Glasgow Math. J. **42** (2000) 1–8.

APPROXIMATING GRAPHS WITH POLYNOMIAL GROWTH

NORBERT SEIFTER

Institut für Mathematik und Angewandte Geometrie, Montanuniversität Leoben, A-8700 Leoben, Austria

and WOLFGANG WOESS

Dipartimento di Matematica e Applicazioni, Universitá di Milano "Bicocca", Via Bicocca degli Arcimboldi 8, 20126 Milano, Italy

Present address: Institut für Mathematik, Technische Universität Graz, Steyrergasse 30, A-8010 Graz, Austria

(Received 19 February, 1997)

Abstract. Let X be an infinite, locally finite, almost transitive graph with polynomial growth. We show that such a graph X is the inverse limit of an infinite sequence of finite graphs satisfying growth conditions which are closely related to growth properties of the infinite graph X.

1991 *Mathematics Subject Classification*. Primary 05C25, Secondary 20F18

1. Introduction and statement of main result. We think of a *graph* X as a set of vertices, equipped with a symmetric, non reflexive neighbourhood relation $E = E(X) \subset X \times X$, the *edge set*. Graphs considered in this paper are assumed to be connected and to have bounded vertex degrees.

The natural *distance* between two vertices $x, y \in X$ (the minimal number of edges on a path from x to y) is denoted by $d(x, y)$. An *automorphism* of X is a self-isometry of X with respect to this metric. The automorphism group of X is denoted by $\mathrm{Aut}(X)$. The graph is called *transitive* if $\mathrm{Aut}(X)$ acts transitively on X, it is *almost transitive* if $\mathrm{Aut}(X)$ acts on X with finitely many orbits.

For a vertex v, we write $B_X(v, n)$ for the subgraph induced by all $x \in X$ with $d(x, v) \leq n$. The *growth function* of X with respect to v is

$$f_X(v, n) = \left|B_X(v, n)\right| \quad (n = 0, 1, 2, \ldots).$$

If X is transitive, $f_X(n) = f_X(v, n)$ is independent of v. We say that X has *polynomial growth* if there are constants c, d such that

$$f_X(v, n) \leq c \cdot n^d, \text{ for all } n$$

and for every vertex v. The results for finitely generated groups (Gromov [**5**]), transitive graphs (Trofimov [**10**]) and locally compact groups (Losert [**8**]) imply that an almost transitive graph with polynomial growth is very similar to a Cayley graph of a finitely generated nilpotent group; see §2 below. In particular, for such a graph there are constants $0 < c_1 \leq c_2 < \infty$ and a nonnegative integer d such that for every vertex v we have

$$c_1 \cdot n^d \leq f_X(v, n) \leq c_2 \cdot n^d, \text{ for all } n. \tag{1}$$

We say that a graph X (finite or infinite, not necessarily almost transitive) has the *doubling property* if there is a number (the *doubling constant*) $A \geq 1$ such that

$$f_X(v, 2n) \leq A \cdot f_X(v, n), \text{ for all } n \tag{2}$$

and for every vertex v. If a locally finite graph X has polynomial growth, then it also has the doubling property. If X is almost transitive then the doubling property also implies that X has polynomial growth.

The purpose of the present note is to provide some new features of almost transitive infinite graphs by use of the doubling property and of coverings.

Given two graphs X_1, X_2, a *covering map* is a surjection $\varphi : X_2 \to X_1$ with the property that for some $r \geq 1$ and each vertex $v \in X_2$, the restriction of φ to $B_{X_2}(v, r)$ is a graph isomorphism onto $B_{X_1}(\varphi v, r)$. We then say that X_2 is a *covering graph* of X_1, and that the covering has *range* $r(\varphi) \geq r$.

Covering properties of transitive graphs with polynomial growth have been studied by Godsil and Seifter [**4**]. Here, we extend and refine some of the results of [**4**].

THEOREM 1. *Let X be an infinite almost transitive graph with polynomial growth. Then there is an increasing sequence $X_1, X_2, X_3, \ldots$ of finite graphs such that*

(a) *each X_k is covered by X via a covering map φ_k, where $r(\varphi_k) \to \infty$;*

(b) *for each $k > l$, there is a covering map $\varphi_{l,k}$ from X_k onto X_l, and one has the relations*

$$\varphi_{l,k} \circ \varphi_k = \varphi_l$$

and, for $k > l > m$,

$$\varphi_{m,l} \circ \varphi_{l,k} = \varphi_{m,k};$$

(c) *each X_k has the doubling property with the same doubling constant $\bar{A}$.*

We could interpret assertions (a) and (b) by saying that the sequence X_k *approximates* the infinite graph X. Thus, X is the projective limit of the X_k.

A finite graph X has *$A - d$ moderate growth* (Diaconis and Saloff-Coste [**3**]) if the positive constants A and d satisfy

$$\frac{f_X(v, n)}{|X|} \geq \frac{1}{A}\left(\frac{n}{\operatorname{diam}(X)}\right)^d, \tag{3}$$

for every $v \in X$ and all n with $1 \leq n \leq \operatorname{diam}(X)$, where $\operatorname{diam}(X)$ is the diameter of X. (Since X is finite, clearly such constants always exist; pick $d \geq 1$ and A large enough.)

As was shown in [**3**, Lemma 5.1], a finite graph X which has the doubling property for some constant A has $A - d$ moderate growth for $d = \log A / \log 2$.

If (a), (b), (c) hold, then of course X has the doubling property with constant $\bar{A}$. However, given that X has doubling constant A, we do not know if in general one can construct the X_k with the same doubling constant. (From our proof, we only get $\bar{A} \geq A$.) As a consequence, we do not know whether the X_k may be constructed so that they have moderate growth with exponent d equal to the degree of growth of X.

Also, we cannot guarantee that the X_k are close to being transitive. However, this is true in the following particular case.

THEOREM 2. *Let X be an infinite graph with polynomial growth and suppose that there is a transitive group $G < \mathrm{Aut}(X)$ with finite vertex stabilizers. Then one can construct the approximating sequence such that, for each k, some factor of G acts transitively on the graph X_k.*

2. Preliminary results. We first gather the necessary material and preliminary results.

Factor graphs and construction of coverings. We first recall the construction of *factor graphs*. Let H be a subgroup of $\mathrm{Aut}(X)$. Then $H\backslash X$ is the graph whose vertices are the orbits of H on X; two different orbits Hx_1, and Hx_2 are neighbours in $H\backslash X$ if and only if there are $y_i \in Hx_i$, $i = 1, 2$, which are neighbours in X.

If G is a subgroup of $\mathrm{Aut}(X)$ and H is a normal subgroup of G, then G acts on $H\backslash X$ by automorphisms; indeed, G/H is a subgroup of $\mathrm{Aut}(H\backslash X)$.

Factor graphs can be used to construct covering maps. We say that an orbit $O = Hx$ is a *covering orbit* if it contains no neighbour of x (that is, O contains no pair of adjacent vertices), and any other orbit contains at most one neighbour of x. Thus, the natural projection $\varphi : X \to H\backslash X$ is a covering map if all orbits under H satisfy these two properties.

More generally, factor graphs can be obtained by the use of block systems. Given $G \leq \mathrm{Aut}(X)$, a *block system* σ is a G-invariant partition of X whose pieces are called *blocks*. It gives rise to the factor graph $\sigma\backslash X$ whose vertices are the blocks, and two blocks are adjacent if they are connected by some edge in X. The action of G gives rise to the homomorphic image $G/\sigma \leq \mathrm{Aut}(\sigma\backslash X)$.

Structure of graphs and groups with polynomial growth. Let G be a discrete group and S a finite, symmetric set of generators (not containing the identity). The *Cayley graph* of G has vertex set $X = G$; two elements g, h are joined by an edge if and only if $g^{-1}h \in S$. The group acts on the Cayley graph by left multiplication. The *growth function* of the group G (with respect to S) is defined as the growth function of the Cayley graph. The property of having polynomial growth is independent of the choice of the generating set. The following fundamental result is due to Gromov [**5**].

PROPOSITION 1. *A finitely generated group has polynomial growth if and only if it has a nilpotent subgroup with finite index.*

In particular, the growth degree d in formula (1) is an integer that is also independent of the generating set and can be calculated by use of a formula due to Bass [**2**]. Besides [**2**] and [**5**], a useful reference for the structure theory of nilpotent groups is Hall [**7**].

Gromov's structure theorem has been extended to vertex transitive graphs with polynomial growth by [**10**]. This can also be seen as a special case of Losert's [**8**] classification of topological groups with polynomial growth (cf. Woess [**12**]). We briefly explain the following slight extension to almost transitive graphs.

PROPOSITION 2. *Let X be an almost transitive graph with polynomial growth. Then there is a normal subgroup K of* $\mathrm{Aut}(X)$ *with the following properties.*

(a) *The orbits of K on X are finite.*

(b) *The factor group* $\mathrm{Aut}(X)/K$ *is a finitely generated group with polynomial growth.*

(c) $\mathrm{Aut}(X)/K$ *acts with finite vertex stabilizers on the factor graph* $K\backslash X$.

Proof. With the topology of pointwise convergence, the group $G = \mathrm{Aut}(X)$ is a locally compact, totally disconnected Hausdorff topological group. Let $Y_1, \ldots, Y_m$ be the orbits of G on X. Let L be the subgroup of G consisting of all $g \in G$ that fix Y_1 pointwise. This is a compact normal subgroup of G.

Let $k = 2m + 1$, and consider the graph $X^{(k)}$ with the same vertex set as X, where two vertices are joined by an edge if $1 \leq d(x, y) \leq k$. Then Y_1 induces a connected subgraph of $X^{(k)}$ that is locally finite and has polynomial growth with the same degree as X. Now, G/L acts transitively on this graph as a closed subgroup of the automorphism group. The argument of [**12**] (in the proof of Theorem 1) shows that G/L is compactly generated and has polynomial growth (in terms of left Haar measure). Consequently, also G has these two properties (see Guivarc'h [**6**, Theorem 1.4]).

Now, by [**8**], G has a compact normal subgroup K such that G/K is a Lie group. As G is totally disconnected, G/K must be zero-dimensional. That is, G/K is a finitely generated, discrete group with polynomial growth that acts on the factor graph $K\backslash X$ as a closed subgroup of the automorphism group. In particular G/K must act with finite vertex stabilizers. $\square$

Formula (1) has been proved in [**2**] for finitely generated groups with a nilpotent subgroup of finite index. By Propositions 1 and 2, this extends to almost transitive graphs with polynomial growth. Indeed, the (integer) growth degree of X in (1) coincides with that of the group $\mathrm{Aut}(X)/K$.

More group theory. Let G be a subgroup $\mathrm{Aut}(X)$. If the stabilizer G_v in G of any vertex v of X consists of the identity only, then we say that G acts *semiregularly* on X. If G in addition acts transitively on X then G is said to act *regularly* on X. In this case we also know that X is a Cayley graph of G.

The so called "Contraction Lemma" of Babai [**1**] will be a useful tool in the study of almost transitive graphs.

LEMMA 1. *If a group G acts semiregularly on a graph X, then X is contractible onto a Cayley graph of G.*

Indeed, below we shall also use the method of Babai's proof. We shall frequently use the following simple and well known group-theoretical lemma.

LEMMA 2. *Let G be a finitely generated group and H a subgroup of G with finite index. Then the intersection of all conjugates of H in G is a normal subgroup of G with finite index.*

The following holds even for polycyclic groups; see e.g. Wolf [**11**].

PROPOSITION 3. *Every finitely generated nilpotent group has a torsion-free subgroup of finite index.*

The next lemma, possibly known to specialists, will be crucial in our construction of coverings. If $g_1, \dots, g_k$ are elements of a group G, then we write (as usual) $\langle g_1, \dots, g_k \rangle$ for the subgroup of G that they generate.

Lemma 3. *Let N be a finitely generated nilpotent group acting semiregularly and almost transitively on the infinite graph X. Then, for each m, there is a torsion-free normal subgroup $N^{(m)}$ of finite index such that*

$$d(x, g(x)) \geq m \text{ for all } x \in X \text{ and all } g \in N^{(m)}, g \neq id.$$

Proof. By Proposition 3, N contains a torsion-free subgroup N' of finite index. If N acts almost transitively on X, then N' also acts almost transitively on X. Hence we can assume that N itself is torsion-free. Since nilpotent groups have a nontrivial center, the center of N then contains elements of infinite order. As N acts semiregularly on X, we conclude that the center of N always contains an infinite cyclic subgroup.

We now proceed by induction on the growth degree d of N (equivalently of X).

If $d = 1$, then we know in addition that N contains an infinite cyclic subgroup of finite index that is central in N (for otherwise $d = 1$ cannot hold). For $N^{(m)}$, we may take a suitable subgroup of the latter.

Let $d > 1$. We choose a central element a of infinite order and write $A = \langle a \rangle$. By semiregularity, each Ax, $x \in X$, is infinite. As a is central and N acts with finitely many orbits, there are only finitely many isomorphism types among the Ax, $x \in X$. Hence, we may choose $q \in \mathbb{N}$ such that $d(x, a^q(x)) \geq m$.

The group $\tilde{N} = N/\langle a^q \rangle$ then acts semiregularly and almost transitively on the factor graph $\tilde{X} = \langle a^q \rangle \backslash X$. Its growth degree is less than d and, by the induction hypothesis, we can find a torsion-free normal subgroup $\tilde{N}^{(m)}$ of $\tilde{N}$ such that $d(\tilde{x}, \tilde{g}(\tilde{x})) \geq m$, for all $\tilde{x} \in \tilde{X}$ and all $\tilde{g} \in \tilde{N}^{(m)}$, $\tilde{g} \neq id$. Let $N^{(m)} = \pi^{-1}(\tilde{N}^{(m)})$, where π is the natural projection of N onto $\tilde{N}$. Let $g \in N^{(m)}$, $g \neq id$ and $x \in X$.

Case 1. $g(x) \in \langle a^q \rangle x$. By semiregularity, $g \in \langle a^q \rangle$ and we are done.

Case 2. Otherwise, $\tilde{g} = \pi(g)$ is different from the identity, and $d(x, g(x)) \geq d(\tilde{x}, \tilde{g}(\tilde{x})) \geq m$. □

3. Proofs. To prove Theorem 1, we proceed in several steps. Theorem 2 will then become obvious from the details of the proof.

Step 1. We assume that $\mathrm{Aut}(X)$ contains a finitely generated nilpotent group G which acts almost transitively.

By Proposition 3 and Lemma 2, G has a torsion-free normal subgroup N with finite index. It acts almost transitively and, as was shown in **[9]**, N must also act semiregularly on X.

We first construct a normal subgroup N_1 of N with finite index, such that X covers $N_1 \backslash X$.

If N itself has this property, then we are done. Otherwise, some among the orbits $O_1 = Nx_1, \dots, O_q = Nx_q$ of N on X are not covering orbits. Consider the (finite) set W consisting of the x_j, $1 \leq j \leq q$, and all their neighbours, and let

https://doi.org/10.1017/S001708950001003X Published online by Cambridge University Press

$$m = \max\{d(v, w) \mid v, w \in W\}. \tag{4}$$

By Lemma 3, we can find a normal subgroup N_1 of N with finite index such that $d(x, g(x)) \geq m$, for all $x \in X$ and $g \in N_1$, $g \neq id$. Consequently, all orbits of N_1 are covering orbits.

Starting with N_1, we shall now construct a descending sequence

$$N_1 \rhd N_2 \rhd \dots$$

of normal subgroups of N with finite index, such that

$$d(x, g(x)) \geq k \cdot m \text{ for all } x \in X \text{ and all } g \in N_k, g \neq id, \tag{5}$$

where m is as in (4).

Inductively, suppose that we already have N_{k-1}. Lemma 3, applied to N_{k-1}, guarantees the existence of a finite index subgroup N'_k which satisfies (5) and is normal in N_{k-1}. In view of Lemma 2, we may set $N_k = \cap_{g \in N} g N'_k g^{-1}$.

Each N_k induces a covering, as this is true for N_1. We define $X_k = N_k \backslash X$, and write φ_k for the natural projection $X \to X_k$. By (5), $r(\varphi_k) \geq k \cdot m$. This proves (a) (under our restricted hypotheses). Also, if $k > l$, then the natural projection $\varphi_{l,k}$ of X_k onto $(N_l/N_k)\backslash X_k \equiv X_l$ satisfies the assertion of (b).

We now prove property (c) and first assume in addition that N acts transitively on X. Then X is a Cayley graph of N with respect to some finite symmetric generating set S. Consider the finite graphs $X_1, X_2, \dots$ constructed as above. By construction, each X_k is the Cayley graph of the group N/N_k with respect to $\varphi_k(S)$. Then we can apply [**6**, Lemma 1.1] to see that every graph X_k has the doubling property with the same constant $A' = A^2$, where A is the doubling constant of X; compare with [**3**, Theorem 5.2]. [The latter theorem says that for a Cayley graph of a nilpotent group, the doubling constant holds with constant A' depending only on the degree nilpotency and the vertex degree. From the proof one sees that one may choose $A' = A^2$.]

Now let X be a graph upon which N acts almost transitively. Then, since N also acts semiregularly on X, the graph X is contractible onto a Cayley graph of N, by Lemma 1. Following the proof of this result, this contraction is done as follows: one can choose a finite tree T_0 which contains exactly one vertex of each orbit of N on X. Then consider the images $T_1, T_2, \dots$ of T_0 under the elements of N. Since N acts semiregularly on X, these trees are pairwise disjoint. Contracting them leads to a Cayley graph $\tilde{X}$ of N.

Let $X_1, X_2, \dots$ be the finite graphs constructed from X as above. Then the groups N/N_k act semiregularly on the X_k, respectively. Since the T_j, $j \geq 0$, contain exactly one vertex of each orbit of N on X, the images of the T_j under the considered covering maps are trees which contain exactly one vertex of each orbit of N/N_k on X_k, respectively. If we now contract these finite trees in the graph X_k, $k \geq 1$, we obtain a graph $\tilde{X}_k$. This coincides with the graph obtained by first contracting the finite trees T_j, $j \geq 0$, in X and then constructing the quotient of $\tilde{X}$ with respect to the group $\tilde{N}_k$; that is, N_k viewed as a subgroup of $\mathrm{Aut}(\tilde{X})$.

Since the $\tilde{X}_k$ have the doubling property with the same constant A', the X_k have the doubling property with the same constant $\bar{A}$, where we may choose $\bar{A} = A'|T_0|$.

Intermediate step: proof of Theorem 2. Under the assumptions of Theorem 2, G must be finitely generated with polynomial growth. From Propositions 1 and 3 we know that G has a finitely generated torsion-free nilpotent subgroup N with finite index. In view of Lemma 2, we can now choose our subgroups N_k such that they are not only normal in N, but also in G. Hence the factor graphs X_k are transitive. □

Step 2. General case. If X is any almost transitive graph, then we know from Proposition 2 that a finitely generated nilpotent group $\mathrm{Aut}(X)/K$ acts with finitely many orbits on $K\backslash X$. Hence our assertions hold for $\tilde{X} = K\backslash X$. To prove that (a), (b) and (c) also hold for X, we set $\tau : \mathrm{Aut}(X) \to \mathrm{Aut}(X)/K$, the natural projection.

Let N again denote a finitely generated torsion-free nilpotent group acting with finitely many orbits on $\tilde{X}$. By $\tilde{X}_1, \tilde{X}_2, \ldots$ we denote the finite graphs which we obtain from $\tilde{X}$ as above.

Let N_i be the normal subgoup of N that leads to $\tilde{X}_i$. By $N_i = N_i^0 \rhd N_i^1 \rhd \ldots \rhd N_i^{s+1} = \{id\}$ we denote the derived series of N_i. Clearly $N_i^j/N_i^{j+1} \cong \mathbb{Z}^{n_j}$ for some n_j and all $j, 0 \le j \le s$. By $\{\tilde{a}_{m_{j+1}}, \ldots, \tilde{a}_{m_{j+1}}\}$ we denote a free generating set of N_i^j/N_i^{j+1}, where $m_0 = 0$, $m_{j+1} = \sum_{l=0}^{l=j} n_l$.

If the group $N_i = N_i^0 \rhd N_i^1 \rhd \ldots \rhd N_i^{r+1} = \{id\}$ leads to the graph $\tilde{X}_i$, then we obtain the same graph $\tilde{X}_i$ if we construct it step by step as follows. If ϵ_r denotes the imprimitivity system induced by the orbits of N_i^r on $\tilde{X}$, then $\tilde{Y}_r$ is the quotient of $\tilde{X}$ with respect to ϵ_r. Of course $M_r = N_i/N_i^r$ acts almost transitively on $\tilde{Y}_r$ and we can repeat the same with $\tilde{Y}_r$ and the orbits of the last nontrivial member of the derived series of M_r on $\tilde{Y}_r$. Then the group $M_{r-1} = N_i/N_i^{r-1}$ acts almost transitively on $\tilde{Y}_{r-1}$ etc. We finally obtain a graph $\tilde{Y}_0$ which is isomorphic to $\tilde{X}_i$.

We now in addition construct the graphs $\tilde{Y}_k, r \ge k \ge 0$, step by step. Let $\tilde{a}_{m_{r+1}}, \ldots \tilde{a}_{m_{r+1}}$ denote the free generators of the abelian group N_i^r. If we now construct the graph $\tilde{W}_1$ as the quotient of $\tilde{X}$ with respect to the orbits of $\tilde{a}_{m_{r+1}}$ on $\tilde{W}_0 = \tilde{X}$, then the graph $\tilde{W}_2$ as the quotient of $\tilde{W}_1$ with respect to the orbits of $\tilde{a}_{m_{r+1}-1}/\tilde{a}_{m_{r+1}}$ on $\tilde{W}_1$ etc. it is again obvious that the graph $\tilde{W}_{n_{r+1}}$ is isomorphic to $\tilde{Y}_r$.

Then we apply the same construction with respect to the orbits of the free generators of N_i^{r-1}/N_i^r on $\tilde{Y}_r$ which leads to $\tilde{Y}_{r-1}$. Repeating this we finally obtain the graph $\tilde{X}_i$ again. Hence there is a sequence of automorphisms $id = \tilde{a}_{m_{r+1}+1}$, $\tilde{a}_{m_{r+1}}, \ldots, \tilde{a}_{m_r}, \ldots, \tilde{a}_{m_{r-1}+1}, \ldots, \tilde{a}_{m_1}, \tilde{a}_{m_1-1}, \ldots, \tilde{a}_1$ and a sequence of graphs $\tilde{X} = \tilde{W}_0$, $\tilde{W}_1, \ldots, \tilde{W}_{m_{r+1}}$ such that by successively reducing the orbits of $\tilde{a}_k/\langle \tilde{a}_{k+1}, \ldots, \tilde{a}_{m_{r+1}+1}\rangle$, $m_{r+1} \ge k \ge 1$, on $\tilde{W}_{m_{r+1}-k}$ to single vertices, we obtain a graph isomorphic to $\tilde{X}_i$ again.

By $a_{m_{r+1}}, \ldots, a_1$ we denote preimages of the $\tilde{a}_{m_{r+1}}, \ldots, \tilde{a}_1$ under τ^{-1} respectively. Of course these preimages are not unique; we simply choose exactly one preimage a_l for each $\tilde{a}_l$, $1 \le l \le m_{r+1}$. We now construct the graphs $X = W_0, W_1, \ldots W_{m_{r+1}} = X_i$ by successively taking quotient graphs with respect to the orbits of the $a_{m_{r+1}+1-l}$, $1 \le l \le m_{r+1}$ on W_{l-1}.

Let ϵ now denote the imprimitivity system induced by the orbits of K on X. The a_l are automorphisms of X whose orbits contain at most one vertex of each block of ϵ. Therefore, and because the $\tilde{a}_l$ induce a covering map from $\tilde{X}$ onto $\tilde{X}_i$, it follows that each W_l is covered by X. Hence the finite graph $X_i = W_{m_{r+1}}$ that we finally obtain is also covered by X. Applying this construction of covering maps to infinitely many nilpotent groups $N_1, N_2, \ldots$, it immediately follows that the range of the covering maps obtained this way tends to infinity.

Note that in our construction the $a_l = a_l^{(i)}$ also depend on N_i. Now we explain how they are chosen inductively with respect to i.

Let X_{i+1}, and X_i be two of the finite graphs obtained in this way; that is X_i corresponds to N_i and X_{i+1}, to N_{i+1}. The orbits of N_{i+1} on $\tilde{X}$ are suborbits of the orbits of N_i on $\tilde{X}$ by the construction of Lemma 3 and Step 1. Furthermore each orbit of N_i on $\tilde{X}$ splits into finitely many orbits of N_{i+1} on $\tilde{X}$. Therefore we can also choose the $a_l^{(i+1)}$ such that their orbits on X are suborbits of the orbits of the $a_l^{(i)}$ which proves (b).

Let m now denote the maximal cardinality of the blocks of ϵ and let $\bar{A}$ be the doubling constant of the graphs $\tilde{X}_1, \tilde{X}_2, \ldots$. Then $A = m\bar{A}$ clearly is a doubling constant for all graphs $X_1, X_2, \ldots$, which proves (c). □

Acknowledgements. The first author was supported by the Austrian-Italian scientific cooperation project Nr. 40 and by DIMANET.

REFERENCES

1. L. Babai, Some applications of graph contractions, *J. Graph Theory* **1** (1977), 125–130.

2. H. Bass, The degree of polynomial growth of finitely generated nilpotent groups, *Proc. London Math Soc. (3)*, **25** (1972), 603–614.

3. P. Diaconis, L. Saloff-Coste, Moderate growth and random walks on finite groups, *Geom. Funct. Anal.* **4** (1994), 1–36.

4. C. D. Godsil, N. Seifter, Graphs with polynomial growth are covering graphs, *Graphs Combin.* **8(3)** (1992), 233–242.

5. M. Gromov, Groups of polynomial growth and expanding maps, *Inst. Hautes Études Sci. Publ. Math.* **53** (1981), 53–78.

6. Y. Guivarc'h, Croissance polynomials et periodes des functions harmoniques, *Bull. Soc. Math. France* **101** (1973), 333–379.

7. P. Hall, Nilpotent Groups, *Collected works of Philip Hall* (Oxford Univ. Press, 1988) 417–462.

8. V. Losert, On the structure of groups with polynomial growth, *Math. Z.* **195** (1987), 109–117.

9. N. Seifter, Groups acting on graphs with polynomial growth, *Discrete Math.* **89** (1991), 269–280.

10. V. I. Trofimov, Automorphism groups of graphs as topological groups, *Math. Notes (Transl. Mat. Zametki)* **38** (1986), 717–720.

11. J. A. Wolf, Growth of finitely generated solvable groups and curvature of Riemannian manifolds, *J. Differential Geom.* **2** (1968), 421–446.

12. W. Woess, Topological groups and infinite graphs, *Discrete Math.* **95** (1991), 373–384.