . Note that $d < \infty$ can be implied by the assumption $\mathbb{P}(Y_1 > x) = o(\bar{F}(x))$. We choose $u > 0$ in (3.8) sufficiently large such that $x^{-u} = o(\bar{F}(x))$; for details of the existence of the mentioned u , please refer to Theorem 2.2.7 in Bingham et al. (1987), in which we can find a useful representation for the tail probability $\bar{F}(x)$ for $F \in \mathcal{D}$. Thus,

$$\begin{aligned} \mathbb{P}(N(1) > x) &= \sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{k=1}^n Y_k > x\right) \mathbb{P}(M(1) = n) \\ &\leq \sum_{n=1}^{\infty} \left(n\mathbb{P}(Y_1 > x/u) + (edn/x)^u\right) \mathbb{P}(M(1) = n) \\ &= \mathbb{E}M(1)\mathbb{P}(Y_1 > x/u) + (ed/x)^u \mathbb{E}M^u(1) \\ &= o(\bar{F}(x)), \end{aligned}$$

where in the last step we have used the fact that

$$\bar{F}(x/u) \asymp \bar{F}(x), \quad \text{for any } u > 0 \tag{3.9}$$

which is implied by the assumption $F \in \mathcal{D}$. This ends the proof.

4. PROOFS OF MAIN RESULTS

4.1. Lemmas

In this section, all the notations are the same as those given at the beginning of Section 2. Before we give the proofs of the main results, we present some important lemmas.

The first lemma below is from Petrov (1975), the two inequalities of which can be regarded as extensions of the classical Lévy’s inequality and Kolmogorov’s inequality respectively. For $0 < q < 1$, we denote by $\gamma_q(X)$, the quantile

of order q of a r.v. X ; that is, the value of $\gamma_q(X)$ can be arbitrarily chosen from the set

$$\Gamma_q(X) = \{x : \mathbb{P}(X \leq x) \geq q, \mathbb{P}(X \geq x) \geq 1 - q\}.$$

Lemma 4.1. 1. For any $0 < q < 1$ and real value of x it holds that

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq \frac{1}{q} \mathbb{P}\left(S_n \geq x - \max_{1 \leq k \leq n} \{\gamma_q(S_k - S_n)\}\right). \tag{4.1}$$

2. Let $\mathbb{E}|X_k| < \infty$ for $1 \leq k \leq n$. Then, for any $0 < q < 1$ and real value of x it holds that

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq \frac{1}{q} \mathbb{P}\left[S_n \geq x - \frac{\sum_{k=1}^n \mathbb{E}|X_k|}{1-q}\right]. \tag{4.2}$$

Next, we put forward another lemma, which shows the closure property of class \mathcal{L} under convolution; for some closely related discussions, we refer to Willekens (1986).

Lemma 4.2. Let the d.f. F_k belong to class \mathcal{L} , $k \geq 1$. Then we have, for any $n \in \mathcal{N}$,

$$V = F_1 * F_2 \cdots * F_n \in \mathcal{L}.$$

Proof. It suffices to show the assertion for the case $n = 2$. From the definition of (2.3), this amounts to showing that $V = F_1 * F_2$ satisfies

$$\bar{V}(x + 1) \sim \bar{V}(x). \tag{4.3}$$

For arbitrarily fixed $0 < M < x$, we write

$$\bar{V}(x) = \int_{-\infty}^{\infty} \bar{F}_1(x-t)F_2(dt) = I_1 + I_2, \tag{4.4}$$

where I_1 and I_2 are the integrals of $\bar{F}_1(x-t)$ with respect to $F_2(dt)$ over $(-\infty, x-M]$ and $(x-M, \infty)$ respectively. By the same method we also divide the tail $\bar{K}(x+1)$ into two parts as $J_1 + J_2$, where J_i is the integral of $\bar{F}_1(x+1-t)$ with respect to $F_2(dt)$ over the same interval as that of I_i , $i = 1, 2$. First we start to compare I_1 with J_1 . Clearly, we have

$$\inf_{t \geq M} \frac{\bar{F}_1(t)}{\bar{F}_1(t+1)} \leq \frac{I_1}{J_1} = \frac{\int_{-\infty}^{x-M} \bar{F}_1(x-t)F_2(dt)}{\int_{-\infty}^{x-M} \bar{F}_1(x+1-t)F_2(dt)} \leq \sup_{t \geq M} \frac{\bar{F}_1(t)}{\bar{F}_1(t+1)}.$$

Therefore, by the definition of (2.3), we see that

$$\lim_{M \rightarrow \infty} \liminf_{x \rightarrow \infty} \frac{I_1}{J_1} = \lim_{M \rightarrow \infty} \liminf_{x \rightarrow \infty} \frac{J_1}{J_1} = 1. \tag{4.5}$$

Now we turn to compare I_2 with J_2 . Again by the fact that $F_2 \in \mathcal{L}$, we can see that, for any fixed $M > 0$,

$$\begin{aligned} I_2 &= \int_{x-M}^x \overline{F}_1(x-t)F_2(dt) + \int_x^\infty (x-t)F_2(dt) \\ &= o(\overline{F}_2(x)) + \overline{F}_1(0)\overline{F}_2(x) + \int_{-\infty}^0 \overline{F}_2(x-t)F_1(dt) \\ &\sim \overline{F}_2(x). \end{aligned}$$

By the same approach we can also derive that $J_2 \sim \overline{F}_2(x)$. Hence, for arbitrarily fixed $0 < M < x$,

$$I_2 \sim J_2. \tag{4.6}$$

We conclude from (4.4), (4.5) and (4.6) that (4.3) holds. This ends the proof. □

Finally we state an inequality for the tail probabilities of sums of i.i.d. r.v.'s with dominatedly varying tails.

Lemma 4.3. *Let G be a d.f. supported on $[0, \infty)$ and belonging to \mathcal{D} with a finite expectation μ . Then, for any $\gamma > \mu$ there exists a constant $C = C(\gamma) > 0$ such that*

$$\overline{G^{*n}}(x) \leq Cn\overline{G}(x) \tag{4.7}$$

for all $n \geq 1$ and all $x \geq \gamma n$.

Proof. The proof of a similar but more precise result can be found in Ng et al. (2001). Here we provide a few key steps of the proof.

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. r.v.'s with a common d.f. G above. We write

$$\widetilde{X}_n = X_n I_{(X_n \leq lx)} \quad \text{and} \quad \widetilde{S}_n = \sum_{i=1}^n \widetilde{X}_i \quad \text{for } n \geq 1,$$

where $0 < l < 1$ is a constant. Obviously,

$$\begin{aligned} \overline{G^{*n}}(x) &\leq P\left(\max_{l \leq i \leq n} X_i > lx\right) + P\left(\max_{l \leq i \leq n} X_i \leq lx, S_n > x\right) \\ &\leq n\overline{G}(lx) + P(\widetilde{S}_n > x). \end{aligned} \tag{4.8}$$

The upper bound, say $C_1(l)n\overline{G}(x)$, for the first term in (4.8) can be obtained by the fact that $G \in \mathcal{D}$ whenever the constant l is fixed.

Let $a = \max \{-\log(n\bar{G}(x))\}$. Clearly, a tends to ∞ uniformly for $x \geq \gamma n$ as $n \rightarrow \infty$. For any $h > 0$, we have

$$\begin{aligned} \frac{P(\bar{S}_n > x)}{n\bar{G}(x)} &\leq \frac{e^{-hx} Ee^{hS_n}}{n\bar{G}(x)} \leq e^{-hx+a} \left\{ \int_0^{lx} (e^{ht} - 1)G(dt) + 1 \right\}^n \\ &\leq \exp \left\{ n \left(\int_0^{lda} + \int_{lda}^{lx} \right) (e^{ht} - 1)G(dt) - hx + a \right\} \quad (4.9) \\ &= \exp \{ n(K_1 + K_2) - hx + a \}. \end{aligned}$$

We can obtain the required upper bound for K_1 by using $e^u - 1 \leq ue^u$ for any real number u . In addition, Proposition 2.2.1 in Bingham et al. (1987) stated that, for $G \in \mathcal{D}$, there exist positive x_0, ρ and B such that

$$\bar{G}(ux) / \bar{G}(x) \leq Bu^{-\rho} \tag{4.10}$$

uniformly for all $x \geq x_0/l$ and all $0 < u < 1$. This result, with u in (4.10) replaced by l/a , can be used to derive the required upper bound for K_2 .

Substituting the upper bounds for K_1 and K_2 into (4.9) and choosing the constants h and l carefully, we can prove that the right-hand side of (4.9) is bounded by a sufficiently large positive constant, say $C_2(l, \gamma) > 0$. Therefore the second term in (4.8) is bounded by $C_2(l, \gamma)n\bar{G}(x)$ uniformly for $n \geq 1$ and $x \geq \gamma n$. From this (4.7) can be finally proven. \square

4.2. Proofs of Theorems

Proof of Theorem 2.1.

In order to prove (2.5) we need to show that

$$\mathbb{P} \left(\max_{1 \leq k \leq n} S_k > x \right) \leq \mathbb{P}(S_n > x) \text{ and } \mathbb{P} \left(\max_{1 \leq k \leq n} S_k > x \right) \geq \mathbb{P}(S_n > x). \tag{4.11}$$

The last relation in (4.11) is obvious. By Lemma 4.2 we know the distribution of S_n belongs to class \mathcal{L} . Then by (4.1), we have, for any $0 < q < 1$,

$$\mathbb{P} \left(\max_{1 \leq k \leq n} S_k > x \right) \leq \frac{1}{q} \mathbb{P} \left(S_n \geq x - \max_{1 \leq k \leq n} \{ \gamma_q (S_k - S_n) \} \right) \sim \frac{1}{q} \mathbb{P}(S_n \geq x).$$

So letting $q \nearrow 1$ in the above yields the first relation in (4.11).

Proof of Theorem 2.2.

Noting the fact $S \subset \mathcal{L}$, we see that $\mathbb{P}(\max_{1 \leq k \leq n} S_k > x) \sim \mathbb{P}(S_n > x)$ is the immediate consequence of Theorem 2.1. The result $\mathbb{P}(S_n > x) \sim C(n)\bar{F}(x)$ is a particular case ($\gamma = 0$) of the Proposition in Sgibnev (1988). In addition, one easily obtains $\mathbb{P}(\max_{1 \leq k \leq n} X_k > x) \sim C(n)\bar{F}(x)$ by induction. This ends the proof of Theorem 2.2.

Proof of Theorem 2.3.

In order to prove (2.7), we consider it as a conjunction of two asymptotic relationships,

$$\mathbb{P}\left(\max_{1 \leq k \leq \tau} S_k > x\right) \sim \mathbb{E}\tau \bar{F}(x) \text{ and } P(S_\tau > x) \sim \mathbb{E}\tau \bar{F}(x). \tag{4.12}$$

We write $X_k^+ = \max\{X_k, 0\}$, $\mu^+ = \mathbb{E}X_k^+$, and let the d.f. G in Lemma 4.3 be the distribution of the r.v. X_k^+ . For a fixed $0 < q < 1$, we choose a sufficiently large constant γ such that

$$\gamma > \mu^+ + \frac{\mathbb{E}|X_1|}{1-q}.$$

Now we derive

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq k \leq \tau} S_k > x\right) &= \sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1 \leq k \leq n} S_k > x\right) \mathbb{P}(\tau = n) \\ &= \left(\sum_{1 \leq n < x/\gamma} + \sum_{n \geq x/\gamma}\right) \mathbb{P}\left(\max_{1 \leq k \leq n} S_k > x\right) \mathbb{P}(\tau = n) \\ &= L_1 + L_2. \end{aligned} \tag{4.13}$$

For $1 \leq n < x/\gamma$, we obtain, successively by (4.2) and (4.7),

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq k \leq n} S_k > x\right) &\leq \frac{1}{q} \mathbb{P}\left(\sum_{k=1}^n X_k^+ \geq x - \frac{n\mathbb{E}|X_1|}{1-q}\right) \\ &\leq \frac{Cn}{q} \mathbb{P}\left(X_1^+ \geq x - \frac{n\mathbb{E}|X_1|}{1-q}\right) \\ &\leq \frac{Cn}{q} \mathbb{P}\left(X_1^+ \geq x\left(1 - \frac{\mathbb{E}|X_1|}{\gamma(1-q)}\right)\right) \\ &\asymp n\mathbb{P}(X_1^+ \geq x) \\ &= n\bar{F}(x), \end{aligned} \tag{4.14}$$

where, in (4.14) we have used (3.9). So by the dominated convergence theorem and Theorem 2.2 we conclude

$$L_1 \sim \sum_{1 \leq n < x/\gamma} n\bar{F}(x) \mathbb{P}(\tau = n) \sim \mathbb{E}\tau \bar{F}(x). \tag{4.15}$$

Additionally, by the condition that $\mathbb{P}(\tau > x) = o(\bar{F}(x))$, we have

$$L_2 \leq \mathbb{P}(\tau \geq x/\gamma) = o(\bar{F}(x/\gamma)) = o(\bar{F}(x)). \tag{4.16}$$

Substituting (4.15) and (4.16) into (4.13), we obtain the first asymptotic relationship in (4.12). Finally, we have

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq k \leq \tau} S_k > x\right) &\geq \mathbb{P}(S_\tau > x) \\ &\geq \sum_{1 \leq n < x\gamma} \mathbb{P}(S_n > x) \mathbb{P}(\tau = n) \\ &\sim \sum_{1 \leq n < x\gamma} n \bar{F}(x) \mathbb{P}(\tau = n) \\ &= \mathbb{E} \tau \bar{F}(x). \end{aligned}$$

This, coupled with the first asymptotic relationship in (4.12), implies the second in (4.12). The proof of Theorem 2.3 is completed.

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