THE N-DIMENSIONAL DIOPHANTINE APPROXIMATION CONSTANTS

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Let C_n and C_n^* denote the n-dimensional (Diophantine approximation) constant and dual constant respectively. Davenport [5], in 1955 showed

$$(*) \qquad C_n^* = C_n \geqslant V_{n,s}/\Delta^{\frac{1}{2}}(F) ,$$

where

(i) $\Delta(F)$ is the (absolute) discriminant of any *real* number field F with $[F : Q] = n+1,s$, that is, of degree $n+1$ with s pairs of complex conjugates (of course $0 \leqslant 2s \leqslant n$); and

(ii) $2^n V_{n,s}$ is the supremum of volumes of n-dimensional 0-centred parallelotopes in the region

$$\left|\prod_{i=1}^{n-2s} x_i\right| \prod_{j=n-2s+1}^{n-s} \tfrac{1}{2}(x_j^2 + x_{s+j}^2) \leqslant 1.$$

$C_1 = 1/\sqrt{5}$ but no exact value of C_n for $n \geqslant 2$ is known.

This thesis consists of three strands.

(a) If $\xi = (\xi_1,\ldots,\xi_n) \in \mathrm{R}^n$, then define the approximation constant and dual approximation constant for ξ , $c(\xi)$ and $c^*(\xi)$, respectively, by

$$c(\xi) = \inf\{c > 0 : \max_{1\leqslant i\leqslant n} |x_0(\xi_i x_0 - x_i)^n| < c$$

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has infinitely many solutions in integers x_i , with $x_0 \neq 0\}$,
and

$$c^*(\xi) = \inf\{c > 0 : |x_0 + \xi_1 x_1 + \ldots + \xi_n x_n| \max_{1 \leqslant i \leqslant n} |x_i|^n < c$$

has infinitely many solutions in integers x_i , x_i not all zero $i=1,\ldots n\}$.

Suppose $1,\xi = 1,\xi_1,\ldots,\xi_n$ is a rational basis of real F , where $[F:Q] = n+1,s$. By considering $\mathfrak{m} = (m_0,\ldots,m_n) \in Z^{n+1}$, $\mathfrak{m} \neq 0$, for which the absolute norm $|N(m_0+m_1\xi_1+\ldots+m_n\xi_n)|$ is "minimal" (amongst 2^{n-1} minimal values, not necessarily distinct) we obtain estimates of $c(\xi)$, $c^*(\xi)$ which lead to

$$C_n(F^n) = \sup\{c(\xi) : \xi \in F^n\} \geqslant V_{n,s}/\Delta^{\frac{1}{2}}(F)$$

$$C_n^*(F^n) = \sup\{c^*(\xi) : \xi \in F^n\} \geqslant V_{n,s}/\Delta^{\frac{1}{2}}(F)$$

with *strict* inequality for *some* F .

By a result of Adams [*1*] these inequalities cannot improve the estimate of C_2 from (*). Whether an explicit improvement for $C_n, n \geqslant 3$ follows is an open question.

The methods developed in obtaining the above are applied to a conjecture by Littlewood [*3*] that for any $X = (x_1,\ldots,x_n) \in R^n$ and $\varepsilon > 0$ there exist integers $m_0 \neq 0$, $m_1,\ldots,m_n$ so that

$$|m_0 \prod_{j=1}^{n} (m_0 x_j - m_n)| < \varepsilon.$$

For any $\xi \in F^n$, where $[F:Q] = n+1,0$ (that is, F is totally real) it is shown subject to a plausible conjecture (trivially true for $n = 2$ and subsumed under a conjecture of Schanuel [*2*]) that for any $\varepsilon > 0$ there exist integers $m_0 \neq 0, m_1,\ldots,m_n$ so that

$$|m_0 \prod_{j=1}^{n} (m_0 \xi_j - m_j)| < \varepsilon .$$

(b) We may write (*) in the form

$$C_n = C_n^* \geqslant V_{n,s}/\Delta_{n,s}^{\frac{1}{2}} ,$$

where $\Delta_{n,s} = \min\{\Delta(F) : [F : Q] = n+1,s\}$

The only known value of $V_{n,s}$ for $n \geqslant 3$ is $V_{3,1} = 2$ due to Cusick [4]. In this work the following estimates (*probably* the best) are obtained.

$$V_{3,0} \geqslant 2.70439\ldots , \quad \text{(rectifying a result in [4])}$$

$$V_{4,2} \geqslant 16/9 ,$$

and

$$V_{5,2} \geqslant 2.3932\ldots .$$

As $\Delta_{4,2} = 1609$, $\Delta_{5,2} = 25^2.53$ (see [6],[7]) we deduce that

$$C_4 \geqslant 0.044319\ldots ,$$

and

$$C_5 \geqslant 0.013149\ldots .$$

More generally we show (in all well defined cases)

$$V_{n+n',s+s'} \geqslant V_{n,s}V_{n',s'} .$$

Improved estimates (not in general the best) follow of $V_{n,s}$ for all $n \geqslant 4$, $0 \leqslant 2s \leqslant n$.

(c) The Szekeres algorithm [8] generates an infinite sequence of simplices with rational vertices. In the case 1, ξ is a rational basis of real F with $[F : Q] = n+1,s$ a formal method is described by which sequences of simplices converging to ("shrinking" onto) ξ are constructed. Properties of these convergent sequences are obtained. Although not shown in this work many of the results of (a) were originally deduced from these properties of the convergent simplex sequences.

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