# APPROXIMATING THE DISTRIBUTION OF A DYNAMIC RISK PORTFOLIO

By William S. Jewell

*Department of Industrial Engineering and Operations Research*
*University of California, Berkeley*

## Abstract

In a previous paper, Jewell and Sundt showed how to approximate a distribution of total losses from a large, fixed, heterogeneous portfolio, using a recursive algorithm developed by Panjer for the distribution of a random sum of random variables (a single casualty contract). This paper extends the approximation procedure to large, dynamic heterogeneous portfolios, in order to model either a portfolio of correlated casualty contracts, or a future portfolio, whose composition is not known with certainty.

## 0. Introduction

The problem of finding the distribution of $\tilde{y} = \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_2 + \cdots + \tilde{x}_N$, where the $(\tilde{x}_i)$ are a *fixed* and large set of independent, *nonidentically* distributed, integer-valued random variables was considered in Jewell and Sundt (1981) (hereinafter referred to as JS). Although, in theory, the discrete density of $\tilde{y}$ is just the $N$-fold convolution of the individual densities, this computation is very time-consuming, and various forms of approximation must be used; moreover, in many risk applications, the use of a normal approximation gives very bad results, even for large $N$, because of the skewness and long tails of the density. However, if the probability $p_i = \Pr\{\tilde{x}_i = 0\}$ is significant for most $i = 1, 2, \ldots, N$, it turns out that a very good approximation can be obtained using newly-developed procedures for the related problem of calculating the distribution of the sum of a *random number* of independent and *identically* distributed random variables.

In many risk applications, especially in insurance and investment management, there are an ever-changing number of risks of different types, and it is of interest to predict the distribution of a portfolio whose future composition is not known with certainty. This paper develops a general model for this situation, and shows how the approximation procedure described in JS can be extended.

## 1. THE DYNAMIC PORTFOLIO MODEL

Let $i = 1, 2, \ldots, N$ index a number of different risk classes (insurance policies or types of investment) in a given portfolio, and let $\tilde{n}_i \in [0, 1, \ldots]$ be the random number of independent risks of type $i$, giving a grand total number of risks in the portfolio

$$\tilde{n}_T = \sum_{i=1}^{N} \tilde{n}_i. \tag{1.1}$$



https://doi.org/10.1017/S0515036100004943 Published online by Cambridge University Press

Risks of type $i$ are *similar*, in the sense that, if $\tilde{x}_{ij}$ is the random monetary gamble from the $j$th risk of type $i$, then its discrete density, $f_i^0(x)$, is the same for all $j$, i.e.:

$$\Pr\{\tilde{x}_{ij}=x\}=f_i^0(x), \qquad (i=1,2,\ldots,N)(j=1,2,\ldots,n_i). \tag{1.2}$$

We shall only consider discrete gambles, with the common range of the $(\tilde{x}_{ij})$ as $[0,1,2,\ldots,R]$. As mentioned above, we assume for the moment that the $(\tilde{x}_{ij})$ are statistically independent of each other and the $(\tilde{n}_i)$, but we do *not* assume that the $(\tilde{n}_i)$ are independent. (But see Appendix A.)

The total monetary gamble for all risks of type $i$ is then the sum of a random number of random variables:

$$\tilde{x}_i=\begin{cases}0, & (\tilde{n}_i=0)\\ \tilde{x}_{il}+\tilde{x}_{i2}+\cdots+\tilde{x}_{i\tilde{n}_i}, & (\tilde{n}_i>0),\end{cases} \qquad (i=1,2,\ldots,N) \tag{1.3}$$

and the grand total monetary risk is then the fixed-term random sum:

$$\tilde{y}=\tilde{x}_1+\tilde{x}_2+\cdots+\tilde{x}_N. \tag{1.4}$$

Note that the $(\tilde{x}_i)$ are now *dependent* random variables, if the $(\tilde{n}_i)$ are.

If $g(y)$ and $\pi(n_1,n_2,\ldots,n_N)$ are the discrete densities of the total risk sum and the number of risks of each type, respectively, we have then the discrete density of $y$ as:

$$\begin{aligned}\Pr\{\tilde{y}=y\}=g(y)=\sum_{n_1}\sum_{n_2}\cdots\sum_{n_N}\pi(n_1,n_2,\ldots,n_N)[f_1^0(y)]^{n_1^*}\\ *[f_2^0(y)]^{n_2^*}*\cdots*[f_N^0(y)]^{n_N^*}, \qquad (y=0,1,2,\ldots)\end{aligned} \tag{1.5}$$

which, of course, is a lengthy and laborious computation. (In JS, the special case of $(\tilde{n}_i)$ deterministic was considered.)

## 2. INTERPRETATIONS

Before describing a method of approximating (1.5), we give some possible practical interpretations of the model.

In insurance applications, the simplest interpretation is that $i$ refers to different, distinguishable types of insurance policies in a given portfolio; for instance, similar policies in personal lines of insurance could refer to ordinary life insurance policies with the same face values issued to persons of the same age. For the current year, we know exactly the number of policies of type $i$ and hence, following JS, can find an approximation to the current $g(y)$. However, an approximation to (1.5) would be necessary to predict total portfolio risk for *next* year, after some policies are withdrawn, some policies have paid out benefits, or new policies have been added, and still others have shifted type. By specifying the stochastic law governing this "drop-add" mechanism, we can get $\pi(n_1,n_2,\ldots,n_N)$ for next year. Possible reasons for leaving correlation between the $(\tilde{n}_i)$ are that we may have a precise idea of how new sales are distributed

https://doi.org/10.1017/S0515036100004943 Published online by Cambridge University Press

among the different types of policies, but may be uncertain about the total new business; or, the new business total may be accurately estimated but the distribution may be uncertain; or, there may be an uncertain number of policies which are shifting type (as in aging of life insurance insured); etc.

A second insurance interpretation is the so-called casualty claim model, in which multiple claims may occur on a policy during a given exposure year. Here $i$ indexes each of a fixed number of policies, $f_i^0(x)$ is the individual claim ("severity") density, $\tilde{n}_i$ is the random number of claims ("frequency"), and $\tilde{x}_i$ is now the total monetary claim on the single policy $i$. Of course, if the $(\tilde{n}_i)$ were independent, then this application could be handled by making $\sum \pi(n_i)[f_i^0(y)]^{n_i^*}$ the basic density used in the procedure described in JS; but this would require prior calculation of this compound law (see also Sections 8 and 9, below). Moreover, external factors, such as weather and economics, often affect the number of claims of all types of contracts in a given portfolio in the same way, thus introducing correlation and the need for a more general model.

In most insurance portfolios, a great deal of effort is used to assure that the $(\tilde{x}_{ij})$ are statistically independent of each other. However, there remains always the possibility that risks of the same type $i$ are influenced by the same exogenous factors. In Appendix A, we consider the case when risks of the same type are *exchangeable random variables*, which leads to a weak form of dependence on the $(\tilde{x}_{ij})$.

In investment portfolios, it is unusual to have independent risks of the same type, i.e., requiring the same investment level, and having the same outcome distribution; instead, we usually have a different amount of money invested in different risks. If we let $\tilde{n}_i$ be the level of investment in type $i$ and $\tilde{x}_i$ the net return from this investment, then (1.3) holds only if the $(\tilde{x}_{ij})$ are perfectly correlated, or what is the same, if (1.3) is replaced by $\tilde{x}_i = \tilde{n}_i \tilde{x}_{il}$. Another limitation on investment modelling is that it is usually possible to have negative net returns, which is discussed in Section 10. It should be remembered also that our approximation is usually successful only if the problem is modelled so that the probability of zero net return is substantial; i.e., all "sure thing" return has been eliminated.

Technological risk applications are based upon the compound law interpretation; for instance, in reliability engineering, $\tilde{n}_i$ may refer to the random number of mechanical, electrical, or thermal shocks of type $i$ which affect a given piece of equipment; in fire damage analysis, $\tilde{n}_i$ is the random number of fires of a given type (size, type of dwelling or land classification) which occur; and so forth. In technology applications, the primary modelling challenge is to express damage in appropriate, additive units for situations where there is no accepted monetary surrogate for the risk.

## 3. NOTATION AND MOMENTS

The success of the approximation procedure to be described depends upon the assumption that most of the total risks, $(\tilde{x}_i)$, have a high probability of being zero; this can occur either because $f_i^0(0)$ is large, or because $\tilde{n}_i$ is often zero. We now change to a traditional notation (see JS) which emphasizes the distribution of risk when it is positive. Let

$$\text{(3.1)} \quad \Pr\{\tilde{x}_{ij}=0\}=f_i^0(0)=p_i=1-q_i, \qquad (i=1,2,\ldots,N)(j=1,2,\ldots,\tilde{n}_i)$$

$$\text{(3.2)} \qquad f_i(x)=\Pr\{\tilde{x}_{ij}=x|\tilde{x}_{ij}>0\}=f_i^0(x)/q_i, \qquad (x=1,2,\ldots,R)$$

and define the first two moments of non-zero risk as:

$$\text{(3.3)} \qquad m_i=\mathscr{E}\{\tilde{x}_{ij}|\tilde{x}_{ij}>0\}=\mathscr{E}\{\tilde{x}_{ij}\}/q_i,$$

$$\text{(3.4)} \qquad v_i=\mathscr{V}\{\tilde{x}_{ij}|\tilde{x}_{ij}>0\}=[\mathscr{V}\{\tilde{x}_{ij}\}/q_i]-p_i(m_i)^2.$$

From the joint counting density, we get the marginal densities:

$$\text{(3.5)} \qquad \pi_i(n)=\Pr\{\tilde{n}_i=n\}, \qquad (i=1,2,\ldots,N)(j=1,2,\ldots,\tilde{n}_i)$$

and the first two moments:

$$\text{(3.6)} \qquad \lambda_i=\mathscr{E}\{\tilde{n}_i\},$$

$$\text{(3.7)} \qquad \gamma_{ik}=\mathscr{C}\{\tilde{n}_i;\tilde{n}_k\}, \qquad (i,k=1,2,\ldots,N).$$

The approximation itself is based upon moment-matching with the first two moments of the exact density (1.5), which we now find in a straightforward manner. First, from (1.3) and the assumptions:

$$\mathscr{E}\{\tilde{x}_i|n_i\}=n_i\mathscr{E}\{\tilde{x}_{ij}\},$$

$$\mathscr{C}\{\tilde{x}_i;\tilde{x}_k|n_i,n_k\}=\begin{cases}n_i\mathscr{V}\{\tilde{x}_{ij}\}, & (i=k)\\ 0, & (i\neq k)\end{cases}$$

so that, unconditioning, we have:

$$\text{(3.8)} \qquad \mathscr{E}\{\tilde{x}_i\}=\mathscr{E}\{\tilde{n}_i\}\mathscr{E}\{\tilde{x}_{ij}\},$$

$$\text{(3.9)} \qquad \mathscr{C}\{\tilde{x}_i;\tilde{x}_k\}=\begin{cases}\mathscr{E}\{\tilde{n}_i\}\mathscr{V}\{\tilde{x}_{ij}\}+\mathscr{V}\{\tilde{n}_i\}[\mathscr{E}\{\tilde{x}_{ij}\}]^2, & (i=k)\\ 0+\mathscr{C}\{\tilde{n}_i;\tilde{n}_k\}\mathscr{E}\{\tilde{x}_{ij}\}\mathscr{E}\{\tilde{x}_{kl}\}, & (i\neq k).\end{cases}$$

Then, using (1.4) and notation defined above, we find the first two moments of total portfolio risk as:

$$\text{(3.10)} \qquad \mathscr{E}\{\tilde{y}\}=\sum_{i=1}^{N}\lambda_i q_i m_i,$$

and

$$\text{(3.11)} \qquad \mathscr{V}\{\tilde{y}\}=\sum_{i=1}^{N}\lambda_i q_i(v_i+p_i m_i^2)+\sum_{i=1}^{N}\sum_{k=1}^{N}\gamma_{ik}q_i q_k m_i m_k.$$

The $(q_i)$, $(m_i)$, and $(v_i)$ are presumed known from past portfolio statistics on each type $i$, and the $(\lambda_i)$ and $(\gamma_{ik})$ are gotten from modelling assumptions regarding the future composition of the portfolio; so, we shall assume that these moments are *given parameters*.

https://doi.org/10.1017/S0515036100004943 Published online by Cambridge University Press

Of course, if the $(\tilde{n}_i)$ are statistically independent, the last term become $\sum \gamma_{ii} q_i^2 m_i^2$. In the static portfolio model in JS, the composition was fixed, with all $\tilde{n}_i = 1$; an equivalent, but slightly generalized model can be gotten from the above, with $\tilde{n}_i = n_i = \lambda_i$, and all $\gamma_{ik} = 0$.

## 4. THE APPROXIMATING RISK COLLECTIVE MODEL

In the approximation, we replace the original portfolio by a *homogeneous* "risk collective", that is, we assume that $\tilde{y}$ is approximately:

$$\tilde{y} = \begin{cases} 0, & (\tilde{n}_e = 0) \\ \tilde{w}_1 + \tilde{w}_2 + \cdots + \tilde{w}_{\tilde{n}_e}, & (\tilde{n}_e > 0) \end{cases} \tag{4.1}$$

where $\tilde{n}_e$ is the random number of *equivalent positive claims* $(\tilde{w}_i)$, assumed to be independent of each other and $\tilde{n}_e$, and *identically distributed*, according to *prototypical counting* and *individual risk* densities:

$$\begin{aligned} \pi(n) &= \Pr\{\tilde{n}_e = n\}, \qquad (n = 0, 1, 2, \ldots); \\ f(w) &= \Pr\{\tilde{w} = w\}, \qquad (w = 1, 2, \ldots), \end{aligned} \tag{4.2}$$

leading to the usual *compound law* of risk theory for the density of $\tilde{y}$:

$$g(y) = \sum_{n=0}^{\infty} \pi(n)[f(y)]^{n^*}. \tag{4.3}$$

As mentioned earlier, the rationale behind this approximation is that, in many applications, the $(\tilde{x}_{ij})$ are zero with high probability; the $(\tilde{w}_i)$ then represent just the positive $(\tilde{x}_{ij})$. (See also JS and GERBER (1979).)

If the prototypical moments are:

$$\lambda = \mathscr{E}\{\tilde{n}_e\}; \qquad \gamma = \mathscr{V}\{\tilde{n}_e\}, \tag{4.4}$$

$$m = \mathscr{E}\{\tilde{w}\}; \qquad v = \mathscr{V}\{\tilde{w}\}, \tag{4.5}$$

then the moments of the random sum in the approximating model will be:

$$\mathscr{E}\{y\} = \lambda m, \tag{4.6}$$

$$\mathscr{V}\{\tilde{y}\} = \lambda v + \gamma m^2. \tag{4.7}$$

For a good approximation, the moments (4.6), (4.7) must be matched as closely as possible with the true values (3.10), (3.11). In addition, the forms of the $\pi(n)$ and $f(w)$ chosen may also be varied.

## 5. THE ADELSON–PANJER RECURSIVE ALGORITHMS

At this point, we should stop and consider whether the computation of the compound law (4.3) can be effected in any efficient manner; otherwise, it is not much improvement over (1.5). A traditional approximation (for the static portfolio problem) used in actuarial circles was to make $\pi(n)$ a Poisson law; this was because (further) approximations to the compound law had been developed in the early risk theory literature (see, e.g., GERBER (1979)).

However, the recent extension by PANJER (1981) of a recursive scheme of ADELSON (1966) now provides an efficient and direct way to compute (4.3). Essentially, if $f(w)$ is discrete over $[1, 2, \ldots]$ and the counting distribution is chosen from a certain $(a, b)$-*family* for which:

$$\pi(n) = \left(a + \frac{b}{n}\right)\pi(n-1), \qquad (n = 1, 2, \ldots) \tag{5.1}$$

then $g(y)$ can be calculated recursively via:

$$\begin{aligned} g(0) &= \pi(0) = \begin{cases} (1-a)^{(a+b)/a}, & (a \neq 0) \\ e^{-b}, & (a = 0) \end{cases} \\ g(y) &= \sum_{x=1}^{\min(y,R)} \left(a + b\frac{x}{y}\right) f(x) g(y-x), \qquad (y = 1, 2, \ldots). \end{aligned} \tag{5.2}$$

This is clearly an efficient computational procedure, provided the $(a, b)$-family is a useful one. As elaborated upon in SUNDT and JEWELL (1981), the only members of this family, apart from the degenerate density, are:

(5.3a) (Poisson) $$\pi(n) = \frac{\lambda^n e^{-\lambda}}{n!}, \qquad (a = 0; b = \lambda);$$

(5.3b) (Binomial) $$\pi(n) = \binom{M}{n} p^n (1-p)^{M-n}, \qquad (a = -p/(1-p); \; b = -a(M+1));$$

(5.3c) (Negative Binomial) $$\pi(n) = \binom{\alpha + n - 1}{n} p^n (1-p)^{\alpha}, \qquad (a = p; b = p(\alpha - 1)).$$

These counting distributions are useful, since they are often used in modelling compound risk laws. Furthermore, since:

$$\lambda = \mathscr{E}\{\tilde{n}_e\} = \frac{a+b}{1-a}; \qquad \gamma = \mathscr{V}\{\tilde{n}_e\} = \frac{a+b}{(1-a)^2}; \tag{5.4}$$

we get:

$$a = 1 - \frac{\lambda}{\gamma}; \qquad b = \frac{\lambda(\lambda+1)}{\gamma} - 1; \tag{5.5}$$

and:

$$\frac{\mathscr{V}\{\tilde{n}_e\}}{\mathscr{E}\{\tilde{n}_e\}} = \frac{\gamma}{\lambda} = \frac{1}{1-a}. \tag{5.6}$$

The importance of the ratio (5.6) in modelling empirical counting processes is well known. From (5.3), we see that this family covers a wide range of such ratios, with the Binomial giving $(\gamma/\lambda) < 1$ and the Negative Binomial (Pascal) giving $(\gamma/\lambda) > 1$; the Poisson $(\gamma = \lambda)$ distribution is the dividing line.

Therefore, for computational simplicity, we propose to use the $(a, b)$-family to model the counting distribution $\pi(n)$ and the recursive procedure (5.2) to compute the approximate density (4.3). Note that, if $a < 0$ in (5.5), we are not completely free in our choice of $b$, since $M$ must be an *integer* in the Binomial law (5.3b); however, this is not usually a serious limitation (see JS).

## 6. THE FINAL APPROXIMATION

Having selected $\pi(n)$ on the basis of computational convenience, we must now choose the prototypical density, $f(w)$. The form which will give the best approximation in all cases is not known. However, a natural way, consistent with the interpretation given in Section 4, is to weight the individual densities (3.2) with weights proportional to the expected number of risks with positive outcome in the corresponding class, i.e., to fix:

$$f(w) = \frac{\sum \lambda_i q_i f_i(w)}{\sum \lambda_j q_j} = \frac{\sum \lambda_i f_i^0(w)}{\sum \lambda_j q_j}, \qquad (w = 1, 2, \ldots, R). \tag{6.1}$$

This choice is consistent with JS for the static risk portfolio model, and also provides the greatest simplification to the formulae below. Using (3.3), (3.4) in (6.1), we find first $m$ and $v$ in (4.5), then substitute into (4.6), (4.7) to find the first two moments of the approximating model; these moments are then equated with the exact results (3.10), (3.11), obtaining finally the *first two moments of the prototypical counting density in terms of the original parameters*:

$$\lambda = \mathscr{E}\{\tilde{n}_e\} = \sum_{i=1}^{N} \lambda_i q_i; \tag{6.2}$$

$$\gamma = \mathscr{V}\{\tilde{n}_e\} = \sum_{i=1}^{N} \lambda_i q_i \left[1 - q_i \left(\frac{m_i}{m}\right)^2\right] + \sum_{i=1}^{N} \sum_{k=1}^{N} \gamma_{ik} q_i q_k \left(\frac{m_i m_k}{m}\right)^2; \tag{6.3}$$

where the mean prototypical severity is:

$$m = \mathscr{E}\{\tilde{w}\} = \frac{\sum \lambda_i q_i m_i}{\sum \lambda_j q_j} \tag{6.4}$$

and the severity variance is:

$$v = \frac{\sum \lambda_i q_i (v_i + m_i^2)}{\sum \lambda_j q_j} - m^2. \tag{6.5}$$

To summarize: In the final approximation, we would first calculate the $f_i(x)$ and the moments of Section 3 using the data, then compute $f(w)$ from (6.1) and use it in the approximating model (4.3), together with one of the $\pi(n)$ of Section 5, with $(a, b)$ selected using (5.5); the approximate density is computed recursively via (5.2).

In the static portfolio case considered in JS, all $\gamma_{ik}$ are identically zero, so that $\gamma < \lambda$, and a Binomial counting law results. This raises the integrality problem for $M$ previously mentioned, and means that the resulting values of $(a, b)$ do not exactly match $\mathscr{V}\{\tilde{y}\}$ in the original and approximating models; however, the resulting error is not serious in the example analyzed in that paper.

In contrast, the dynamic portfolio model of this study can give $\gamma/\lambda > 1$, and hence Negative Binomial $\pi(n)$, if the $(\gamma_{ik})$ are large enough. To see this, consider the case of *independent*, but still random, $(\tilde{n}_i)$. (6.3) then becomes:

$$\gamma = \mathscr{V}\{\tilde{n}_e\} = \sum_{i=1}^{N} \lambda_i q_i + \sum_{i=1}^{N} \left(\frac{q_i m_i}{m}\right)^2 (\gamma_{ii} - \lambda_i). \tag{6.6}$$

Thus, we see that, if a sufficient number of (marginal) counting densities (3.5) have $\gamma_{ii}/\lambda_i > 1$, then also $\gamma/\lambda > 1$, a most reasonable result.

## 7. THE COMPOUND MULTINOMIAL COUNTING DISTRIBUTION

One natural way in which the number of risks in the different classes, $\tilde{\mathbf{n}} = (\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_N)$, might be generated in a predictive, dynamic model is from a Multinomial law, with given total number of risks, $n_T$, and a set of *selection probabilities*, $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_N)$, viz:

$$\Pr\{\tilde{\mathbf{n}} = \mathbf{n} \mid n_T; \boldsymbol{\pi}\} = \binom{n_T}{n_1, n_2, \ldots, n_N} \prod_{i=1}^{N} \pi_i^{n_i}, \qquad (\textstyle\sum n_i = n_T)(\sum \pi_i = 1). \tag{7.1}$$

With fixed $n_T$ and $\boldsymbol{\pi}$, there are already correlations between the counts in different classes, as:

$$\mathscr{E}\{\tilde{n}_i \mid n_T; \boldsymbol{\pi}\} = \pi_i n_T; \qquad (i = 1, 2, \ldots, N) \tag{7.2}$$

$$\mathscr{C}\{\tilde{n}_i; \tilde{n}_k \mid n_T; \boldsymbol{\pi}\} = \begin{cases} \pi_i n_T - \pi_i^2 n_T, & (i = k) \\ -\pi_i \pi_k n_T, & (i \neq k). \end{cases} \tag{7.3}$$

However, to give more modelling flexibility, we now permit both $n_T$ and $\boldsymbol{\pi}$ to be a random scalar and random vector, respectively, but require that they be *independent* of each other, for simplicity. This "collective" model dependency gives a more complex covariance structure.

Define:

$$\mathscr{E}\{\tilde{n}_T\} = \lambda_T = \sum_{i=1}^{N} \lambda_i; \qquad \mathscr{V}\{\tilde{n}_T\} = \gamma_T = \sum_{i=1}^{N} \sum_{k=1}^{N} \gamma_{ik}; \tag{7.4}$$

then, unconditioning (7.2), (7.3), we obtain the moments for use in (6.2), (6.3):

$$\mathscr{E}\{\tilde{n}_i\} = \lambda_i = \lambda_T \mathscr{E}\{\pi_i\}; \tag{7.5}$$

https://doi.org/10.1017/S0515036100004943 Published online by Cambridge University Press

$$
(7.6) \qquad \mathscr{C}\{\tilde{n}_i;\tilde{n}_k\} = \gamma_{ik} = \begin{cases} \lambda_T\mathscr{E}\{\tilde{\pi}_i\} + (\gamma_T - \lambda_T + \lambda_T^2)\mathscr{V}\{\tilde{\pi}_i\} & (i = k) \\ \qquad + (\gamma_T - \lambda_T)\mathscr{E}^2\{\tilde{\pi}_i\}, & \\ (\gamma_T - \lambda_T + \lambda_T^2)\mathscr{C}\{\tilde{\pi}_i;\tilde{\pi}_j\} & \\ \qquad + (\gamma_T - \lambda_T)\mathscr{E}\{\tilde{\pi}_i\}\mathscr{E}\{\tilde{\pi}_j\}, & (i \neq k). \end{cases}
$$

It is easy to show that these satisfy (7.4), by using $\sum \tilde{\pi}_i = 1$.

It seems to the author that practical modelling variations might fall into one of two extremes: either (1) the $(\pi_i)$ might be known rather precisely, and forecasting uncertainty might be associated with the total number of risks or, (2) there would be a relatively stable number of risks, but prediction uncertainty would remain about their distribution over the different risk classification types. (For the casualty claim model, only the first variation would probably be relevant.)

An interesting special case of the compound Multinomial coccurs when the $(\pi_i)$ are fixed, and $\gamma_T = \lambda_T$. It then follows from (7.5), (7.6) that $\lambda_i = \gamma_{ii}$ $(i = 1, 2, \ldots, N)$ and $(\tilde{n}_i, \tilde{n}_k; i \neq k)$ are *uncorrelated*. This then simplifies (6.2), (6.3), (6.6) to $\lambda = \gamma$, that is, $a = 0$, $b = \lambda$, and a Poisson counting distribution would be used in the approximation of Section 5! One obvious way in which this could happen is if $\tilde{n}_T$ were Poisson with parameter, say $\mu$; it is then well known that the $(\tilde{n}_i)$ must be statistically mutually independent, with marginal densities that are Poisson with parameters $(\pi_i\mu)$.

## 8. AN EXACT RESULT

There is one case in which the proposed procedure gives an *exact* result. Consider a risk portfolio of fixed size $N$, with each contract $i = 1, 2, \ldots, N$ having an individual claim density $f_i^0(x)$, with parameters $q_i$, $m_i$, $v_i$, and an *independent* claim number density that is Poisson, with parameter $\mu_i$. This is the basic model used in casualty insurance.

Following the procedure in Sections 5 and 6, we get the same special results described in the previous section, namely, $\gamma_{ii} = \lambda_i$, $\gamma_{ik} = 0$, $(i \neq k)$ and $\lambda = \gamma = \sum \mu_i q_i$. In other words, once $f(w)$ is determined from (6.1), the recursive algorithm (5.2) is used with the Poisson density (5.3a) to find the approximate $g(y)$.

However, it is easy to show, using generating functions, that the exact form (1.5) reduces to a compound Poisson law with parameter $\lambda$, and a severity density $f(w)$. Thus, the dynamic portfolio approximation is, in fact, *exact* for independent Poisson claims. This is true even if $p_i = 0$ for all $i$!!

Unfortunately, the same line of proof shows also that independent Binomial or Negative Binomial claim densities (with different parameters for each $i$) can only lead to an *approximation* of the true $g(y)$. However, it follows from Section 6 that the approximating law for $\tilde{n}_e$ would be Binomial or Negative Binomial, respectively.

## 9. MODELLING WITH FIXED AND RANDOM NUMBER OF COUNTS

To highlight the differences between the model and procedure of this paper and the static portfolio model in JS, it is instructive to re-examine how the independent

Poisson casualty claim model of Section 8 would be handled according to the JS procedure. We use *primes to* designate the equivalent parameters of this paper, in terms of the given model parameters $\mu_i$, $q_i$, $m_i$, $v_i$.

First of all, since all $\tilde{n}_i \equiv 1$ in the JS model, we would have to estimate or calculate separately the $N$ individual *total severity* densities for each contract risk, $\tilde{x}_i$:

(9.1) $$f_i^{0\prime}(x) = g_i(x) = \sum_{n=0}^{\infty} \frac{(\mu_i)^{n-\mu_i}}{n!} [f_i^0(x)]^{n*}.$$

(This could be done by $N$ applications of the Adelson algorithm, or might be approximated from real total severity data.)

Then, in terms of the parameters of this paper, we would get:

(9.2) $$\lambda_i' = 1; \qquad q_i' = 1 - e^{-\mu_i q_i}; \qquad m_i' = \left(\frac{q_i}{q_i'}\right)\mu_i m_i$$
$$v_i' = \left(\frac{q_i}{q_i'}\right)\mu_i v_i + (m_i - p_i' m_i') m_i'.$$

Thus, the static portfolio approach of JS would use the Panjer recursive algorithm with:

(9.3) $$f'(w) = \left(\sum f_i^{0\prime}(w)\right) / \left(\sum q_j'\right), \qquad (w = 1, 2, \ldots)$$

and a *Binomial* counting density with moments:

(9.4) $$\lambda' = \sum q_i' < \lambda;$$
$$\gamma' = \sum q_i'\left[1 - q_i'\left(\frac{m_i'}{m'}\right)^2\right].$$

The resulting $g(y)$ would then only approximate the true density, which could be obtained exactly in this case. Thus, one might be tempted to dismiss the JS procedure in compound claims applications. However, we can imagine situations in practice where the actuary has used empirical data to estimate the densities, $g_i(x)$ and $\pi(n_i)$. Then the question of the best approximation procedure is still open.

We remind the reader that, if the $(\tilde{n}_i)$ are, in fact, deterministic, then the procedures of the two papers are equivalent; conversely, if the $(\tilde{n}_i)$ are correlated, only the procedure described here applies.

## 10. OTHER VARIATIONS

In JS, an improved approximation for the example considered was obtained by modifying the $\pi(0)$ of the Binomial (5.3b) to enable an exact match of $\mathcal{V}\{\tilde{y}\}$, together with an integral value of $M$. This modification could be used with the model of this paper whenever $(\gamma/\lambda) < 1$, and requires only a trivial change in the recursive algorithm. But this refinement is not necessary in the other cases,

https://doi.org/10.1017/S0515036100004943 Published online by Cambridge University Press

as $\mathscr{V}\{\tilde{y}\}$ is matched exactly. Of course, one might try matching other moments or values of the exact distribution by modifying the initial values of the prototypical counting density (see the discussion in JS).

It would also be desirable, particularly in investment applications, to extend the range of permitted $(\tilde{x}_{ij})$ to negative values. The difficulty then is that the relationship (5.2) is no longer recursive, and must be solved by other means, such as iterative methods. This point is discussed in SUNDT and JEWELL (1981), where possible procedures for the Binomial and Poisson cases are suggested; exact recursion with negative values in the Negative Binomial case $(\gamma/\lambda)>1$ does not seem to be possible.

## 11. COMPUTATIONAL CONCLUSIONS AND ACKNOWLEDGEMENT

The limited computations carried out thus far indicate that the same general kinds of approximation error result as in JS; in other words, the underlying severity density should not be too "lumpy" if there are only a few risk types. Errors also seem higher in strongly correlated cases, as expected. A future paper will explore computational results in more detail.

The author would like to thank the referee who found several errors in the original formulae.

## APPENDIX A
## DEPENDENT RISKS

In Section 1, it was assumed that the individual risk severities $(\tilde{x}_{ij})$ were statistically independent of each other and of the counts $(\tilde{n}_i)$. In this appendix, we consider the modifications necessary if the risks are *exchangeable random variables* within each type $i$, but still independent of the counts. As is well known, this weak dependency is equivalent to assuming that, for each type $i=1,2,\ldots,N$, there exists a random *parameter*, $\tilde{\theta}_i$, such that the individual risks are independent if $\tilde{\theta}_i=\theta_i$ is known, and depend in the same way upon $\theta_i$. Thus, the basic density (1.1) is replaced by:

$$\text{(A.1)} \qquad \Pr\{\tilde{x}_{ij}=x\mid\theta_i\}=f_i^0(x\mid\theta_i), \qquad (i=1,2,\ldots,N),(j=1,2,\ldots,n_i)$$

giving a joint density within type $i$, given $\tilde{n}_i=n_i$ similar risks, of:

$$\text{(A.2)} \qquad \Pr\left\{\bigcap_{j=1}^{n_i}\tilde{x}_{ij}=x_{ij}\mid n_i\right\}=\mathscr{E}\prod_{j=1}^{n_i}f_i^0(x_{ij}\mid\tilde{\theta}_i),$$

and a common marginal density for any risk of type $i$:

$$\text{(A.3)} \qquad \Pr\{\tilde{x}_{ij}=x\}=\mathscr{E}f_i^0(x\mid\tilde{\theta}_i)=f_i^0(x).$$

(Expectations in the above are over the random values of $\tilde{\theta}_i$.) Exchangeable random variables thus have the property that they have the same marginal density (and self moments), their arguments may be permuted in any fashion in their joint density (A.2), and they have common cross moments.

In addition to the dependency between different types introduced by the correlation between different counts, we will also permit the different parameters in $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_N)$ to be statistically dependent, with arbitrary joint d.f. $U(\boldsymbol{\theta})$. In short, our new model substitutes for (1.5) the general form:

$$\text{(A.4)} \qquad \begin{aligned} g(y) = \int_{\theta_1} \int_{\theta_2} \cdots \int_{\theta_N} dU(\boldsymbol{\theta}) \sum_{n_1} \sum_{n_2} \cdots \sum_{n_N} \pi(\boldsymbol{n}) [f_i^0(y|\theta_1)]^{n_1^*} \\ *[f_2^0(y|\theta_2)]^{n_2^*} * \cdots * [f_N^0(y|\theta_N]^{n_N^*}. \end{aligned}$$

Intuitively, we can think of $\theta_i$ as representing *exogenous factors*, such as the economy, weather, political factors, etc. that influence the random outcome of all risks of type $i$ jointly. This type of "collective behaviour" model is often used in casualty insurance, where it is recognized that all risk classification schemes are imperfect, and that residual correlations still exists among risks of a given type due to the unexplained inhomogeneity still present within the class $i$. Further, there might be common factors between the different classes, which would account for the dependency between $\tilde{\theta}_i$ and $\tilde{\theta}_k$ $(i \neq k)$.

Proceeding in a manner similar to Section 3, we define the positive risk densities $f_i(x|\theta_i)$, the probabilities $p_i(\theta_i)$ and $q_i(\theta_i)$, and the first two moments, $m_i(\theta_i)$ and $v_i(\theta_i)$, all dependent upon the risk parameter. (3.8), (3.9) still have the same form, except that they express only the conditional mean total risk, $\mathscr{E}\{\tilde{x}_i|\theta_i\}$, and conditional covariance of total risks between different classes, $\mathscr{C}\{\tilde{x}_i; \tilde{x}_k|\theta_i; \theta_k\}$ in terms of the conditional moments of individual risk, and the (non-$\tilde{\boldsymbol{\theta}}$-dependent) moments (3.6), (3.7) of the counts.

Now all that remains is to uncondition these moments, using the relationships:

$$\text{(A.5)} \qquad \mathscr{E}\{\tilde{y}\} = \sum_{i=1}^{N} \mathscr{E}\mathscr{E}\{\tilde{x}_i|\tilde{\theta}_i\},$$

$$\text{(A.6)} \qquad \mathscr{V}\{\tilde{y}\} = \sum_{i=1}^{N} \sum_{k=1}^{N} [\mathscr{E}\mathscr{C}\{\tilde{x}_i; \tilde{x}_k|\tilde{\theta}_i; \tilde{\theta}_k\} + \mathscr{C}\{\mathscr{E}\{\tilde{x}_i|\tilde{\theta}_i\}; \mathscr{E}\{\tilde{x}_k|\tilde{\theta}_k\}\}].$$

(Innermost operators are over the total risks $(\tilde{x}_i)$; outermost operators are over the risk parameters $(\tilde{\theta}_i)$.)

We define the unconditional versions of $q_i(\theta)$, $m_i(\theta)$, $v_i(\theta)$ as:

$$\text{(A.7)} \qquad q_i = \mathscr{E}\{q_i(\tilde{\theta})\}; \qquad \bar{m}_i = \mathscr{E}\{m_i(\tilde{\theta}_i)\}; \qquad \bar{v}_i = \mathscr{E}\{v_i(\tilde{\theta}_i)\}.$$

By the theorem of conditional expectation, $q_i = \Pr\{\tilde{x}_{ij} > 0\}$ is the same as in (3.1). However, as the referee reminds us, $\bar{m}_i$ and $\bar{v}_i$ are *not* the same as $m_i$ and $v_i$ in (3.3), (3.4) unless the variation due to $\tilde{\theta}_i$ vanishes; hence, the different notation. In fact, in the current notation, we see that:

$$\text{(A.8)} \qquad m_i = \mathscr{E}\{\tilde{x}_{ij}|\tilde{x}_{ij} > 0\} = \bar{m}_i + \mathscr{C}\{q_i(\tilde{\theta}_i); m_i(\tilde{\theta}_i)/q_i\}.$$

In addition to correlations, we shall also need higher-order cross-moments, so we define:

$$\text{(A.9)} \quad Q_i(\theta_i) = q_i(\theta_i) - q_i; \qquad M_i(\theta_i) = m_i(\theta_i) - \bar{m}_i; \qquad V_i(\theta_1) = v_i(\theta_i) - \bar{v}_i;$$

and use notation like:

$$\text{(A.10)}\qquad \begin{aligned} \overline{Q_iQ_k} &= \mathscr{E}\{Q_i(\tilde{\theta}_i)Q_k(\tilde{\theta}_k)\} = \mathscr{C}\{q_i(\tilde{\theta}_i);\, q_k(\tilde{\theta}_k)\};\\ \overline{Q_iM_i} &= \mathscr{E}\{Q_i(\tilde{\theta}_i)M_i(\tilde{\theta}_k)\};\\ \overline{Q_iM_iM_k} &= \mathscr{E}\{Q_i(\tilde{\theta}_i)M_i(\tilde{\theta}_i)M_k(\tilde{\theta}_k)\}; \end{aligned}$$

and so forth.

In place of (3.10), we have:

$$\text{(A.11)}\qquad \mathscr{E}\{\tilde{y}\} = \sum_i \lambda_i q_i \bar{m}_i + \left\{\sum_i \lambda_i \overline{(Q_iM_i)}\right\},$$

and, in place of (3.11), we obtain:

$$\text{(A.12)}\qquad \begin{aligned} \mathscr{V}\{\tilde{y}\} = {} & \sum_i \lambda_i q_i(\bar{v}_i + p_i(\bar{m}_i)^2) + \sum_i\sum_k \gamma_{ik} q_i q_k \bar{m}_i \bar{m}_k \\ & \times \sum_i \lambda_i (q_i p_i \overline{M_i^2} - (\bar{m}_i)^2 \overline{Q_i^2}) \\ & + \sum_i \lambda_i [\overline{Q_iV_i} + 2\bar{m}_i(p_i - q_i)\overline{Q_iM_i} + (p_i - q_i)\overline{Q_iM_i^2} - 2\bar{m}_i\overline{Q_i^2M_i} - \overline{Q_i^2M_i^2}] \\ & + \sum_i\sum_k 2\gamma_{ik} q_i \bar{m}_i \overline{Q_kM_k} \\ & + \sum_i\sum_k (\gamma_{ik} + \lambda_i\lambda_k)[q_iq_k\overline{M_iM_k} + \bar{m}_i\bar{m}_k\overline{Q_iQ_k} + 2q_i\bar{m}_k\overline{Q_iM_k} \\ & + 2q_i\overline{Q_kM_iM_k} + 2\bar{m}_i\overline{Q_iQ_kM_k} + \overline{Q_iQ_kM_iM_k}] \\ & - \left[\sum_i \lambda_i \overline{Q_iM_i}\right]^2. \end{aligned}$$

The term in braces in (A.11) gives a correction term to the calculation of $\lambda$ in (6.2) (with, of course, $m_i$ and $v_i$ replaced by $\bar{m}_i$ and $\bar{v}_i$); similarly, the terms in braces in (A.11) and (A.12) give two correction terms to the calculation of $\gamma$ in (6.3).

In many applications, these corrections simplify because either the probability of a claim *or* the moments are independent of $\tilde{\theta}_i$. For instance, in life insurance, $m_i = \bar{m}_i$ and $v_i = \bar{v}_i$ are the moments of the face value of policies of type *i*, which do not usually change with exogenous conditions, while the expiration probability, $q_i(\theta)$, would probably vary with external effects; this would eliminate *all* terms in (A.11), (A.12) with $M_i$, $M_k$, or $V_i$! Conversely, in casualty insurance, the probability of a claim, $q_i$, might be relatively fixed several years in a row, but the severity moments, $m_i(\theta_i)$ and $v_i(\theta_i)$, might be relatively uncertain in view of inflation, etc.; in this case, all terms in (A.11), (A.12) involving $Q_i$ and $Q_k$ can be eliminated!

A more complex model can also be developed by permitting the $(\tilde{n}_i)$ to depend upon $\tilde{\boldsymbol{\theta}}$; details are left to the reader.

## REFERENCES

ADELSON, R. M. (1966) Compound Poisson Distributions, *Operations Research Quarterly*, **17**, 73–75.

GERBER, H. (1979) *An Introduction to Mathematical Risk Theory*, Huebner Foundation Monograph, Irwin, R. D.: Homewood, Ill.

JEWELL, W. S. and SUNDT, B. (1981) Improved Approximations for the Distribution of a Heterogeneous Risk Portfolio, *Bulletin of the Association of Swiss Actuaries*, 221–240.

PANJER, H. H. (1981) Recursive Evaluation of a Family of Compound Distributions, *ASTIN Bulletin*, **12**, 22–26.

SUNDT, B. and JEWELL, W. S. (1981) Further Results on Recursive Evaluation of Compound Distributions, *ASTIN Bulletin*, **12**, 27–39.