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Moments of the central L-values of the Asai lifts

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Abstract. We study some analytic properties of the Asai lifts associated with cuspidal Hilbert modular forms, and prove sharp bounds for the second moment of their central L-values.

1 Introduction

Let $\mathbf{F}$ be a fixed real quadratic field over $\mathbf{Q}$, with ring of integers $O = O_{\mathbf{F}}$ and the real imbeddings $\sigma_1 = 1,\ \sigma_2$. For simplicity, we assume the narrow class number of $\mathbf{F}$ is 1, so the totally positive units are squares of units and every ideal has a totally positive generator. Let $SL(2, O)$ be the Hilbert modular group. For any ideal $\mathcal{C} \subset O$, the Hecke congruence subgroups $\Gamma_0(\mathcal{C}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, O), \quad c \equiv 0 \pmod{\mathcal{C}}\right\}$, act discontinuously on the upper half-space $\mathbf{H}^2$ in the usual way with finite co-volumes, i.e., for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathcal{C}) \text{ and } z = (z_1,\ z_2) \in \mathbf{H}^2,$$

we have

$$\gamma(z) = \left(\frac{\sigma_1(a)z_1 + \sigma_1(b)}{\sigma_1(c)z_1 + \sigma_1(d)},\ \frac{\sigma_2(a)z_1 + \sigma_2(b)}{\sigma_2(c)z_1 + \sigma_2(d)}\right).$$

Denote by $M_k(\Gamma_0(\mathcal{C}))(k \in 2\mathbf{Z}$ and $\geq 2)$, the space of Hilbert modular forms of parallel even weight (k, k), level $\mathcal{C}$ with trivial character, i.e., the space of holomorphic functions $f(z)$ on $\mathbf{H}^2$ such that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathcal{C})$, $f(\gamma(z)) = N(cz + d)^k f(z)$, where for $z = (z_1,\ z_2) \in \mathbf{H}^2$,

$$N(cz + d)^k = (\sigma_1(c)z_1 + \sigma_1(d))^k \cdot (\sigma_2(c)z_2 + \sigma_2(d))^k.$$

Any $f(z)$ in $M_k(\Gamma_0(\mathcal{C}))$ has the following Fourier expansion (we assume that the different of $\mathbf{F}$ is generated by $\delta = \delta_{\mathbf{F}} > 0$, where and henceforth $\xi > 0$ for $\xi \in \mathbf{F}$ means that ξ is a totally positive element in $\mathbf{F}$, and denote $\nu^{(i)} = \sigma_i(\nu)$, the ith conjugate of ν

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for $i = 1, 2$):

(1)
$$f(z) = \sum_{\nu \in O,\ \nu \geq 0} a(\nu) \exp(2\pi i \mathrm{Tr}(\nu z)),$$

where

$$\mathrm{Tr}(\nu z) = \sum_{i=1}^{2} \nu^{(i)} z_i \delta^{(i)-1}.$$

Since any $f(z)$ in $M_k(\Gamma_0(\mathcal{C}))$ is invariant under $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$, where ε is an unit in O, we have $a(\varepsilon^2 \nu) = a(\nu)$.

$f(z) \in M_k(\Gamma_0(\mathcal{C}))$ is called a Hilbert modular cusp form if the Fourier expansion of $f(g(z))N(cz+d)^{-k}$ (see [Lu, p. 130]) has no constant term for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{F})$. Space of all such cusp forms is denoted by $S_k(\Gamma_0(\mathcal{C}))$.

It is well-known (see [Ga]) that $\dim_{\mathbf{C}} S_k(\Gamma_0(\mathcal{C}))$ is finite, and (see [Sh]) $J =: \dim_{\mathbf{C}} S_k(\Gamma_0(\mathcal{C})) \sim \frac{\mathrm{vol}(\Gamma_0(\mathcal{C})\backslash \mathbf{H}^2)}{(4\pi)^2}(k-1)^2$ as $k \to \infty$. Moreover,

$$\begin{aligned} \mathrm{vol}(\Gamma_0(\mathcal{C})\backslash \mathbf{H}^2)) &= [SL(2,O) : \Gamma_0(\mathcal{C})]\ \mathrm{vol}(SL(2,O)\backslash \mathbf{H}^2) \\ &= 2N(\mathcal{C}) \prod_{\mathcal{P}|\mathcal{C}} (1 + N(\mathcal{P})^{-1}) \times \pi^{-2} \zeta_{\mathbf{F}}(2) D^{3/2}, \end{aligned}$$

where $\zeta_{\mathbf{F}}(s)$ is the Dedekind zeta-function of $\mathbf{F}$ and $D = D_{\mathbf{F}}$ is the discriminant. The Petersson inner product on $S_k(\Gamma)$ is defined by

$$< g_1,\ g_2 >= \int_{\Gamma \backslash \mathbf{H}^2} g_1(z)\overline{g_2(z)} \prod_{i=1}^{2} y_i^{k-2} dx_i dy_i,$$

where $z = (z_1,\ z_2)$ with $z_i = x_i + y_i\sqrt{-1},\ i = 1, 2$.

Now, let f be a cuspidal Hilbert modular form of parallel weight (k, k) for even $k \geq 2$ and with respect to $GL^+(2, O) \supset SL(2, O)$. We assume f is a normalized Hecke eigenform with Fourier coefficients $a_f(\nu) = a_f(1)\lambda_f(\nu)N(\nu)^{(k-1)/2},\ \nu \in O$, where $\lambda_f(\mu)$ is the eigenvalue of $f(z)$ for the Hecke operator $T_{(\mu)}$ (see, e.g., [Ga]). We have

$$\lambda_f(\mu)\lambda_f(\nu) = \sum_{(d),\ d|(\mu,\nu),\ d>0} \lambda_f\left(\frac{\mu\nu}{d^2}\right).$$

The standard L-function associated with f is defined, for $\Re(s) > 1$, by

$$L(s,\ f) = \sum_{(\mu),\ \mu>0} \lambda_f(\mu) N(\mu)^{-s},$$

which has Euler product

$$\prod_{(\pi),\ \pi>0} (1 - \lambda_f(\pi)N(\pi)^{-s} + N(\pi)^{-2s})^{-1},$$

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where π stands for prime element of O. It is well-known that $L(s, f)$ has analytic continuation to the whole complex plane as an entire function. Let

$$\Lambda(s, f) = (2\pi)^{-2s}\Gamma^2(s + (k-1)/2)L(s, f).$$

We then have the functional equation

$$\Lambda(s, f) = \varepsilon_f D^{1-2s}\Lambda(1-s, f),$$

where ε_f is the root number of absolute value 1.

Asai [As] defined a new Dirichlet series by restricting the coefficients on rational integers,

$$L(s, \mathrm{As}(f)) = \zeta(2s)\sum_{m=1}^{\infty}\lambda_f(m)m^{-s},\ \Re(s) > 1.$$

He showed that the function

$$\Lambda(s, \mathrm{As}(f)) = D^{s/2}(2\pi)^{-2s}\Gamma(s+k-1)\Gamma(s)L(s, \mathrm{As}(f))$$

admits analytic continuation to the whole s-plane with possible simple poles at $s = 0, 1$, and satisfies the functional equation

$$\Lambda(s, \mathrm{As}(f)) = \Lambda(1-s, \mathrm{As}(f)).$$

Moreover, if

$$\begin{aligned} L(s, f) &= \prod_{(\pi),\ \pi>0}(1 - \lambda_f(\pi)N(\pi)^{-s} + N(\pi)^{-2s})^{-1} \\ &= \prod_{(\pi),\ \pi>0}[(1-\alpha_f(\pi)N\pi^{-s})(1-\beta_f(\pi)N\pi^{-s})]^{-1}, \end{aligned}$$

then we have

$$L(s, \mathrm{As}(f)) = \prod_p L_p(s),$$

where

$$L_p^{-1}(s) = \begin{cases} (1-\alpha_f(\pi_1)\alpha_f(\pi_2)p^{-s})(1-\alpha_f(\pi_1)\beta_f(\pi_2)p^{-s}) & \\ (1-\beta_f(\pi_1)\alpha_f(\pi_2)p^{-s})(1-\beta_f(\pi_1)\beta_f(\pi_2)p^{-s}), & \text{if} \quad p = \pi_1\pi_2, \pi_1 \neq \pi_2; \\ (1-\alpha_f(\pi)p^{-s})(1-\beta_f(\pi)p^{-s})(1-p^{-2s}), & \text{if} \quad p = \pi; \\ (1-\alpha_f^2(\pi)p^{-s})(1-\beta_f^2(\pi)p^{-s})(1-p^{-s}), & \text{if} \quad p = \pi^2. \end{cases}$$

Ramakrishnan [Ra] and Krishnamurthy [Kr] proved that $\Lambda(s, \mathrm{As}(f))$ is in fact the L-function associated with an automorphic form on $GL(4, A_{\mathbf{Q}})$, the Asai lift $\mathrm{As}(f)$ of f. Then, in view of the Splitting Formula in [As] and assuming $D = D_{\mathbf{F}}$ is odd, we have

$$L(s, f\otimes f^t) = L(s, \mathrm{As}(f))\ L(s, \mathrm{As}(f)\otimes\chi_D),$$

where

$$\chi_D(\cdot) = \left(\frac{D}{\cdot}\right)$$

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is the Kronecker symbol, and

$$f^t(z_1, z_2) = f(z_2, z_1).$$

If f is a base change from an Hecke eigenform $h \in S_k(SL_2(\mathbf{Z}))$, then f is symmetric, i.e., $f = f^t$, and

$$L(s, \mathrm{As}(f)) = L(s, \mathrm{sym}^2(h))\, L(s, \chi_D),$$

while if f is a base change from an Hecke eigenform $h \in S_k(\Gamma_0(D), \chi_D)$, then also $f = f^t$, and

$$L(s, \mathrm{As}(f)) = L(s, \mathrm{sym}^2(h))\, \zeta(s)$$

(see [As, Section 5]).

Moreover, Prasad and Ramakrishnan [PR] established the following (special case of) cuspidal criterion for $\mathrm{As}(f)$.

Theorem 1.1 (Prasad and Ramakrishnan) *With the same notation as above. If f is non-dihedral, then $\mathrm{As}(f)$ is non-cuspidal iff f and f^t are twist-equivalent; if f is dihedral, then $\mathrm{As}(f)$ is non-cuspidal iff f is induced from a quadratic extension K of F which is biquadratic over* $\mathbf{Q}$.

Choosing an orthonormal basis $\{f_j(z)\}_{j=1}^J$ of $S_k(\Gamma_0(\mathcal{C}))$ and denote the Fourier coefficients of $f_j(z)$ by $a_j(\cdot)$. We normalize the Fourier coefficients $a_j(\mu)$ by

$$\psi_j(\mu) = \left(\frac{N(\mathcal{C})((k-1)!)^2 D^{k+1}}{((4\pi)^2 N(\mu))^{k-1}} \right)^{1/2} a_j(\mu).$$

We then have the Petersson formula for Hilbert modular forms as proved in [Lu],

$$\sum_{j=1}^{J} \bar{\psi}_j(\nu)\psi_j(\mu) = \chi_\nu(\mu) D^{3/2} N(\mathcal{C})(k-1)^2 + N(\mathcal{C})(k-1)^2 D(2\pi)^2 \sum_{\varepsilon \in U} \sum_{c \in \mathcal{C}^\times/U}$$

$$\frac{1}{|N(c)|} S(\nu, \mu\varepsilon^2; c)\; NJ_{k-1}(4\pi\sqrt{\mu\nu}|\varepsilon|/|c|), \tag{2}$$

where χ_ν is the characteristic function of the set $\{\nu\varepsilon^2,\ \varepsilon \in U\}$, U is the unit group of $\mathbf{F}$,

$$S(\nu, \mu; c) = \sum_{h \pmod{c}}{}^{*}\, e\left(\frac{\nu h + \mu \bar{h}}{c}\right)$$

is the generalized Kloosterman sum, and $e(x) = \exp(2\pi i \mathrm{Tr}(x))$ for $x \in \mathbf{F}$. We will assume that in the above formula, the c's are chosen among their associates the representatives satisfying $|N(c)|^{1/2} \ll |c^{(i)}| \ll |N(c)|^{1/2}$, $i = 1, 2$.

If the L^2-normalized basis element $f_j = \tilde{f}_j/\|\tilde{f}_j\|$ is a newform, where $\tilde{f}_j$ is the corresponding *arithmetically* normalized newform with the first Fourier coefficient 1, then $\psi_j(\mu) = \psi_j(1)\, \lambda_j(\mu)$, where $\lambda_j(\cdot)$ denotes the (normalized) Hecke eigenvalues of f_j as noted above. For $\mathcal{C} = (1)$, from the integral representation for $L(s,\ \tilde{f}_j \otimes \overline{\tilde{f}_j})$,

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and the factorization $L(s,\ \tilde{f}_j \otimes \overline{\tilde{f}_j}) = \zeta_{\mathbf{F}}(s)\ L(s,\ \mathrm{ad}(\tilde{f}_j))$, we have

$$|a_j(1)|^{-2} = \|\tilde{f}_j\|^2 = 16D^{1+k}(4\pi)^{-2k-2}\Gamma^2(k)\ L(1,\ \mathrm{ad}(\tilde{f}_j))/L(1,\ \chi_D).$$

Thus for $\mathfrak{C} = (1)$,

$$\bar{\psi}_j(\nu)\ \psi_j(\mu) = \frac{(4\pi)^4 L(1,\ \mathrm{ad}(\tilde{f}_j))}{16L(1,\ \chi_D)}\lambda_j(\nu)\ \lambda_j(\mu).$$

For each $j,\ 1 \le j \le J$ and any $\varepsilon > 0$, we have (see [Ta])

$$\lambda_j(\mu) \ll N(\mu)^{\varepsilon},$$

and by a straightforward extension of results of [Iw] and [HL] that

$$k^{-\varepsilon} \ll L(1,\ \mathrm{ad}(\tilde{f}_j)) \ll k^{\varepsilon}.$$

In [Lu], we proved an asymptotic formula for the mean value of the linear form in $\psi_j(\cdot)$ in the level aspect. In this paper, we establish an analogous result for the weight aspect as well in the context of the quadratic field **F**, with an application to the second moment of $L(1/2, \mathrm{As}(f))$. The generalization of Theorem 1.2 to the general totally real fields is straightforward.

Theorem 1.2 *Let $b(\cdot)$ be an arbitrary complex numbers such that $b(\varepsilon^2\mu) = b(\mu)$ for $\varepsilon \in U$, and $\eta > 0$. Then for $S_k(\Gamma_0(\mathfrak{C}))$, we have as $k \to \infty$,*

$$\sum_{j=1}^{J}\left|\sum_{\mu} b(\mu)\psi_j(\mu)\right|^2 \ll (N(\mathfrak{C})k^2 + X)(kXN(\mathfrak{C}))^{\eta}\sum_{\mu}|b(\mu)|^2,$$

where the summation over μ's is restricted to $\mu \in O^{\times}/U^2,\ \mu > 0,\ N(\mu) \le X$, and the implicit constant only depends on the quadratic field **F** *and η.*

Assume $As(f)$ is cuspidal. From [IK, p. 98], we have a series representation for the central L-value of $L(s, As(f))$,

$$L(1/2,\ As(f)) = 2\sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^{1/2}}V_{1/2}\left(\frac{n}{\sqrt{D}}\right), \tag{3}$$

where

$$V_{1/2}(y) = \frac{1}{2\pi i}\int_{(2)}(4\pi^2 y)^{-u}\ \zeta(1+2u)\frac{\Gamma(1/2+u)\ \Gamma(k+u-1/2)}{\Gamma(1/2)\ \Gamma(k-1/2)}\frac{du}{u}.$$

Since

$$\frac{\Gamma(k+u-1/2)}{\Gamma(k-1/2)} \ll k^{\Re(u)}$$

by Stirling's formula, we see that $V_{1/2}(y) \ll k^{-A}$ for any $A \ge 1$, if $y > k^{1+\eta}$ for any $\eta > 0$. Thus, we have

$$L(1/2,\ As(f)) = 2\sum_{n \le k^{1+\eta}}\frac{\lambda_f(n)}{n^{1/2}}V_{1/2}\left(\frac{n}{\sqrt{D}}\right) + O(1).$$

From Theorem 1.2 and the above formula for $L(1/2, As(f))$, and by extending the orthonormal Hecke basis of $S_k(GL_2^+(O))$ to an orthonormal (Hecke) basis of $S_k(SL(2, O))$ and the positivity, we obtain the following theorem.

Theorem 1.3 *For the orthonormal Hecke basis $\{f_j\}$ of $S_k(GL_2^+(O))$ and any $\eta > 0$, we have*

$$\sum_{1\le j\le J}{}^{*}\, |L(1/2,\ As(f_j))|^2 \ll k^{2+\eta},$$

where the $*$ means that the summation is restricted to cuspidal Asai lifts $As(f_j)$, and the constant implicit only depends on the quadratic field $\mathbf{F}$ and η.

It remains to prove Theorem 1.2, which is the goal of the next section.

2 Proof of the Theorem 1.2

From the Poisson integral representation [GR, p. 953, (8)], we have

$$J_{k-1}(x) = \frac{\left(\frac{x}{2}\right)^{k-1}}{\sqrt{\pi}\,\Gamma(k-1/2)} \int_{-1}^{1} (1-t^2)^{k-3/2}\cos(xt)\,dt$$

$$\ll \left(\frac{ex}{2k}\right)^{k-1}, \tag{4}$$

where the implicit constant is absolute.

To prove Theorem 1.2, we may assume that μ's are chosen among their associates mod U^2 the representatives satisfying $N(\nu)^{1/2} \ll \nu^{(i)} \ll N(\nu)^{1/2},\ i = 1, 2$. We have by the Petersson formula (2),

$$\begin{aligned}
&\sum_{j=1}^{J} |\sum_{\mu} b(\mu)\psi_j(\mu)|^2\\
&= \sum_{\mu,\,\nu} b(\mu)\bar{b}(\nu) \sum_{j=1}^{J} \psi_j(\mu)\bar{\psi}_j(\nu)\\
&= \sum_{\mu} |b(\mu)|^2) D^{3/2}(k-1)^2 N(\mathfrak{C})\\
&\quad + (k-1)^2 D N(\mathfrak{C})(2\pi)^2 \sum_{\varepsilon\in U}\ \sum_{c\in\mathfrak{C}^\times/U}\\
&\quad \times \frac{1}{|N(c)|} \sum_{\mu,\,\nu} b(\mu)\bar{b}(\nu) S(\nu, \mu\varepsilon^2; c)\ N J_{k-1}(4\pi\sqrt{\mu\nu}|\varepsilon|/|c|)\\
&= \textstyle\sum_1 + \sum_2, \quad \text{say.}
\end{aligned}$$

We first prove Theorem 1.2 under the condition that $k^2 N(\mathfrak{C}) \ge 8(4\pi)^2 X$. In view of (4) and bound $|J_{k-1}(y)| \le 1$, we have $J_{k-1}(y) \ll \left(\frac{ey}{2k}\right)^{k-1-\eta'} \ll \left(\frac{2y}{k}\right)^{k-1-\eta'}$, for $y > 0$ and $0 \le \eta' < 1/2$, we have (choosing η' to be 0 or η, $0 < \eta < 1/2$ depending upon whether $|\varepsilon^{(i)}| \geq 1$ or not)

$$NJ_{k-1}(4\pi\sqrt{\mu\nu}|\varepsilon|/|c|)) \ll (4(4\pi)^2\sqrt{(N\mu)(N\nu)}/k^2|N(c)|)^{k-1}(k^2|N(c)|)^\eta \prod_{1\leq j\leq 2,\ |\varepsilon^{(j)}|\geq 1} |\varepsilon^{(j)}|^{-\eta}$$
$$\ll \left(\frac{1}{2|N(c_1)|}\right)^{k-1} (k^2|N(c)|)^\eta \prod_{1\leq j\leq 2,\ |\varepsilon^{(j)}|\geq 1} |\varepsilon^{(j)}|^{-\eta},$$

where we write $c = c_1\mathcal{C}$.

Also we have trivially

$$|S(\nu, \mu\varepsilon^2; c)| \leq N(c).$$

Hence, the partial sum of $\sum_2$ with the condition $*$ on U that $\varepsilon^{(0)} =: \max(|\varepsilon^{(1)}|, |\varepsilon^{(2)}|) \geq \exp(\log^2 N(\mathcal{C}))$, is bounded by

$$k^{2+2\eta}(N(\mathcal{C}))^{1+\eta} \sum_{\varepsilon\in U}{}^{*} |\varepsilon^{(0)}|^{-\eta} \sum_{c_1\in O^\times/U} \frac{2^{-k}X}{|N(c_1)|^{k-1-\eta}} \sum_\mu |b(\mu)|^2 \ll X\sum_\mu |b(\mu)|^2,$$

where we use the fact that the number of units ε satisfying $x \leq \log\varepsilon^{(0)} < 2x$, is $O(x)$ since U is cyclic and generated by a fundamental unit of O.

It remains to deal with the remaining sum Σ_2' with the sum over the units ε in U satisfying the condition #: $\log\varepsilon^{(0)} < \log^2 N(\mathcal{C})$. Note the above method clearly also works in this case if $N(\mathcal{C}) \leq 2^{k/2}$. Hence, we may assume $N(\mathcal{C}) > 2^{k/2}$ and thus $k \ll \log N(\mathcal{C})$. We will apply the following lemma proved in [Lu].

Lemma *Let c_1, $c_2 > 0$ be constants, $X \geq 1$, $d(\cdot)$ arbitrary complex numbers, and $c \in O$. Then we have*

$$\sum_{a \pmod{c}} \left| \sum_{N(\nu)\leq X,\ \nu\in O}{}' d(\nu)e\left(\frac{\nu a}{c}\right)\right|^2 = (|N(c)| + O(X)) \sum_{N(\nu)\leq X,\ \nu\in O}{}' |d(\nu)|^2,$$

where "′" means that the summation is restricted to those ν's such that $\nu > 0$, $c_1 N(\nu)^{1/2} \leq \nu^{(i)} \leq c_2 N(\nu)^{1/2}$.

Using the Mellin–Barnes integral representation [MOS, Section 3.6.3, p. 82],

$$J_{k-1}\left(\frac{4\pi\sqrt{\mu^{(i)}\nu^{(i)}}|\varepsilon^{(i)}|}{|c^{(i)}|}\right)$$
$$=\frac{1}{4\pi i}\int_{(2+\eta)} \left(\frac{2\pi\sqrt{\mu^{(i)}\nu^{(i)}}|\varepsilon^{(i)}|}{|c^{(i)}|}\right)^s \Gamma\left(\frac{k-1}{2} - \frac{s}{2}\right)\left[\Gamma\left(1 + \frac{k-1}{2} + \frac{s}{2}\right)\right]^{-1} ds,$$

opening the Kloosterman sum, and by Cauchy's inequality, we infer that for $c \in \mathcal{C}^\times/U$ and with $s_i = 2 + \eta + \sqrt{-1}t_i$ $(i = 1, 2)$ and $0 < \eta < 1/2$,

$$\sum_{\mu,\nu} b(\mu)\bar{b}(\nu)S(\nu, \mu\varepsilon^2; c)\ NJ_{k-1}(4\pi\sqrt{\mu\nu}|\varepsilon|/|c|)$$
$$\ll \int_{(2+\eta)} |ds_1| \int_{(2+\eta)} |ds_2| \left|\frac{\Gamma\left(\frac{k-1}{2} - \frac{s_1}{2}\right)}{\Gamma\left(1 + \frac{k-1}{2} + \frac{s_1}{2}\right)}\right| \cdot \left|\frac{\Gamma\left(\frac{k-1}{2} - \frac{s_2}{2}\right)}{\Gamma\left(1 + \frac{k-1}{2} + \frac{s_2}{2}\right)}\right|$$

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$$\times \max_{s_1,s_2} \sum_{h \pmod{c}} \left| \sum_{\mu,\,\nu} b(\mu)\,\bar{b}(\nu) \left(4\pi^2 \sqrt{N(\mu)N(\nu)}/|N(c)|\right)^{2+\eta} \prod_{i=1}^{2} (\sqrt{\mu^{(i)}\nu^{(i)}})^{\sqrt{-1}t_i} e\left(\frac{\mu h}{c}\right) \right|$$
$$\ll N(c)^{-(2+\eta)} \int_{(2+\eta)} \frac{|ds_1|}{k+|s_1|} \int_{(2+\eta)} \frac{|ds_2|}{k+|s_2|} \left| \frac{\Gamma\left(\frac{3}{2}-\frac{s_1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{s_1}{2}\right)} \right| \left| \frac{\Gamma\left(\frac{3}{2}-\frac{s_2}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{s_2}{2}\right)} \right|$$
$$\times \max_{s_1,s_2} \sum_{h \pmod{c}} \left| \sum_{\mu} b(\mu)\,(N(\mu))^{1+\eta/2} \prod_{i=1}^{2} (\mu^{(i)})^{\sqrt{-1}t_i/2} e\left(\frac{\mu h}{c}\right) \right|^2$$
$$\ll N(c_1)^{-(2+\eta)}\,(|N(c)|+X)\,(N(\mathcal{C}))^{\eta} \sum_{\mu} |b(\mu)|^2,$$

since $k \ll \log N(\mathcal{C})$, where as before, we write $c = c_1\mathcal{C}$.

Thus the partial sum Σ_2' is bounded by

$$k^2 (N(\mathcal{C}))^{\eta} \sum_{\varepsilon\in U}^{\#} \sum_{c_1 \in O^{\times}/U} \frac{1}{|N(c_1)|^{2+\eta}} (|N(c_1\mathcal{C})| + X) \sum_{\mu} |b(\mu)|^2$$
$$\ll (N(\mathcal{C}) + X) N(\mathcal{C})^{\eta} \sum_{\mu} |b(\mu)|^2,$$

since

$$\sum_{\varepsilon\in U}^{\#} 1 \ll \log^2 N(\mathcal{C}).$$

Hence, Theorem 1.2 is true if $k^2 N(\mathcal{C}) \geq 8(4\pi)^2 X$.

In the case $k^2 N(\mathcal{C}) < 8(4\pi)^2 X$, we reduce it to the previous case by the famous embedding trick of Iwaniec. Choosing a prime ideal $\mathcal{P} \subset O$ such that $N(\mathcal{P})k^2 N(\mathcal{C}) \asymp X$ and $N(\mathcal{P})k^2 N(\mathcal{C}) \geq 8(4\pi)^2 X$. Note that $[\Gamma_0(\mathcal{C}) : \Gamma_0(\mathcal{P}\mathcal{C})] \leq N(\mathcal{P}) + 1$. Let $H_k(\mathcal{C})$ denote an orthonormal basis of $S_{2k}(\Gamma_0(\mathcal{C}))$, and write

$$S_{\mathcal{C}}(b) = \sum_{f \in H_k(\mathcal{C})} |\sum_{\mu} b(\mu)\psi_f(\mu)|^2.$$

We deduce that

$$S_{\mathcal{C}}(b) \leq (1 + N(\mathcal{P})^{-1}) S_{\mathcal{P}\mathcal{C}}(b)$$
$$\ll (N(\mathcal{P}\mathcal{C})k^2 + X)(kXN(\mathcal{C}))^{\eta} \sum_{\mu} |b(\mu)|^2$$
$$\ll X(kXN(\mathcal{C}))^{\eta} \sum_{\mu} |b(\mu)|^2,$$

and this completes our proof.

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