BULL. AUSTRAL. MATH. SOC. 41A05, 41A10, 41A25
VOL. 12 (1975), 267-276.

A convergent quasi-Hermite-Féjer interpolation process

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D.L. Berman has proved several divergence theorems about "extended" Hermite-Fejér interpolation on the Chebyshev nodes of the first kind. These are surprising in light of the classical convergence theorem of L. Fejér concerning ordinary Hermite-Fejér interpolation on these nodes. However there is one case which has been neglected so far: the case of quasi-Hermite-Fejér interpolation on these nodes. In this paper it is proved that quasi-Hermite-Fejér interpolation polynomials on the Chebyshev nodes converge uniformly to the continuous function being interpolated. In addition, an estimate for the rate of convergence is established.

1. Introduction

The following result proved by Fejér [3] is now classical:

THEOREM 1 (Fejér). *Let* $f(x)$ *be continuous on the interval* $[-1, 1]$ *and let* $H_n(f, x)$ *be the polynomial of degree* $2n - 1$ *uniquely determined by the conditions*

$$H_n(f, x_{kn}) = f(x_{kn}) \ , \quad k = 1, 2, \ldots, n \ ,$$
$$H_n'(f, x_{kn}) = 0 \ , \quad k = 1, 2, \ldots, n \ ,$$

where

$$x_{kn} = \cos\left((2k-1)\pi/2n\right) \ , \quad k = 1, 2, \ldots, n \ ,$$

Received 5 December 1974.

and the dash in $H_n'(f, x)$ *denotes differentiation with respect to* x *. Then* $H_n(f, x)$ *converges to* $f(x)$ *uniformly on the interval* $[-1, 1]$ *as* n *tends to infinity.*

Throughout this paper x_{kn} will be defined by (1) and denoted by x_k where there is no confusion.

In 1969, Berman [1], considered a related interpolation process. Let $F_n(f, x)$ be the polynomial of degree $2n + 3$ uniquely determined by the conditions

$$F_n(f, 1) = f(1) \ ; \ F_n(f, -1) = f(-1) \ ;$$

$$F_n'(f, 1) = 0 \ ; \ F_n'(f, -1) = 0 \ ;$$

$$F_n(f, x_k) = f(x_k) \ ; \ F_n'(f, x_k) = 0 \quad \text{for} \quad k = 1, 2, \ldots, n \ .$$

One of his results is as follows:

THEOREM 2 (Berman). *If* $f(x) = x^2$ *, then the sequence* $(F_n(f, x))$ *diverges for every* x *in the open interval* $(-1, 1)$.

In a later paper, Berman [2], considered the polynomial $A_n(f, x)$ of degree $2n + 2$ uniquely determined by the conditions

$$A_n(f, 1) = f(1) \ ; \ A_n(f, -1) = f(-1) \ ;$$

$$A_n'(f, 1) = 0 \ ;$$

$$A_n(f, x_k) = f(x_k) \ ; \ A_n'(f, x_k) = 0 \quad \text{for} \quad k = 1, 2, \ldots, n \ .$$

Concerning this process he proved another divergence theorem:

THEOREM 3 (Berman). *If* $f(x) = x^2$ *, then the sequence* $(A_n(f, x))$ *diverges for every* x *in the open interval* $(-1, 1)$.

In this paper we shall consider the polynomial $V_n(f, x)$ of degree $2n + 1$ uniquely determined by the conditions

$$
(2) \qquad \begin{aligned}
V_n(f, 1) &= f(1) , \\
V_n(f, -1) &= f(-1) , \\
V_n(f, x_k) &= f(x_k) , \quad k = 1, 2, \ldots, n , \\
V_n'(f, x_k) &= 0 \qquad , \quad k = 1, 2, \ldots, n .
\end{aligned}
$$

Such processes were called quasi-Hermite-Fejér interpolation processes by Szász [5]. We shall prove the following estimate which shows that if f is continuous on $[-1, 1]$ then $V_n(f, x)$ converges to f uniformly on the closed interval $[-1, 1]$.

THEOREM 4. *Let $f(x)$ be continuous on the interval $[-1, 1]$ and let $\omega(f; \delta)$ be the modulus of continuity of f. Then*

$$\|V_n(f, x)-f(x)\| \le c_1\omega\left(f; n^{-\frac{1}{2}}\right) .$$

Here c_1 (and later $c_2, c_3, \ldots$) is an absolute constant independent of f and n and $\|\cdot\|$ is the uniform norm on $[-1, 1]$.

2. Proof of Theorem 4

We shall prove the theorem by using a series of lemmas which will be proved in the next section.

LEMMA 1. *(V_n) is a sequence of uniformly bounded linear operators.*

LEMMA 2. *Let $m = \left[n^{\frac{1}{2}}\right]$ and let $p_m(x)$ be the best approximating polynomial of degree m to $f(x)$ in $[-1, 1]$. Then,*

$$\|V_n(p_m, x)-p_m(x)\| \le c_2\omega\left(f; n^{-\frac{1}{2}}\right) .$$

The proof of the theorem is now quite straight forward. By the fundamental approximation theorem of Jackson,

$$\|f(x)-p_m(x)\| \le c_3\omega\left(f; n^{-\frac{1}{2}}\right) .$$

Hence,

https://doi.org/10.1017/S0004972700023868 Published online by Cambridge University Press

$$\|V_n(f, x)-f(x)\| \le \|V_n(f, x)-V_n(p_m, x)\| + \|V_n(p_m, x)-p_m(x)\| + \|p_m(x)-f(x)\|$$
$$\le (\|V_n\|c_3+c_2+c_3)\omega(f; n^{-\frac{1}{2}})$$
$$\le c_4\omega(f; n^{-\frac{1}{2}})$$

and the theorem follows.

3. Proofs of the lemmas

Proof of Lemma 1. From Szász' paper we know that

$$V_n(f, x) = f(1)\frac{(1+x)}{2}T_n^{\,2}(x) + f(-1)\frac{(1-x)}{2}T_n^{\,2}(x) + \sum_{k=1}^{n} f(x_k)\frac{1-x^2}{1-x_k^{\,2}}v_k(x)l_k^{\,2}(x)$$

where

$$v_k(x) = 1 + \frac{x_k(x-x_k)}{1-x_k^{\,2}}, \quad k = 1, 2, \ldots, n,$$

and

$$l_k(x) = \frac{T_n(x)}{T_n'(x_k)(x-x_k)}, \quad k = 1, 2, \ldots, n,$$

and

$$T_n(x) = \cos(n(\text{arc cos } x)) .$$

Let us set

$$V_n(f, x) = \sum_{k=0}^{n+1} f(x_k)h_k(x)$$

where $x_0 = 1$ and $x_{n+1} = -1$. Then

$$\|V_n\| \le \sup \sum_{k=0}^{n+1} |h_k(x)|$$
$$\le 2 + \sup \sum_{k=1}^{n} |h_k(x)| ,$$

where the supremum is taken over all x in $[-1, 1]$.

Now let $x \in (-1, 1)$ and suppose that j is an integer satisfying $1 \le j \le n$ and

(3) $$|x-x_j| \le |x-x_k| , \quad k = 1, 2, \ldots, n .$$

Naturally $j = j(n)$. Should there be two such integers then pick either one. Since $V_n(f, x_j) = f(x_j)$ we may assume that $x \ne x_j$.

To estimate $\|V_n\|$ consider the expression

(4) $$2 + \sum_{k=1}^{j-1} |h_k(x)| + |h_j(x)| + \sum_{k=j+1}^{n} h_k(x)$$

and estimate each part in turn. If $j = 1$ or n then one of these parts will not occur.

Now

$$h_j(x) = \frac{1-x^2}{1-x_j^2} \left(1 + \frac{x_j(x-x_j)}{1-x_j^2} \right) l_j^2(x) .$$

Furthermore

$$\frac{|x_j(x-x_j)|}{1-x_j^2} \le \frac{|t-t_j|}{\sin t_j} \cdot \frac{\sin r_j}{\sin t_j} \le c_5 ,$$

where $x = \cos t$, $x_j = \cos t_j$, and r_j is some number between t and t_j. Hence

$$|h_j(x)| \le c_6 \frac{(1-x^2) l_j^2(x)}{1-x_j^2} .$$

But Varma has shown in [6] that

$$\sum_{k=1}^{n} \frac{1-x^2}{1-x_k^2} l_k^2(x) \le 8$$

and so we have

(5) $$|h_j(x)| \le c_7 .$$

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Now we estimate $\sum_{k=1}^{j-1} |h_k(x)|$. By decomposing $h_k(x)$ into partial fractions we get

$$h_k(x) = \frac{(1-x^2)T_n^{\,2}(x)}{n^2(x-x_k)^2} + \frac{xT_n^{\,2}(x)}{n^2(x-x_k)} - \frac{(1+x)T_n^{\,2}(x)}{2n^2(1-x_k)} - \frac{(1-x)T_n^{\,2}(x)}{2n^2(1+x_k)} .$$

Thus

$$(6) \qquad |h_k(x)| \le \frac{(1-x^2)T_n^{\,2}(x)}{n^2(x-x_k)^2} + \frac{1}{n^2|x-x_k|} + \frac{1}{n^2(1-x_k)} + \frac{1}{n^2(1+x_k)}$$

$$= A_k + B_k + C_k + D_k .$$

It is known that

$$\sum_{k=1}^{n} C_k = \sum_{k=1}^{n} D_k = 1 .$$

Hence

$$(7) \qquad \sum_{k=1}^{j-1} C_k \le 1$$

and

$$(8) \qquad \sum_{k=1}^{j-1} D_k \le 1 .$$

To estimate B_k , let $k = j - i$ where $i \ge 1$ and note that

$$\begin{aligned} \sin\big((t+t_k)/2\big) &= \sin t/2 \cos t_k/2 + \cos t/2 \sin t_k/2 \\ &\ge |\sin t/2 \cos t_k/2 - \cos t/2 \sin t_k/2| \\ &= \sin\big(|t-t_k|/2\big) \\ &\ge |t-t_k|/\pi \\ &\ge c_8\, i/n . \end{aligned}$$

Hence

$$\begin{aligned} B_k &= \left(n^2|x-x_k|\right)^{-1} \\ &= \left(2n^2 \sin\left((t+t_k)/2\right)\sin\left(|t-t_k|/2\right)\right)^{-1} \\ &\le \left(2n^2\sin^2\left(|t-t_k|/2\right)\right)^{-1} \\ &\le c_9 i^{-2} . \end{aligned}$$

So we obtain

$$\sum_{k=1}^{j-1} B_k \le c_9 \sum_{i=1}^{j-1} i^{-2} \le c_{10} . \tag{9}$$

Finally let us consider A_k :

$$\begin{aligned} T_n^{\,2}(x) &= \cos^2 nt \\ &= \left(\cos nt - \cos nt_k\right)^2 \\ &= 4 \sin^2\left(n(t+t_k)/2\right)\sin^2\left(n(t-t_k)/2\right) . \end{aligned}$$

Therefore,

$$\begin{aligned} A_k &= \frac{\left(1-x^2\right)T_n^{\,2}(x)}{n^2\left(x-x_k\right)^2} \\ &\le \frac{\sin^2 t}{\sin^2\left((t+t_k)/2\right)} \cdot \frac{1}{n^2} \cdot \frac{\sin^2\left(n(t-t_k)/2\right)}{\sin^2\left((t-t_k)/2\right)} . \end{aligned}$$

From the inequalities

$$\begin{aligned} \sin t &\le \sin t + \sin t_k \\ &\le 2 \sin\left((t+t_k)/2\right) \end{aligned}$$

and

$$n^{-2} \sum_{k=1}^{n} \frac{\sin^2\left(n(t-t_k)/2\right)}{\sin^2\left((t-t_k)/2\right)} \le c_{11} ,$$

it follows that

(10) $$\sum_{k=1}^{j-1} A_k \le c_{12} .$$

By (7), (8), (9), and (10) we now have

(11) $$\sup \sum_{k=1}^{j-1} |h_k(x)| \le c_{13} .$$

Similarly,

(12) $$\sup \sum_{k=1+j}^{n} |h_k(x)| \le c_{14} .$$

From (4), (5), (11), and (12), Lemma 1 now follows.

Proof of Lemma 2. From Szász' work we know that since $p_m(x)$ is a polynomial of degree $m < 2n + 1$,

$$p_m(x) = V_n(p_m, x) + Q_n(p_m, x) ,$$

where

$$Q_n(p_m, x) = \sum_{k=1}^{n} p_m'(x_k) \cdot \frac{(1-x^2)T_n^{\,2}(x)}{n^2(x-x_k)} .$$

Hence

$$|V_n(p_m, x)-p_m(x)| \le \sum_{k=1}^{n} |p_m'(x_k)| \frac{(1-x^2)T_n^{\,2}(x)}{n^2|x-x_k|} .$$

Now a recent result of Szabados [4] states that

$$|p_m'(x)| \le c_{15} \frac{m\omega(f;m^{-1})}{(1-x^2)^{\frac{1}{2}}} , \quad |x| < 1 .$$

Consequently

(13) $$|V_n(p_m, x)-p_m(x)| \le c_{16}\omega(f; m^{-1}) \sum_{k=1}^{n} u_k(x) ,$$

where

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$$u_k(x) = \frac{\left(1-x^2\right)T_n^{\,2}(x)}{n^{3/2}\left(1-x_k^{\,2}\right)^{\frac{1}{2}}|x-x_k|} .$$

Once again let j be defined by (3). Then

$$(14) \qquad \sum_{k=1}^{n} u_k(x) = \sum_{k=1}^{j-1} u_k(x) + u_j(x) + \sum_{k=j+1}^{n} u_k(x) .$$

We begin by estimating $u_j(x)$:

$$(15) \qquad u_j(x) \le \frac{n}{n^{3/2}} \cdot \frac{1-x^2}{1-x_j^{\,2}} \cdot l_j(x)$$

$$\le 4n^{-\frac{1}{2}} .$$

Now we shall estimate $n^{3/2} \sum_{k=1}^{j-1} u_k(x)$. Writing

$$1 - x^2 = 1 - x_k^{\,2} + \left(x-x_k\right)^2 - 2x\left(x-x_k\right) ,$$

we obtain

$$(16) \qquad n^{3/2}u_k(x) \le \left(1-x_k^{\,2}\right)^{\frac{1}{2}} \frac{T_n^{\,2}(x)}{x-x_k} + |x-x_k| \frac{T_n^{\,2}(x)}{\left(1-x_k^{\,2}\right)^{\frac{1}{2}}} + |x| \frac{T_n^{\,2}(x)}{\left(1-x_k^{\,2}\right)^{\frac{1}{2}}}$$

$$\le n|l_k(x)| + 3/\left(1-x_k^{\,2}\right)^{\frac{1}{2}} .$$

Now it is known that

$$(17) \qquad \sum_{k=1}^{n} \left(1-x_k^{\,2}\right)^{-\frac{1}{2}} \le c_{17} n \ln n$$

and

$$(18) \qquad \sum_{k=1}^{n} |l_k(x)| \le c_{18} \ln n .$$

Hence by (16), (17), and (18),

$$(19) \qquad \sum_{k=1}^{j-1} u_k(x) \le c_{19} .$$

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Similarly,

$$\sum_{k=j+1}^{n} u_k(x) \le c_{20} . \tag{20}$$

By (15), (19), and (20),

$$\sum_{k=1}^{n} u_k(x) \le c_{21} .$$

Thus, returning to (13) we have

$$\|V_n(p_m, x)-p_m(x)\| \le c_{21}\omega(f; m^{-1}) ,$$

which proves Lemma 2.

References

[1] D.L. Berman, "A study of the Hermite-Fejér interpolation process", *Soviet Math. Dokl.* **10** (1969), 813-816.

[2] D.L. Berman, "Extended Hermite-Fejér interpolation processes diverging everywhere", *Soviet Math. Dokl.* **11** (1970), 830-833.

[3] Leopold Fejér, "Ueber Interpolation", *Nachr. K. Ges. Wiss. Göttingen Math.-Phys. Kl.* **1916**, 66-91.

[4] J. Szabados, "On the convergence of Hermite-Fejér interpolation based on the roots of the Legendre polynomials", *Acta Sci. Math.* **34** (1973), 367-370.

[5] Paul Szász, "On quasi-Hermite-Fejér interpolation", *Acta Math. Acad. Sci. Hungar.* **10** (1959), 413-439.

[6] A.K. Varma, "On a problem of P. Turán on lacunary interpolation", *Canad. Math. Bull.* **10** (1967), 531-557.

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