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APÉRY SEQUENCES AND LEGENDRE TRANSFORMS

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Abstract

A lower bound for the minimal length of the polynomial recurrence of a binomial sum is obtained.

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A sequence a_n satisfies a *polynomial recurrence of length r and degree m* if there exist r polynomials $P_0, P_1, \ldots, P_{r-1}$, with degree at most m such that

$$P_0(n)a_n + P_1(n)a_{n-1} + \cdots + P_{r-1}(n)a_{n-r} = 0 \tag{1}$$

for $n \geq r$. For a sequence a_n the recurrence (1) is called *minimal* if it has minimal length and minimal degree.

It is well known (see [1, 8]) that the Apéry sequence

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfies the three term polynomial recurrence

$$n^3 a_n - (34n^3 - 51n^2 + 27n - 5)\, a_{n-1} + (n-1)^3 a_{n-2} = 0$$

for $n \geq 2$, where as usual $\binom{p}{q}$ denotes a binomial coefficient. Since the characteristic polynomial $x^2 - 34x + 1$ has roots $(1 \pm \sqrt{2})^4$, it follows that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \left(1 + \sqrt{2}\right)^4$$

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is irrational and that a_n cannot satisfy a two term recurrence. Apéry used these facts in his celebrated proof of the irrationality of $\zeta(3)$ (see [8]) and stimulated much interest in recursive sequences.

Wilf and Zeilberger [9] and others have shown that certain hypergeometric sums, including the binomial sum

$$(2)\quad a(n) = \sum_{k=0}^{n} \binom{n}{k}^{r_0} \binom{n+k}{k}^{r_1} \binom{n+2k}{k}^{r_2} \cdots \binom{n+tk}{k}^{r_t} = \sum_{k=0}^{n} \prod_{i=0}^{t} \binom{n+ik}{k}^{r_i},$$

where $r_0, r_1, r_2, \ldots, r_t$ are nonnegative integers, satisfy polynomial recurrences, without however any bounds on their lengths and degrees. It is not easy to find recurrences even for $a_r(n) = \sum_{k=0}^{n} \binom{n}{k}^r$ and

$$(3)\qquad a_{r,s}(n) = \sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^s,$$

and at present no nontrivial lower bounds for the minimal lengths of the recurrences for $a_{r,s}(n)$ exist.

The sums $a_r(n)$ (see above) for $n \geq 0$ have been studied by many people. Apart from the trivial recurrences

$$a_1(n+1) - 2a_1(n) = 0, \quad \text{and} \quad (n+1)a_2(n+1) - (4n+2)a_2(n) = 0$$

with $n \geq 0$, Franel [2, 3] was the first to obtain recurrences for $a_3(n)$ and $a_4(n)$, namely

$$P_0(n)a_r(n+1) + P_1(n)a_r(n) + P_2(n)a_r(n-1) = 0$$

for $n \geq 1$, where, for $r = 3$

$$P_0(n) = (n+1)^2, \quad P_1(n) = -(7n^2 + 7n + 2), \quad P_2(n) = -8n^2$$

and for $r = 4$

$$P_0(n) = (n+1)^3, \quad P_1(n) = -2(2n+1)(3n^2 + 3n + 1),$$

and

$$P_2(n) = -4n(4n+1)(4n-1).$$

For $r = 5$ and 6, Perlstadt [4] found recurrences of length 4 while Schmidt and Yuan [6] showed that the recurrences stated for $r = 3, 4, 5$ and 6 are minimal and that the minimal lengths for $r > 6$ are at least 3. In this paper a nontrivial lower bound for the minimal length of the sequence (2) is obtained. We prove the following result.

THEOREM 1. *Let $r_0, r_1 \geq 1$ and $m, r_2, \ldots, r_t$ be nonnegative integers. Then there exist no nontrivial integer polynomials*

$$P_0(n) = c_0 + c_1 n + \cdots + c_m n^m, \quad P_1(n) = d_0 + d_1 n + \cdots + d_m n^m$$

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such that

$$P_0(n)\,a(n+1) + P_1(n)\,a(n) = 0 \tag{4}$$

for $n \geq 0$.

Every sequence (c_k) has an associated Legendre transform $L(n)$ defined by

$$L(n) = \sum_{k=0}^{n} c_k \binom{n}{k}\binom{n+k}{k}.$$

For $r \in \mathbb{Z}$, $r \geq 2$ numerical evidence indicates that each of the sequences $a_{r,r}(n)$ defined as in (3) is the Legendre transform of an integer sequence $\left(c_k^{(r)}\right)$. Schmidt [5] and Strehl [7] proved independently that

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3,$$

that is, $c_k^{(2)} = \sum_{j=0}^{k} \binom{k}{j}^3$.

The next theorem, proved later, shows that this is the only case of this form.

THEOREM 2. *Let $r, s \geq 1$ be integers. There exists an integer $l \geq 1$ such that the sequence $a_{r,s}(n)$ (defined as in* (3)*) for $n \geq 0$ is the Legendre transform of the integer sequence*

$$c_k = \sum_{j=0}^{k} \binom{k}{j}^l,$$

if and only if $s = 2$, $r = 2$ and $l = 3$.

Before the theorems are proved the congruence properties of $a(n)$, defined as in (2) are determined.

LEMMA 1. *For any prime $p > t$, $r_0, r_1 \in \mathbb{N}$, and $r_2, \dots, r_t$ nonnegative integers, the following hold*:

(i) $a(p-1) \equiv 1 \pmod{p}$;
(ii) $a(jp) \equiv a(j) \pmod{p}$;
(iii) $a(p+1) \equiv a(1)^2 \pmod{p}$;
(iv) $a(2p-1) \equiv a(1) \pmod{p}$.

PROOF. Firstly case (i) is considered. If $1 \leq i \leq p-1$, then $p \mid (p-1+i)!$ but $p \nmid i!(p-1)!$ implying that

$$p \mid \binom{p-1+i}{i}.$$

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Using this we have that

$$a(p-1)=\sum_{k=0}^{p-1}\prod_{i=0}^{t}\binom{p-1+ik}{k}^{r_i}\equiv\prod_{i=0}^{t}\binom{p-1}{0}^{r_i}\equiv 1 \pmod p.$$

To prove (ii), write

$$\begin{aligned} a(jp) &= \sum_{k=0}^{jp}\prod_{i=0}^{t}\binom{jp+ik}{k}^{r_i} \\ &= \sum_{k=0}^{j}\prod_{i=0}^{t}\binom{jp+ikp}{kp}^{r_i}+\sum_{\substack{0\le k<j\\1\le l<p}}\prod_{i=0}^{t}\binom{jp+i(kp+l)}{kp+l}^{r_i}. \end{aligned} \tag{5}$$

For $0\le k<j$, $1\le l<p$ we have

$$p^{k+1}\mid (jp)(jp-1)\cdots(jp-(kp+l)+1)$$

but $p^{k+1}\nmid (kp+l)!$, so

$$\binom{jp}{kp+l}\equiv 0 \pmod p. \tag{6}$$

It is readily verified, using the fact that $\prod_{l=1}^{p-1} l\equiv -1 \pmod p$, that

$$\prod_{\substack{0\le m<k\\0<l<p}}\frac{jp-(mp+l)}{kp-(mp+l)}\equiv 1 \pmod p.$$

Hence, as

$$\binom{jp}{kp}=\prod_{0\le m<k}\frac{(j-m)p}{(k-m)p}\prod_{\substack{0\le m<k\\0<l<p}}\frac{jp-(mp+l)}{kp-(mp+l)}=\binom{j}{k}\prod_{\substack{0\le m<k\\0<l<p}}\frac{jp-(mp+l)}{kp-(mp+l)}$$

we have

$$\binom{jp}{kp}\equiv\binom{j}{k} \pmod p. \tag{7}$$

Using (5), (6) and (7) we obtain

$$a(jp)\equiv\sum_{k=0}^{j}\prod_{i=0}^{t}\binom{j+ik}{k}^{r_i}=a(j) \pmod p.$$

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To prove case (iii), the formulae

$$\binom{n+1}{j+1} = \binom{n}{j+1} + \binom{n}{j} \tag{8}$$

and

$$\binom{jp+l}{m} = \sum_{k=0}^{m} \binom{l}{k}\binom{jp}{m-k} \tag{9}$$

are used. The latter equation follows from the identity $(1+z)^{jp+l} = (1+z)^l(1+z)^{jp}$. By (6), (7), (8), (9) and the fact that $\binom{p}{j} \equiv 0 \pmod p$, for $1 \le j \le p-1$ it is readily verified that

$$\begin{aligned}
a(p+1) &= \sum_{k=0}^{p+1} \prod_{i=0}^{t} \binom{p+1+ik}{k}^{r_i} \\
&\equiv \sum_{k=0}^{p+1} \left[\binom{p}{k} + \binom{p}{k-1}\right]^{r_0} \prod_{i=1}^{t} \binom{p+1+ik}{k}^{r_i} \\
&\equiv \binom{p}{0}^{r_0} \prod_{i=1}^{t} \binom{p+1}{0}^{r_i} + \binom{p}{0}^{r_0} \prod_{i=1}^{t} \binom{p+1+i}{1}^{r_i} \\
&\quad + \binom{p}{p}^{r_0} \prod_{i=1}^{t} \binom{p+1+ip}{p}^{r_i} + \binom{p}{p}^{r_0} \prod_{i=1}^{t} \binom{p+1+i(p+1)}{p+1}^{r_i} \\
&\equiv 1 + 2^{r_1+1}3^{r_2}\cdots(1+l)^{r_t} + 2^{2r_1}3^{2r_2}\cdots(l+1)^{2r_t} \\
&= (1 + 2^{r_1}3^{r_2}\cdots(l+1)^{r_t})^2 = a(1)^2\,.
\end{aligned}$$

Finally, case (iv) is considered. For $1 \le k \le p-1$,

$$p \mid \binom{p+(p-1)+k}{k} \quad \text{and} \quad p \mid \binom{p+(p-1)+(p+k)}{p+k} \tag{10}$$

since $p^2 \mid [2p+(k-1)]!$ and $p^3 \mid [3p+(k-1)]!$ but $p^2 \nmid k!(2p-1)!$ and $p^3 \nmid (p+k)!(2p-1)!$.

By definition

$$a(2p-1) = \sum_{k=0}^{p+(p-1)} \prod_{i=0}^{t} \binom{p+(p-1)+ik}{k}^{r_i}$$

and by (10) the terms $k = 1, \ldots, p-1$ and $k = p+1, \ldots, 2p-1$ are congruent to zero, leaving the terms $k = 0$ and $k = p$. From (6), (7) and (9) it follows that

$$a(2p-1) \equiv 1 + \binom{1}{1}^{r_0}\binom{2}{1}^{r_1}\cdots\binom{l+1}{1}^{r_t} = a(1) \pmod p$$

proving the lemma. □

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PROOF (of Theorem 1). We prove the theorem using induction on m. Assume first that there is a recurrence relation with $m = 1$, then for any prime p

$$\begin{cases} (c_0 + c_1)\, a(2) + (d_0 + d_1)\, a(1) = 0 \\ (c_0 + c_1(p-1))\, a(p) + (d_0 + d_1(p-1))\, a(p-1) = 0 \\ (c_0 + c_1 p)\, a(p+1) + (d_0 + d_1 p)\, a(p) = 0 \\ (c_0 + c_1(2p-1))\, a(2p) + (d_0 + d_1(2p-1))\, a(2p-1) = 0. \end{cases}$$

Therefore, using Lemma 1, for any prime $p > t$ we have

$$\left.\begin{aligned} (c_0 + c_1)a(2) + (d_0 + d_1)a(1) &\equiv 0 \\ (c_0 - c_1)a(1) + (d_0 - d_1) &\equiv 0 \\ c_0 a(1)^2 + d_0 a(1) &\equiv 0 \\ (c_0 - c_1)a(2) + (d_0 - d_1)a(1) &\equiv 0 \end{aligned}\right\} \pmod{p}$$

It is readily verified that

$$a(2) \neq a(1)^2 \tag{11}$$

and with some manipulation it follows from (11) that $c_0 = c_1 = d_0 = d_1 = 0$, which proves the claim for $m = 1$.

Now suppose that the claim is true for $\deg(P_0) \leq m - 1$ and $\deg(P_1) \leq m - 1$ and assume that there exists a recurrence with $\deg(P_0) = m$ and $\deg(P_1) = m$. Therefore, (4) holds for all $n \geq 0$, and in particular, for $n = p - 1$ and $n = 2p - 1$, where $p > t$ is any prime. Then by Lemma 1

$$(c_0 - c_1 + \cdots + (-1)^m c_m)\, a(1) + d_0 - d_1 + \cdots + (-1)^m d_m \equiv 0 \pmod{p}$$

and

$$(c_0 - c_1 + \cdots + (-1)^m c_m)\, a(2) + (d_0 - d_1 + \cdots + (-1)^m d_m)\, a(1) \equiv 0 \pmod{p}.$$

Hence, as this holds for all $p > t$

$$(c_0 - c_1 + \cdots + (-1)^m c_m)\, a(1) + d_0 - d_1 + \cdots + (-1)^m d_m = 0$$

and

$$(c_0 - c_1 + \cdots + (-1)^m c_m)\, a(2) + (d_0 - d_1 + \cdots + (-1)^m d_m)\, a(1) = 0.$$

Using (11) it is not difficult to show that

$$c_0 - c_1 + \cdots + (-1)^m c_m = d_0 - d_1 + \cdots + (-1)^m d_m = 0,$$

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that is, -1 is a root of P_0 and P_1. Whence there exist integer polynomials $\tilde{P}_0$ and $\tilde{P}_1$ with degree $m-1$ such that

$$P_0 = (n+1)\tilde{P}_0(n) \quad \text{and} \quad P_1(n) = (n+1)\tilde{P}_1(n),$$

and

$$\tilde{P}_0(n)\, a(n+1) + \tilde{P}_1(n)\, a(n) = 0$$

for $n \geq 0$. By the induction hypothesis $\tilde{P}_0 = \tilde{P}_1 = 0$, which implies that $P_0 = P_1 = 0$ and completes the proof of the theorem. □

PROOF (of Theorem 2). Assume that the sequence

$$a_{r,s}(n) = \sum_{j=0}^{n} \binom{n}{j}^r \binom{n+j}{j}^s$$

is the Legendre transform of the integral sequence

$$c_j = \sum_{k=0}^{j} \binom{j}{k}^l.$$

Then

$$a_{r,s}(n) = \sum_{j=0}^{n} c_j \binom{n}{j}\binom{n+j}{j}.$$

Therefore, for any prime $p > 2$

$$a_{r,s}(p) = \sum_{j=0}^{p} c_j \binom{p}{j}\binom{p+j}{j}$$

and hence by (7) and since $p \mid \binom{p}{j}$ for $1 \leq j \leq p-1$,

$$a_{r,s}(p) \equiv 1 + 2^s \equiv c_0 + 2c_p \pmod{p}.$$

As

$$c_0 = 1 \quad \text{and} \quad c_p = \sum_{k=0}^{p} \binom{p}{k}^l \equiv 2 \pmod{p},$$

it follows that $1 + 2^s \equiv 1 + 4 \pmod{p}$ and therefore, as p is an arbitrary prime, that $1 + 2^s = 5$ implying that $s = 2$.

Since $c_0 = 1$, $c_1 = 2$, $c_2 = 2 + 2^l$, $s = 2$,

$$a_{r,s}(2) = \sum_{k=0}^{2} \binom{2}{k}^r \binom{2+k}{k}^s = 1 + 2^r 3^s + 6^s$$

and

$$a_{r,s}(2) = \sum_{k=0}^{2} c_k \binom{2}{k}\binom{2+k}{k} = c_0 + 6c_1 + 6c_2$$

we get $2^{l+1} = 4 + 3.2^r$ and it is easy to show that this equation has only one solution, namely $r = 2$ and $l = 3$ which completes the proof. □

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