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ON SOME SUBCLASSES OF HARMONIC MAPPINGS

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Abstract

Let $\mathcal{P}^0_{\mathcal{H}}(M)$ denote the class of normalised harmonic mappings $f = h + \overline{g}$ in the unit disk $\mathbb{D}$ satisfying $\operatorname{Re}(zh''(z)) > -M + |zg''(z)|$, where $h'(0) - 1 = 0 = g'(0)$ and $M > 0$. Let $\mathcal{B}^0_{\mathcal{H}}(M)$ denote the class of sense-preserving harmonic mappings $f = h + \overline{g}$ in the unit disk $\mathbb{D}$ satisfying $|zh''(z)| \leq M - |zg''(z)|$, where $M > 0$. We discuss the coefficient bound problem, the growth theorem for functions in the class $\mathcal{P}^0_{\mathcal{H}}(M)$ and a two-point distortion property for functions in the class $\mathcal{B}^0_{\mathcal{H}}(M)$.

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1. Introduction and definitions

Harmonic mapping techniques are very useful in the study of fluid flow problems (see [1]). In particular, univalent harmonic mappings having special geometric properties such as convexity, starlikeness and close-to-convexity arise naturally in planar fluid dynamical problems. Properties of univalent harmonic maps were used in [1] to give explicit solutions to the incompressible two-dimensional Euler equations and in [5] to give a complete classification of all two-dimensional fluid flows. The harmonic maps derived from such problems can be very complicated. In this connection, it is interesting to consider the problem of approximating harmonic maps by harmonic polynomials, which motivates us to consider partial sums of the series representing univalent harmonic mappings.

Let $\mathcal{H}$ be the class of complex-valued harmonic functions f defined in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, normalised by $f(0) = 0 = f_z(0) - 1$. It is well known that every function $f \in \mathcal{H}$ has the canonical representation $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1.1}$$

Both h and g are analytic in $\mathbb{D}$ and are called the analytic and co-analytic parts of f, respectively. When $g(z) = 0$, $\mathcal{H}$ reduces to the class $\mathcal{A}$ of analytic functions in $\mathbb{D}$

with $f(0) = 0$ and $f'(0) = 1$. If $f = h + \overline{g}$, then the Jacobian $J_f(z)$ of f is defined by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ and we say that f is sense preserving if $J_f(z) > 0$ in $\mathbb{D}$. Let $\mathcal{S}_{\mathcal{H}}$ be the subclass of $\mathcal{H}$ consisting of univalent and sense-preserving harmonic mappings. If $g(z) = 0$ in $\mathbb{D}$, then the class $\mathcal{S}_{\mathcal{H}}$ reduces to the class $\mathcal{S}$ of univalent and analytic functions in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. Furthermore, if $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$, then $|g'(0)| = |b_1| < 1$ (since $J_f(0) = 1 - |g'(0)|^2 = 1 - |b_1|^2 > 0$). The function F, given by

$$F(z) = \frac{f(z) - \overline{b_1 f(z)}}{1 - |b_1|^2},$$

belongs to $\mathcal{S}_{\mathcal{H}}$. Since F is an affine map of f, it is clear that F is univalent in $\mathbb{D}$. A simple observation shows that $F_{\overline{z}}(0) = 0$. Thus, we may restrict our attention to the following subclass:

$$\mathcal{S}_{\mathcal{H}}^0 := \{f \in \mathcal{S}_{\mathcal{H}} : b_1 = \overline{f_{\overline{z}}(0)} = 0\}.$$

For any function $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}^0$, its analytic and co-analytic parts can be represented by

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n. \tag{1.2}$$

The family $\mathcal{S}_{\mathcal{H}}^0$ is known to be compact and normal, whereas $\mathcal{S}_{\mathcal{H}}$ is normal, but not compact.

In 1984, Clunie and Sheil-Small [4] investigated the class $\mathcal{S}_{\mathcal{H}}$ and its geometric subclasses, which have subsequently been extensively studied (see, for example, [3, 7, 15]). A domain Ω is called *starlike* with respect to a point $z_0 \in \Omega$ if the line segment joining z_0 to any point in Ω lies in Ω. In particular, if $z_0 = 0$, then Ω is simply called starlike. A complex-valued harmonic mapping $f \in \mathcal{H}$ is said to be starlike if $f(\mathbb{D})$ is starlike. We denote the class of harmonic starlike functions in $\mathbb{D}$ by $\mathcal{S}_{\mathcal{H}}^*$. Starlikeness is a hereditary property in that if an analytic function maps the unit disk $\mathbb{D}$ univalently onto a starlike domain, then it also maps each concentric circle onto a starlike domain. On the other hand, in general, starlike harmonic mappings do not have this property. The failure of this hereditary property of starlike harmonic mappings led to the introduction of fully starlike harmonic mappings.

THEOREM 1.1 [9]. *A sense-preserving harmonic function $f = h + \overline{g}$ is fully starlike in $\mathbb{D}$ if the analytic function $h + \epsilon g$ is starlike in $\mathbb{D}$ for each ϵ with $|\epsilon| = 1$.*

For $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}^0$ with the series representation given by (1.2), the sections or partial sums $s_{n,m}(f)$ are defined by

$$s_{n,m}(f)(z) = s_n(h)(z) + \overline{s_m(g)(z)},$$

where $n \geq 1$ and $m \geq 2$, $s_n(h) = \sum_{k=1}^{n} a_k z^k$ and $s_m(g) = \sum_{k=2}^{m} b_k z^k$. From the definition, it is clear that a partial sum of f can be thought of as an approximation of f by complex-valued harmonic polynomials. For recent results on the partial sums of univalent harmonic maps, we refer to [10, 11, 13]. We recall a result from [10] which is the motivation for the problem that we are considering in this paper.

THEOREM 1.2 [10]. *Let $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}^0$ with the series representation (1.2). Suppose that f belongs to $C_{\mathcal{H}}^0$, the class of close-to-convex harmonic mappings. Then the sections $s_{n,m}(f)$ are univalent in the disk $|z| < r_{n,m}$. Here $r_{n,m}$ is the unique positive root of the equation $\mu(n, m, r) = 0$, where*

$$\mu(n,m,r) = \frac{1}{12r}\left(\frac{1-r}{1+r}\right)\left(1-\left(\frac{1-r}{1+r}\right)^6\right) - R_n - T_m$$

with

$$R_n = \sum_{k=n+1}^{\infty} A_k r^{k-1}, \quad T_m = -\sum_{k=m+1}^{\infty} A_{-k} r^{k-1}, \quad A_k = \frac{k(k+1)(2k+1)}{6}.$$

In particular, every section $s_{n,n}(f)(z)$ is univalent in the disk $|z| < r_{n,n}$, where

$$r_{n,n} \geq r_{n,n}^l = 1 - \frac{7\log n - 4\log(\log n)}{n} \quad \textit{for } l = \min\{n, m\} \geq 15.$$

We now define the subclasses $\mathcal{P}_{\mathcal{H}}^0(M)$ and ${\mathcal{B}_{\mathcal{H}}}^0(M)$ of harmonic univalent maps. For $M > 0$,

$$\mathcal{P}_{\mathcal{H}}^0(M) := \{f = h + \overline{g} \in \mathcal{H} : \mathrm{Re}\,(zh''(z)) > -M + |zg''(z)| \text{ and } g'(0) = 0\}$$

and, following [6],

$${\mathcal{B}_{\mathcal{H}}}^0(M) := \{f = h + \overline{g} \in \mathcal{H}_0 : |zh''(z)| \leq M - |zg''(z)| \text{ for } z \in \mathbb{D}\}.$$

The organisation of the paper is as follows. In Section 2, we will show that functions in the class $\mathcal{P}_{\mathcal{H}}^0(M)$ are univalent harmonic maps. We then obtain sharp coefficient bounds for functions in $\mathcal{P}_{\mathcal{H}}^0(M)$. Growth results for functions in $\mathcal{P}_{\mathcal{H}}^0(M)$ will also be discussed in Section 2. In Section 3, we prove a two-point distortion theorem for functions in the class ${\mathcal{B}_{\mathcal{H}}}^0(M)$.

2. Main results

THEOREM 2.1. *The harmonic map $f = h + \overline{g}$ belongs to $\mathcal{P}_{\mathcal{H}}^0(M)$ if and only if the function $F_\epsilon = h + \epsilon g$ belongs to $\mathcal{P}(M)$ for $|\epsilon| = 1$, where $\mathcal{P}(M)$ is defined by*

$$\mathcal{P}(M) := \{\phi \in \mathcal{A} : \mathrm{Re}\,(z\phi''(z)) > -M \textit{ for } M > 0\}.$$

PROOF. Suppose that $f = h + \overline{g} \in \mathcal{P}_{\mathcal{H}}^0(M)$. For $|\epsilon| = 1$,

$$\begin{aligned}\mathrm{Re}\,(zF_\epsilon''(z)) &= \mathrm{Re}\,(z(h''(z) + \epsilon g''(z)) \\ &> \mathrm{Re}\,(zh''(z)) - |zg''(z)| > -M \quad \text{for } z \in \mathbb{D}.\end{aligned}$$

Hence, $F_\epsilon = h + \epsilon g \in \mathcal{P}(M)$ for each $|\epsilon| = 1$. Conversely, if $F_\epsilon \in \mathcal{P}(M)$, then

$$\mathrm{Re}\,(zh''(z)) + \mathrm{Re}(\epsilon z g''(z)) = \mathrm{Re}\,(z(h''(z) + \epsilon g''(z))) > -M \quad \text{for } z \in \mathbb{D},$$

Since ϵ ($|\epsilon| = 1$) is arbitrary, for an appropriate choice of ϵ,

$$\mathrm{Re}\,(zh''(z)) - |zg''(z)| > -M \quad \text{for } z \in \mathbb{D}$$

and so $f \in \mathcal{P}_{\mathcal{H}}^0(M)$. □

Note that for $0 < M < 1/\log 4$, each function in the class $\mathcal{P}(M)$ is univalent and starlike. Thus, by Theorem 1.1, $\mathcal{P}^0_{\mathcal{H}}(M) \subset \mathcal{S}^{*\,0}_{\mathcal{H}}$ for $0 < M < 1/\log 4$. In particular, when $0 < M < 1/\log 4$, functions in $\mathcal{P}^0_{\mathcal{H}}(M)$ are fully starlike.

Theorem 2.2. *Let $f = h + \overline{g} \in \mathcal{P}^0_{\mathcal{H}}$ for $M > 0$ be of the form (1.2). Then, for $n \geq 2$,*

$$|b_n| \leq \frac{2M}{n(n-1)}.$$

The result is sharp for the function f given by $f(z) = z - M\overline{z^n}/n(n-1)$.

Proof. Let $f \in \mathcal{P}^0_{\mathcal{H}}(M)$. Then

$$\operatorname{Re}(zh''(z)) - |zg''(z)| > -M \quad \text{for } z \in \mathbb{D}$$

and so

$$|zg''(z)| < M + \operatorname{Re}(zh''(z)) \quad \text{for } z \in \mathbb{D}. \tag{2.1}$$

Using the series representation of $g(z)$ in (2.1),

$$\begin{aligned} r^{n-1}n(n-1)|b_n| &\leq \frac{1}{2\pi}\int_0^{2\pi} |g''(re^{i\theta})|\,d\theta \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} M + \operatorname{Re}(re^{i\theta}h''(re^{i\theta}))\,d\theta = M. \end{aligned}$$

Letting $r \to 1^-$ gives the desired bound. It is easy to see that $f(z) = z - M\overline{z^n}/n(n-1)$ belongs to $\mathcal{P}^0_{\mathcal{H}}(M)$ and $|b_n(f)| = 2M/n(n-1)$. □

Theorem 2.3. *Let $f = h + \overline{g} \in \mathcal{P}^0_{\mathcal{H}}(M)$ for $M > 0$ be of the form (1.2). Then, for $n \geq 2$:*

(i) $|a_n| + |b_n| \leq \dfrac{2M}{n(n-1)}$;

(ii) $||a_n| - |b_n|| \leq \dfrac{2M}{n(n-1)}$;

(iii) $|a_n| \leq \dfrac{2M}{n(n-1)}$.

The result is sharp for the function f given by $f'(z) = 1 - 2M\ln(1-z)$.

Proof. Let $f \in \mathcal{P}^0_{\mathcal{H}}(M)$ for $M > 0$. Then, by Theorem 2.1, $F_\epsilon = h + \epsilon g$ belongs to $\mathcal{P}(M)$ for each $|\epsilon| = 1$. Thus, for each $|\epsilon| = 1$,

$$\operatorname{Re}(zF''_\epsilon(z)) = z(h''(z) + \epsilon g''(z)) > -M.$$

This implies that there exists an analytic function p of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ with $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$ such that

$$\frac{zF''_\epsilon(z) + M}{M} = p(z). \tag{2.2}$$

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Comparing coefficients on both sides of (2.2),

$$n(n+1)(a_{n+1}+\epsilon b_{n+1}) = Mp_n \quad \text{for } n \geq 1. \tag{2.3}$$

Since $|p_n| \leq 2$ for $n \geq 1$ and ϵ ($|\epsilon| = 1$) is arbitrary, it follows from (2.3) that

$$n(n+1)(|a_{n+1}| + |b_{n+1}|) \leq 2M,$$
$$n(n+1)||a_{n+1}| - |b_{n+1}|| \leq 2M,$$

which implies that

$$|a_n| \leq \frac{2M}{n(n-1)} \quad \text{for } n \geq 2.$$

It is easy to see that the function f defined by $f'(z) = 1 - 2M\ln(1-z))$ belongs to the class $\mathcal{P}^0_{\mathcal{H}}(M)$ and that all three inequalities are sharp. □

Theorem 2.4. *Let $f = h + \overline{g} \in \mathcal{P}^0_{\mathcal{H}}(M)$ for $M > 0$ be of the form (1.2). Then*

$$|z| - 2M\sum_{n=2}^{\infty}\frac{|z|^n}{n(n-1)} \leq |f(z)| \leq |z| + 2M\sum_{n=2}^{\infty}\frac{|z|^n}{n(n-1)}.$$

The right-hand inequality is sharp for the function f given by $f'(z) = 1 - 2M\ln(1-z)$.

Proof. Let $f = h + \overline{g}$ be in the class $\mathcal{P}^0_{\mathcal{H}}(M)$. Then $F_\epsilon = h + \epsilon g$ belongs to $\mathcal{P}(M)$ and, for each ϵ with $|\epsilon| = 1$,

$$\operatorname{Re}(zF''_\epsilon(z)) = z(h''(z) + \epsilon g''(z)) > -M \quad \text{for } z \in \mathbb{D}.$$

Thus, there exists an analytic function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{D}$ such that

$$\frac{zF''_\epsilon(z) + M}{M} = \frac{1+\omega(z)}{1-\omega(z)}. \tag{2.4}$$

Simplification of (2.4) gives

$$F''_\epsilon(z) = \frac{2M\omega(z)}{z(1-\omega(z))}. \tag{2.5}$$

Let

$$\frac{\omega(z)}{1-\omega(z)} = \sum_{n=1}^{\infty} w_n z^n$$

map the unit disk $\mathbb{D}$ onto a convex domain. By subordination, $|w_n| \leq 1$ for $n \geq 1$. Therefore, from (2.5),

$$\begin{aligned}|F'_\epsilon(z)| &= \left|1 + 2M\int_0^z \frac{\omega(\xi)}{\xi(1-\omega(\xi))}\,d\xi\right| = \left|1 + 2M\int_0^z \sum_{n=1}^{\infty} w_n\xi^{n-1}\,d\xi\right| \\ &= \left|1 + 2M\int_0^z \sum_{n=1}^{\infty} w_n t^{n-1}e^{i(n-1)\theta}e^{i\theta}\,dt\right| \\ &\leq 1 + 2M\int_0^{|z|}\sum_{n=1}^{\infty}|w_n|t^{n-1}\,dt = 1 + 2M\sum_{n=1}^{\infty}\frac{|z|^n}{n}.\end{aligned}$$

Thus,

$$|F'_\epsilon(z)| = |h'(z) + \epsilon g'(z)| \le 1 + 2M \sum_{n=1}^{\infty} \frac{|z|^n}{n}. \tag{2.6}$$

Since ϵ ($|\epsilon| = 1$) is arbitrary, it follows from (2.6) that

$$|h'(z)| + |g'(z)| \le 1 + 2M \sum_{n=1}^{\infty} \frac{|z|^n}{n}.$$

Let Γ be the radial segment from 0 to z. Then

$$\begin{aligned}|f(z)| &= \left| \int_\Gamma \frac{\partial f}{\partial \xi}\, d\xi + \frac{\partial f}{\partial \bar{\xi}}\, d\bar{\xi} \right| \le \int_\Gamma (|h'(\xi)| + |g'(\xi)|)\, |d\xi| \\ &\le \int_0^{|z|} \left(1 + 2M \sum_{n=1}^{\infty} \frac{|t|^n}{n}\right) dt = |z| + 2M \sum_{n=2}^{\infty} \frac{|z|^n}{n(n-1)}.\end{aligned}$$

From (2.5),

$$|F'_\epsilon(z) - 1| = \left| 2M \int_0^z \frac{\omega(\xi)}{\xi(1-\omega(\xi))}\, d\xi \right| = \left| 2M \int_0^z \sum_{n=1}^{\infty} w_n \xi^{n-1} \right| d\xi \le 2M \sum_{n=1}^{\infty} \frac{|z|^n}{n}. \tag{2.7}$$

From (2.7),

$$||F'_\epsilon(z)| - 1| \le |F'_\epsilon(z) - 1| \le 2M \sum_{n=1}^{\infty} \frac{|z|^n}{n}$$

and so

$$|F'_\epsilon(z)| = |h'(z) + \epsilon g'(z)| \ge 1 - 2M \sum_{n=1}^{\infty} \frac{|z|^n}{n}. \tag{2.8}$$

Since ϵ ($|\epsilon| = 1$) is arbitrary, it follows from (2.8) that

$$|h'(z)| - |g'(z)| \ge 1 - 2M \sum_{n=1}^{\infty} \frac{|z|^n}{n}. \tag{2.9}$$

In view of (2.9),

$$\begin{aligned}|f(z)| &= \int_\Gamma \left| \frac{\partial f}{\partial \xi}\, d\xi + \frac{\partial f}{\partial \bar{\xi}}\, d\bar{\xi} \right| \ge \int_\Gamma (|h'(\xi)| - |g'(\xi)|)\, |d\xi| \\ &\ge \int_0^{|z|} \left(1 - 2M \sum_{n=1}^{\infty} \frac{|t|^n}{n}\right) dt = |z| - 2M \sum_{n=2}^{\infty} \frac{|z|^n}{n(n-1)}. \qquad \square\end{aligned}$$

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3. Two-point distortion theorem

In this section, we concentrate on a two-point distortion theorem for functions in the class $\mathcal{B}^0_{\mathcal{H}}(M)$. The basic analytic and geometrical properties of functions in the class $\mathcal{B}^0_{\mathcal{H}}(M)$ were explored in [6].

The following result due to Bazilevich [2] gives a necessary and sufficient condition for a normalised analytic function to be univalent in $\mathbb{D}$.

THEOREM 3.1 [2]. *Let $\phi(z) = z + \sum_{k=2}^{\infty} a_k z^n$ be analytic on $\mathbb{D}$. Then ϕ is univalent on $\mathbb{D}$ if and only if for each $z \in \mathbb{D}$ and for each $t \in [0, \pi/2]$,*

$$\frac{\phi(re^{i\zeta}) - \phi(re^{i\psi})}{re^{i\zeta} - re^{i\psi}} := \sum_{k=1}^{\infty} a_k \frac{\sin kt}{\sin t} z^{k-1} \neq 0,$$

where $t = (\zeta - \psi)/2$, $z = re^{i(\zeta+\psi)/2}$ and $(\sin kt/\sin t)|_{t=0} = k$.

Starkov [14] generalised Theorem 3.1 to normalised sense-preserving harmonic maps on $\mathbb{D}$.

THEOREM 3.2 [14]. *A sense-preserving harmonic mapping $f = h + \overline{g}$ defined on $\mathbb{D}$ determined by (1.1) is univalent if and only if for each $z \in \mathbb{D} \setminus \{0\}$ and for each $t \in (0, \pi/2]$,*

$$\frac{f(re^{i\zeta}) - f(re^{i\psi})}{re^{i\zeta} - re^{i\psi}} := \sum_{k=1}^{\infty} (a_k z^k - \overline{b_k z^k}) \frac{\sin kt}{\sin t} \neq 0, \tag{3.1}$$

where $t = (\zeta - \psi)/2$ and $z = re^{i(\zeta+\psi)/2}$.

The following two-point distortion theorem plays a vital role in the proof of Theorem 3.4.

THEOREM 3.3 [8]. *If $\phi \in \mathcal{S}$, $r \in (0, 1)$ and $t, \psi \in \mathbb{R}$, then*

$$\left| \frac{\phi(re^{it}) - \phi(re^{i\psi})}{re^{it} - re^{i\psi}} \right| \geq \frac{1 - r^2}{r^2} |\phi(re^{it})||\phi(re^{i\psi})|.$$

We now give our result on functions in $\mathcal{B}^0_{\mathcal{H}}(M)$.

THEOREM 3.4. *Let $f = h + \overline{g} \in \mathcal{B}^0_{\mathcal{H}}(M)$, $0 \leq M \leq 1$. Then*

$$\left| \frac{f(re^{it}) - f(re^{i\psi})}{re^{it} - re^{i\psi}} \right| \geq (1 - r^2)\left(1 - Mr + \frac{M^2 r^2}{4}\right).$$

PROOF. Let $f = h + \overline{g}$ be in $\mathcal{B}^0_{\mathcal{H}}(M)$ for $M > 0$. Consider $F = h + \epsilon g \in \mathcal{B}(M)$ with $|\epsilon| = 1$, where the class $\mathcal{B}(M)$ (see [12]) is defined by

$$\mathcal{B}(M) := \{\phi \in \mathcal{A} : |z\phi''(z)| \leq M \text{ for } z \in \mathbb{D}\}.$$

We note that the functions in $\mathcal{B}(M)$ are univalent for $M > 0$. For every pair of points re^{it} and $re^{i\psi}$, we can find an ϵ with $|\epsilon| = 1$ such that

$$(h(re^{it}) - h(re^{i\psi})) + \overline{(g(re^{it}) - g(re^{i\psi}))} = (h(re^{it}) - h(re^{i\psi})) + \epsilon(g(re^{it}) - g(re^{i\psi})).$$

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Therefore,

$$\left|\frac{f(re^{it}) - f(re^{i\psi})}{re^{it} - re^{i\psi}}\right| = \left|\frac{F(re^{it}) - F(re^{i\psi})}{re^{it} - re^{i\psi}}\right|. \tag{3.2}$$

From Theorem 3.3, (3.2) implies that

$$\left|\frac{f(re^{it}) - f(re^{i\psi})}{re^{it} - re^{i\psi}}\right| = \left|\frac{F(re^{it}) - F(re^{i\psi})}{re^{it} - re^{i\psi}}\right| \geq \frac{1 - r^2}{r^2}|F(re^{it})||F(re^{i\psi})|. \tag{3.3}$$

In order to complete the proof, we need to find a lower bound for $|F(z)|$.

Since $F \in \mathcal{B}(M)$, it follows from the definition that

$$|zF''(z)| \leq M.$$

Thus, there exists a Schwarz function $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that

$$zF''(z) = M\omega(z). \tag{3.4}$$

From (3.4),

$$F''(z) = \frac{M\omega(z)}{z}$$

and so

$$|F'(z) - 1| = \left|\int_0^z \frac{M\omega(\xi)}{\xi}\, d\xi\right|. \tag{3.5}$$

Since ω is a Schwarz function, $|\omega(z)| \leq |z|$ for $z \in \mathbb{D}$. Hence, from (3.5),

$$|F'(z) - 1| \leq M|z|$$

and therefore

$$||F'(z)| - 1| \leq M|z|,$$

which implies that

$$|F'(z)| \geq 1 - M|z|.$$

Let Γ be the radial segment from 0 to $z = re^{i\theta}$. Then

$$\begin{aligned} |F(z)| = \int_\Gamma |F'(\xi)||\, d\xi| &\geq \int_\Gamma (1 - M|\xi|)\,|d\xi| \\ &= \int_0^r (1 - Mt)\, dt = r - M\frac{r^2}{2} = |z| - M\frac{|z|^2}{2}. \end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.3) gives

$$\begin{aligned} \left|\frac{f(re^{it}) - f(re^{i\psi})}{re^{it} - re^{i\psi}}\right| &= \left|\frac{F(re^{it}) - F(re^{i\psi})}{re^{it} - re^{i\psi}}\right| \\ &\geq \frac{1 - r^2}{r^2}\Big(r - M\frac{r^2}{2}\Big)^2 = (1 - r^2)\Big(1 - Mr + \frac{M^2r^2}{4}\Big). \end{aligned}$$

□

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THEOREM 3.5. *Let $f = h + \overline{g} \in \mathcal{B}^0(M)$ be of the form (1.2). Then $s_{n,m}(f)$ is univalent in $|z| < r_{n,m}$, where $r_{n,m}$ is the unique positive root of the equation $\nu(n, m, r, M) = 0$ in* $(0, 1)$ *and*

$$\nu(n, m, r, M) = A(r, M) - \sum_{k=n+1}^{\infty} \frac{M}{k-1} r^{k-1} - \sum_{k=m+1}^{\infty} \frac{M}{k-1} r^{k-1} \tag{3.7}$$

with

$$A(r, M) = (1 - r^2)\Big(1 - Mr + \frac{M^2 r^2}{4}\Big).$$

PROOF. Suppose that $f = h + \overline{g} \in \mathcal{B}^0(M)$ with series representation given by (1.2). Set

$$F_{r,\epsilon} = \frac{h(rz)}{r} + \epsilon \frac{g(rz)}{r}$$

for $0 < r < 1$ and $|\epsilon| = 1$. Then

$$F_{r,\epsilon} = z + \sum_{k=2}^{\infty} a_k r^{k-1} z^k + \epsilon\Big(\sum_{k=2}^{\infty} b_k r^{k-1} z^k\Big).$$

To prove that $s_{n,m}(f)$ is univalent in $|z| < r_{n,m}$, it is sufficient to show that $s_{n,m}(F_{r,\epsilon})$ is univalent in $\mathbb{D}$ for all ϵ with $|\epsilon| = 1$. Using Theorem 3.1, we see that $s_{n,m}(F_{r,\epsilon})$ is univalent if and only if the associated section $P_{n,m,r,M}$ has the property that

$$P_{n,m,r,M} = \sum_{k=1}^{M} (a'_k z^{k-1} + b'_k \overline{z}^{k-1}) \frac{\sin kt}{\sin t} \neq 0 \quad \text{for all } z \in \mathbb{D} \setminus \{0\} \text{ and } t \in [0, \pi/2],$$

where $M = \max\{n, m\}$, $l = \min\{n, m\}$ and $a'_k = a_k r^{k-1}$, $b'_k = \epsilon b_k r^{k-1}$ for $k \le l$,

$$a'_k = \begin{cases} a_k r^{k-1} & \text{for all } k > l \text{ if } M = n, \\ 0 & \text{for all } k > l \text{ if } M > n \end{cases}$$

and

$$b'_k = \begin{cases} \epsilon b_k r^{k-1} & \text{for all } k > l \text{ if } M = m, \\ 0 & \text{for all } k > l \text{ if } M > m. \end{cases}$$

By letting $t = (\zeta - \psi)/2$, $z = re^{i(\zeta+\psi)/2}$ in (3.1) and, using the univalence of $F_{r,\epsilon}$ for $0 < r < 1$,

$$\begin{aligned} \left| \frac{F_{r,\epsilon}(re^{i\zeta}) - F_{r,\epsilon}(re^{i\psi})}{re^{i\zeta} - re^{i\psi}} \right| &= \left| \sum_{k=1}^{\infty} (a_k z^{k-1} + \epsilon b_k \overline{z}^{k-1}) r^{k-1} \frac{\sin kt}{\sin t} \right| \\ &\ge (1 - r^2)\Big(1 - Mr + \frac{M^2 r^2}{4}\Big). \end{aligned} \tag{3.8}$$

In order to find the lower bound for $|P_{n,m,r,M}|$, we need to find an upper bound for

$$|Q_{n,m,r,M}(z)| = \left| \sum_{k=n+1}^{\infty} a_k r^{k-1} z^{k-1} \frac{\sin kt}{\sin t} + \sum_{k=m+1}^{\infty} \epsilon b_k r^{k-1} \overline{z}^{k-1} \frac{\sin kt}{\sin t} \right|.$$

It follows from [6, Theorem 2.2] that

$$
\begin{aligned}
|Q_{n,m,r,M}(z)| &\le \sum_{k=n+1}^{\infty} |a_k||r^{k-1}|\left|\frac{\sin kt}{\sin t}\right| + \sum_{k=m+1}^{\infty} |b_k||r^{k-1}|\left|\frac{\sin kt}{\sin t}\right| \\
&\le \sum_{k=n+1}^{\infty} \frac{M}{k(k-1)} r^{k-1} k + \sum_{k=m+1}^{\infty} \frac{M}{k(k-1)} r^{k-1} k \\
&= \sum_{k=n+1}^{\infty} \frac{M}{k-1} r^{k-1} + \sum_{k=m+1}^{\infty} \frac{M}{k-1} r^{k-1} \\
&= R_n(M) + T_m(M).
\end{aligned} \tag{3.9}
$$

From (3.8) and (3.9),

$$
|P_{n,m,r,M}| \ge (1-r^2)\left(1 - Mr + \frac{M^2 r^2}{4}\right) - R_n(M) - T_m(M) = \nu(n,m,r,M).
$$

Therefore, $P_{n,m,r,M} \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$ provided $\nu(n,m,r,M) > 0$, where $\nu(n,m,r,M)$ is defined by (3.7). This shows that $\nu(n,m,r,M) > 0$ for all $r \in (0, r_{n,m})$, where $r_{n,m}$ is the positive root of the equation $\nu(n,m,r,M) = 0$ which lies in $(0,1)$. Thus, $s_{n,m}(F_\epsilon(r,z))$ is univalent in the disk $|z| < r_{n,m}$ for all ϵ with $|\epsilon| = 1$, which implies that $s_{n,m}(f)$ is univalent in $|z| < r_{n,m}$. □

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