BULL. AUSTRAL. MATH. SOC. 06C15
VOL. 34 (1986) 219-223

THE FREE ORTHOMODULAR WORD PROBLEM IS SOLVABLE

GUDRUN KALMBACH

It is shown that the free orthomodular word problem is solvable. Since the free orthomodular lattice L_o on countably many generators has, as a partial subalgebra, every finite partial orthomodular lattice P, which is contained in some orthomodular lattice as a partial subalgebra, it is sufficient to prove Evans embedding property for these P only. The construction of the finite orthomodular lattice containing P as a partial subalgebra has and can be done outside of L_o. It uses the coatom extension for ortholattices.

In order to prove that the free orthomodular word problem is solvable, it is sufficient, by [1], to prove that every finite "partial" orthomodular lattice S can be embedded in a finite orthomodular lattice.

We can assume that S is a finite suborthoposet of the free orthomodular lattice L_o on countably many generators and we shall prove that there exists a finite suborthoposet $M(S)$ of L_o which is generated by S and $M(S)$ is, with the induced structure, an orthomodular lattice (Theorem 4).

Received 12 November 1985. I thank M. Dichtl, who worked for three years on this problem, for numerous discussions. I am sorry that it was not he, but I, who found a solution.

Theorem 4 holds for all m-generated suborthoposets of L_o with $m \leq 2$ since the free orthomodular lattice on two generators is finite [2;3.9]. Our inductive hypothesis is:

(A) For every m-generated suborthoposet $T \subseteq L_o$ with $m < n$ there exists a finite orthomodular lattice $M(T)$ such that

(i) T is a generating suborthoposet of $M(T)$,

(ii) joins and meets in L_o of elements in T, which exist in T, are preserved in $M(T)$,

(iii) $M(T)$ can be embedded into L_o such that T is mapped identically onto itself and every chain $E \subseteq T$ generates the Boolean subalgebra $\Gamma E \subseteq L_o$ in $M(T)$.

Here ΓA is, for $A \subseteq L_o$, the subalgebra of L_o generated by A. For condition (iii) we observe that, in general, the condition on L_o implies that, for any finite orthomodular lattice N generated by T, there exists an isomorphic copy of N in L_o. We shall write $x^0 = x$ and $x^1 = x'$ for elements x of an ortholattice.

LEMMA 1. *Let L be a finite ortholattice and let $a \in L - \{0,1\}$ be such that:*

(i) $M = L - \{a, a^1\}$, with the induced structure, is an orthomodular lattice

(ii) there exist $b_0, b_1 \in L$ such that

$$[0, a^\varepsilon) = [0, b_\varepsilon] \subseteq L \quad \text{for } \varepsilon = 0,1 .$$

Then there exist a finite orthomodular lattice N containing M as a subalgebra and L as a generating suborthoposet.

Proof. By [2; p. 310] we construct for the quasi-ideal $D = [0, b_o \vee b_1] \underline{C} M$ a coatom-extension $N = M \cup (D \times \{c'\}) \cup (D' \times \{c\})$ where $(1, c)$ is a new coatom above the elements of D. If we identify $a^\varepsilon \in L$ with $(b_1^{1-\varepsilon}, c^\varepsilon)$ then M is a subalgebra of N and L is embedded as a generating suborthoposet in N.

LEMMA 2. *Let* S *be a finite* n*-generated suborthoposet of* L_o *with an element* $a \in S$ *such that* $T = S - \{a,a'\}$ *is* $(n-1)$*-generated. Then there exists a finite orthomodular lattice* $M(S)$ *such that (i)-(iii) of (A) hold for* $S \subseteq M(S)$.

Proof. By the inductive hypothesis there exists a suborthoposet $M(T)$ of L_o with the properties (i)-(iii) of (A) for $T \subseteq M(T)$. In T there exist finitely many elements $a_1,\ldots,a_r < a$ and $d_1,\ldots,d_t > a$. Define $b_o = a_1 \vee \ldots \vee a_r$ and $b_1' = d_1 \wedge \ldots \wedge dt$. We can assume $b_o, b_1 \in M(T)$ since $T \cup \{b_o, b_1, b_o', b_1'\}$ is $(n-1)$-generated. By 1. there exists a finite orthomodular lattice $N = M(S)$ containing the ortholattice $M(T) \cup \{a,a'\}$,-with $x \leq a \leq y$ for $x,y \in M(T)$ iff $x \leq b_o$ and $b_1' \leq y$,- as a generating suborthoposet and $M(T)$, as a subalgebra. The definition of N and the properties of $M(T)$, S and L_o imply that (i)-(iii) of (A) holds for $S \subseteq M(S)$.

It follows from Lemma 2 that we can make the new inductive assumption:

(B) For every n-generated suborthoposet $S \subseteq L_o$ which contains an $(n-1)$-generated suborthoposet T with $2 \leq |S - T| < 2r$ there exists a finite orthomodular lattice $M(S)$ such that (i)-(iii) of (A) holds for $S \subseteq M(S)$.

Let $S \subseteq L_o$ be an n-generated suborthoposet with an $(n-1)$-generated suborthoposet T such that $|S - T| = 2r$. Let $a \in S - T$ be such that S is generated by $T \cup \{a\}$. We measure the length of an element in S in terms of the generating set D and choose an element $b \in S - T$ of maximal length. Then $E = S - \{b,b'\}$ is a suborthoposet of L_o generated by D for which the inductive hypothesis (B) applies. Let L_1 be a finite orthomodular lattice which is embedded into L_o such that (i)-(iii) hold for $E \subseteq M(E) = L_1$. We can assume that $b \notin L_1$ in the following lemma since otherwise $M(E) = M(S)$ satisfies (i)-(iii) of (B) for $S \subseteq M(S)$.

LEMMA 3. *Assume that whenever* $z \in E$ *covers* b*, and* b *covers two elements* $x,y \in E$*, then* $u < b < v$ *for* $u,v \in E$ *implies* $z \leq v$ *and* $u \leq x$ *or* $u \leq y$. *Then there exists a finite orthomodular lattice*

$N = M(S)$, containing $L_1 \cup \{b,b'\}$ as a suborthoposet, such that (i)-(iii) of (A) holds for $S \subseteq M(S)$.

Proof. We assume $b \notin L_1$. In particular, by (iii), $x \not\leq y'$. In L_1 we consider the interval $[x \wedge y, z]_1$ and its extension to the orthoposet $N_1 = [x \wedge y, z]_1 \cup \{b,b'\}$ such that $[0,b) \cap N_1 = ([0,x] \cup [0,y]) \cap N_1$ is a quasi-ideal A in $N_1 \cap L_1$ and where $a < b < c$ in N_1 implies $c = z$ and $a \leqq x$ or $a \leqq y$. By using the coatom-extension for the new coatom b and the quasi-ideal A there exists a finite orthomodular lattice N_2 containing N_1 as a generating suborthoposet. We can assume that N_2 is embedded in L_0 and we replace in L_1 the interval $[0, z \wedge (x' \vee y')]_1$ by an isomorphic copy N_3 of N_2. In the product L_2 of $[0, z' \vee (x \wedge y)]_1 \subseteq L_1$ with N_3 we identify the element $(x \wedge y, b \wedge (x' \vee y'))^{\varepsilon}$ with b^{ε}. We paste the ortholattices L_2 and $L_3 = L_1 - ([0, z \wedge (x' \vee y')]_1 \cup [z' \vee (x \wedge y), 1]_1)$ to the ortholattice N along the common segment $[0, z' \vee (x \wedge y)]_1 \times \{0,1\}$. A new element in $N - L_1$ is only comparable with an element $x \in L_3$ if $x \in L_2$ holds. Therefore if, for $c,d \in L_3$, we have both $c < d'$ and $c' \wedge d' \notin L_3$, then $c' \wedge d' < z \wedge (x' \vee y')$ and we extend the partial order on N for these elements and their orthocomplements by $u \leq c', d'$ for all $u \leq c' \wedge d'$ in N. This way we obtain from N an orthomodular lattice $M(S)$ such that (i)-(iii) of (B) holds for $S \subseteq M(S)$.

Some remarks additional to Lemma 3 are: If there exists $A \subseteq S$ with $|A| \geq 2$ such that z covers b and b covers a for all $a \in A$ and $z \in S$, and if $u < b < v$ for $u,v \in E$ implies $z \leq v$ and $u \leqq a$ for some $a \in A$, then we can use the same arguments for the quasi-ideal $\cup \{[0,a] | a \in A\} \cap N_1$ replacing $([0,x] \cup [0,y]) \cap N_1$ in Lemma 3. We also observe that, for the case where two elements u,v (or more) cover b and b covers two elements x,y (or more) in S such that $a < b < c$ for $a,b \in S$ implies $a \leqq x$ or $a \leqq y$ and $u \leqq c$ or $v \leqq c$, there exists an element z in L_1 such that for the E-generated element b either $b < z \leq u,v$ or $x,y \leq z < b$ holds. In the first case we apply the construction of Lemma 3. to the interval $N_1 = [x \wedge y, z]_1 \subseteq L_1$. The other case is dual and for the case where more than two elements in S cover b or are covered by b we apply the procedure just described (several times if necessary). We conclude that the assertion of Lemma 3 holds without the additional assumptions on E.

THEOREM 4. *Let $S \subseteq L_0$ be a finite suborthoposet. Then there exists a finite suborthoposet $M(S) \subseteq L_0$ containing S as a generating set which, together with its induced structure, is an orthomodular lattice.*

References

[1] T. Evans, "Embeddability and the word problem," *J. London Math. Soc.* 28 (1953), 76-80.

[2] G. Kalmbach, *Orthomodular lattices.* (Academic Press, London, 1983).

Abt.Math.III, O.E.
University Ulm
D-7900 ULM
West Germany.