Bull. Aust. Math. Soc. **94** (2016), 353–361
doi:10.1017/S0004972716000575

THE DIMENSION OF CENTRALISERS OF MATRICES OF ORDER n

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(Received 3 May 2016; accepted 7 June 2016; first published online 26 September 2016)

Abstract

In this paper, we study the integer sequence $(E_n)_{n\geq 1}$, where E_n counts the number of possible dimensions for centralisers of $n\times n$ matrices. We give an example to show another combinatorial interpretation of E_n and present an implicit recurrence formula for E_n, which may provide a fast algorithm for computing E_n. Based on the recurrence, we obtain the asymptotic formula $E_n = \frac{1}{2}n^2 - \frac{2}{3}\sqrt{2}n^{3/2} + O(n^{5/4})$.

2010 *Mathematics subject classification*: primary 05A05; secondary 05A16, 05A17.

Keywords and phrases: centraliser, asymptotic enumeration, partition, commutator.

1. Introduction

Throughout this paper, $M_{n\times n}(\mathbb{C})$ denotes the algebra of $n\times n$ matrices over the complex field $\mathbb{C}$, $n\in\mathbb{N}^+$. For a given $A\in M_{n\times n}(\mathbb{C})$, the set $C(A) = \{B\in M_{n\times n}(\mathbb{C}) : AB = BA\}$ is called the *centraliser* (or *commutator*) of A, which could be regarded as a linear space over $\mathbb{C}$. An unusual but efficient theory to study centralisers is the Weyr structure, which could be seen as a modified Jordan form (see [4, Ch. 2]). Utilising a result which states that for a nilpotent matrix of order n, $\dim C(A) = n_1^2 + n_2^2 + \cdots + n_r^2$, where $(n_1, n_2, \ldots, n_r)$ is the Weyr structure of A (see [4, Proposition 3.2.2]), one can readily obtain the following result.

THEOREM 1.1. *For any $n\in\mathbb{N}^+$,*

$$\{\dim C(A) \mid A\in M_{n\times n}(\mathbb{C})\} = \mathcal{U}_n, \tag{1.1}$$

where

$$\mathcal{U}_n = \Big\{\sum_{i=1}^{k} n_i^2 \,\Big|\, \sum_{i=1}^{k} n_i = n,\ \text{where } n_1,\ldots,n_k\in\mathbb{N},\ k = 1,2,\ldots,n\Big\}.$$

Following the notation in [7], we write

$$E_n = \#\mathcal{U}_n, \tag{1.2}$$

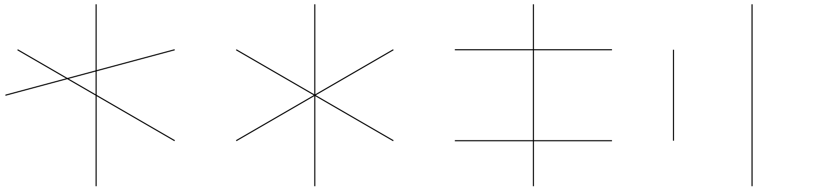

FIGURE 1. Partitions via three lines.

where $\#\mathcal{U}_n$ counts the number of elements of $\mathcal{U}_n$, $n = 1, 2, \ldots$. The sequence $(E_n)_{n\geq 1}$ is A069999 in the OEIS (Online Encyclopaedia of Integer Sequences) [6] but is only defined there in terms of commutators of matrices via (1.1) and (1.2).

Theorem 1.1 loads E_n with significance and we will give a new proof which only involves the Jordan form in Section 2. In fact, $(E_n)_{n\geq 1}$ reflects the partitions of n to some extent and it is an interesting sequence which has plenty of contact with other mathematical problems. In [1], Brouder *et al.* showed the application of the partitions of n to the enumeration of connected Feynman graphs in quantum field theory. We give an example to show the combinatorial interpretations of E_n.

PROPOSITION 1.2. *Consider the partition number set*

$$\mathcal{P}_n := \{m \in \mathbb{N}^+ \mid \exists\ n \text{ lines in general position dividing the plane into } m \text{ parts}\},$$

where general position means that no three lines pass through a common point, $n = 1, 2, \ldots$. *Then*

$$\mathcal{P}_n = \left\{\frac{n^2 + 2n + 2 - k}{2} \;\middle|\; k \in \mathcal{U}_n\right\} \quad \text{and} \quad \#\mathcal{P}_n = E_n.$$

REMARK 1.3. The phrase 'general position' means no three lines intersect at a common point but we do allow lines to be parallel. Thus, two lines can partition the plane into three or four parts (depending on whether or not they are parallel). So, $\#\mathcal{P}_2 = 2$. If we further exclude parallel lines, then we get the 'lazy caterer's sequence' (OEIS A000124) [5].

Figure 1 shows all the possible positional relationships and partitions for three lines. The second is not admitted in our setting. We can easily see that $\mathcal{P}_3 = \{7, 6, 4\}$.

REMARK 1.4. If we remove the condition 'general position', then the cardinality of the corresponding set $\overline{\mathcal{P}}_n := \{m \in \mathbb{N}^+ :$ there exist n lines dividing the plane into m parts$\}$ may not equal E_n. For example, Figure 2 shows a set of six lines which divide the plane into 17 parts, that is, $17 \in \overline{\mathcal{P}}_6$. However, from Proposition 1.2, $17 \notin \mathcal{P}_6$ (see Table A.1). Since $\mathcal{P}_n \subset \overline{\mathcal{P}}_n$, we have $\mathcal{P}_6 \subsetneq \overline{\mathcal{P}}_6$ and thus $\#\overline{\mathcal{P}}_6 > \#\mathcal{P}_6 = E_6$. In fact, the minimal n with $\mathcal{P}_n \neq \overline{\mathcal{P}}_n$ is 6.

QUESTION 1.5. Some open problems about the partition problem are:

(1) how to characterise $\overline{\mathcal{P}}_n$ and calculate $\#\overline{\mathcal{P}}_n$;
(2) what is the set of possible partition numbers for partitions of $\mathbb{R}^3$ via n planes?

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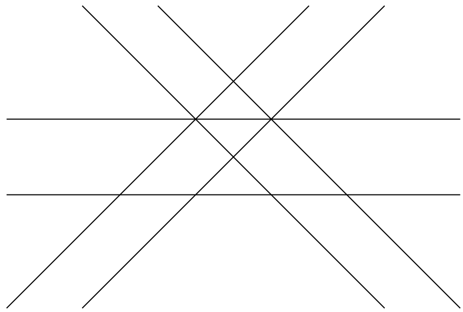

FIGURE 2. An exception based on a configuration of six lines.

Since we have already seen the importance of E_n, it is natural to ask whether E_n can be calculated accurately. Both the generating function and an explicit recurrence formula are unknown. There is a program which finds all the unordered partitions of n, calculates the quadratic sums and then counts the number of different sums [6]. However, the number of unordered partitions of n is asymptotic to $(4n\sqrt{3})^{-1}e^{\pi\sqrt{2n/3}}$ (see [2]), so the program is inefficient when n is sufficiently large. To overcome these problems, we give an implicit recurrence formula for E_n (Theorem 1.6). Based on this recurrence formula, we obtain a program which is much more efficient.

As we cannot find an exact formula of E_n, we look instead for an asymptotic estimate. Savitt and Stanley [7] showed that the dimension of the space spanned by characters of the symmetric powers of the standard n-dimensional representation of S_n is asymptotic to $\frac{1}{2}n^2$ and indirectly gave a lower bound of $\frac{1}{2}n^2 - cn^{3/2}$ for E_n, where c is a positive constant. O'Donovan [3] extended their work and demonstrated the wide applicability of the methods used in [7]. One can easily see that $E_n \le \frac{1}{2}n(n-1)+1$. In this paper, we give a more precise bound for E_n in Theorem 1.7. This also gives an asymptotic formula for E_n.

In order to prove Theorem 1.7 using the results in [7], we introduce the notation

$$C_n = \Big\{ \sum_{i=1}^{k} \binom{n_i}{2} \;\Big|\; \sum_{i=1}^{k} n_i = n, \text{ where } n_1,\dots,n_k \in \mathbb{N},\ k=1,2,\dots,n \Big\},$$

where we adopt the standard symbol $\binom{m}{2} = \frac{1}{2}m(m-1)$ for $m \in \mathbb{N}$. It is easy to see that $\#C_n = \#\mathcal{U}_n = E_n$, $n = 1,2,\dots$.

THEOREM 1.6. *Let $\mathcal{U}_0 = \{0\}$. Then we have the identities*

$$\mathcal{U}_n = \bigcup_{i=1}^{\lfloor n/2 \rfloor} (\mathcal{U}_i + \mathcal{U}_{n-i}) \cup \{n^2\} = \bigcup_{i=0}^{n-1} (\mathcal{U}_i + \{(n-i)^2\}) = \bigcup_{i=1}^{n} (\{i^2\} + \mathcal{U}_{n-i})$$

and

$$\mathcal{U}_n = \bigcup_{i=1}^{\lfloor n/2 \rfloor} (\{i^2\} + \mathcal{U}_{n-i}) \cup \{n^2\}, \tag{1.3}$$

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where $\lfloor n/2 \rfloor$ *means the largest integer which does not exceed* $n/2$, *and* $\mathcal{U}_i + \mathcal{U}_{n-i} := \{v + u : v \in \mathcal{U}_i,\ u \in \mathcal{U}_{n-i}\}$, $\{i^2\} + \mathcal{U}_{n-i} := \{i^2 + u : u \in \mathcal{U}_{n-i}\}$ *use the notion of the sum of two sets of integers,* $n \in \mathbb{N}$. *Furthermore, the same equalities hold for* C_n.

Based on Theorem 1.6, we not only get some implicit recurrence formulae for E_n, but also obtain a quick algorithm to compute E_n. Moreover, we provide a Mathematica program (see Appendix) using (1.3) in Theorem 1.6.

Finally, our main result is a sharp estimation for E_n.

Theorem 1.7. *There exist* $c_1, c_2 > 0$ *such that*

$$\frac{n^2}{2} - \frac{2}{3}\sqrt{2}n^{3/2} + c_1 n \geq E_n \geq \frac{n^2}{2} - \frac{2}{3}\sqrt{2}n^{3/2} - c_2 n^{5/4} \tag{1.4}$$

for any $n \in \mathbb{N}^+$. *In particular,* $E_n = \frac{1}{2}n^2 - \frac{2}{3}\sqrt{2}n^{3/2} + O(n^{5/4})$.

2. Proofs

Proof of Proposition 1.2. It can be easily proved that the number of parts via k sets of parallel lines in which the ith set contains n_i parallel lines is $\sum_{1\leq i<j\leq k} n_i n_j + \sum_{i=1}^{k} n_i + 1$. Since $\sum_{i=1}^{k} n_i = n$,

$$\sum_{1\leq i<j\leq k} n_i n_j + \sum_{i=1}^{k} n_i + 1 = \frac{1}{2}\Big(n^2 - \sum_{i=1}^{k} n_i^2\Big) + n + 1.$$

Consequently, $\mathcal{P}_n = \{\frac{1}{2}(n^2 + 2n + 2 - m) \mid m \in \mathcal{U}_n\}$. □

Proof of Theorem 1.1. Let $D(A, B) = \{K \in M_{n\times n}(\mathbb{C}) \mid AK = KB\}$, $A, B \in M_{n\times n}(\mathbb{C})$. We can easily verify that $C(A)$ and $D(A, B)$ are linear spaces.

Claim 2.1. *Let* $A = \operatorname{diag}(A_1, A_2, \ldots, A_n)$ *and* $B = (B_{ij})_{n\times n}$. *Then* $AB = BA$ *if and only if* $B_{ij} \in D(A_i, A_j)$ *for* $i, j \in \{1, 2, \ldots, n\}$. *Moreover,* $\dim C(A) = \sum_{i=1}^{n}\sum_{j=1}^{n} \dim D(A_i, A_j)$.

Proof of Claim 2.1. An elementary calculation shows that

$$AB = BA \Leftrightarrow A_i B_{ij} = B_{ij} A_j,\ i, j = 1, 2, \ldots, n \Leftrightarrow B_{ij} \in D(A_i, A_j),\ i, j = 1, 2, \ldots, n.$$

Let $\{B_{ij,k}\}_{k=1}^{\dim D(A_i,A_j)}$ be a base of $D(A_i, A_j)$. Let

$$E(B_{ij,k}) = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & B_{ij,k} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

be the block matrix whose (i, j)th block is $B_{ij,k}$ and all others are zero. Then

$$\{E(B_{ij,k}) \mid i, j = 1, 2, \ldots, n;\ k = 1, \ldots, \dim D(A_i, A_j)\}$$

is a base of $C(A)$ and thus $\dim C(A) = \sum_{i=1}^{n}\sum_{j=1}^{n} \dim D(A_i, A_j)$. □

Claim 2.2. *If $AX = XB$, then $g(A)X = Xg(B)$ for any polynomial g. Furthermore, if A and B have no common eigenvalues, then the solution of $AX = XB$ is $X = 0$.*

Proof of Claim 2.2. By $AX = XB$, $A^nX = A^{n-1}XB = \cdots = XB^n$ and so $g(A)X = Xg(B)$. Let the eigenvalues of A be $a_1, a_2, \ldots, a_n$, and let the eigenvalues of B be $b_1, b_2, \ldots, b_n$. Assume that f is a characteristic polynomial of B. Then $f(a_1), f(a_2), \ldots, f(a_n)$ and $f(b_1), f(b_2), \ldots, f(b_n)$ are the eigenvalues of $f(A)$ and $f(B)$, respectively. Note that $f(b_i) = 0$ and $f(a_i) \neq 0$, $i = 1, 2, \ldots, n$. Consequently, $f(A)$ is invertible and then $X = 0$. □

Claim 2.3. *Let J_1 be a $p \times p$ Jordan block and let J_2 be a $q \times q$ Jordan block. Then*

$$\dim D(J_1, J_2) = \begin{cases} \min\{p, q\} & \text{if } \lambda_1 = \lambda_2, \\ 0 & \text{if } \lambda_1 \neq \lambda_2, \end{cases}$$

where λ_1 and λ_2 are the eigenvalues of J_1 and J_2, respectively.

Proof of Claim 2.3 . Since $\dim D(J_1, J_2) = \dim D(J_2, J_1)$, without loss of generality, we can assume that $p \leq q$. If $\lambda_1 = \lambda_2$, the following $p \times q$ matrices form a basis of $D(J_1, J_2)$:

$$\begin{pmatrix} & 0 & 0 & 0 & 0 & 1 \\ & & 0 & 0 & 0 & 0 \\ 0 & & & \vdots & \vdots & \vdots \\ 0 & 0 & & & 0 & 0 \end{pmatrix}, \begin{pmatrix} & 0 & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 0 & 1 \\ 0 & & & \vdots & \vdots & \vdots \\ 0 & 0 & & & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} & 0 & 1 & 0 & 0 & 0 \\ & & 0 & 1 & 0 & 0 \\ 0 & & & \vdots & \vdots & \vdots \\ 0 & 0 & & & 0 & 1 \end{pmatrix}.$$

If $\lambda_1 \neq \lambda_2$, then we can easily get $D(J_1, J_2) = 0$ by Claim 2.2. □

Suppose that J is a matrix of order n and eigenvalues all the same which is constituted by Jordan blocks. Assume that there are k_i blocks, J_i, of order i, for $i = 0, 1, \ldots, n$. By Claims 2.1 and 2.3,

$$\begin{aligned}
\dim C(J) &= \sum_{i=1}^{n} \sum_{j=1}^{n} k_i k_j \dim D(J_i, J_j) \\
&= \sum_{i=1}^{n} k_i^2 \dim D(J_i, J_i) + 2 \sum_{1 \leq i < j \leq n} k_i k_j \dim D(J_i, J_j) \\
&= \sum_{i=1}^{n} i(k_i^2 + 2k_i k_{i+1} + \cdots + 2k_i k_n) \\
&= \sum_{i=1}^{n-1} i((k_i + k_{i+1} + \cdots + k_n)^2 - (k_{i+1} + \cdots + k_n)^2) + nk_n^2 \\
&= \sum_{i=1}^{n} (k_i + k_{i+1} + \cdots + k_n)^2 = \sum_{i=1}^{n} n_i^2,
\end{aligned}$$

where $n = \sum_{i=1}^{n} ik_i = \sum_{i=1}^{n} (k_i + k_{i+1} + \cdots + k_n) = \sum_{i=1}^{n} n_i$ and $n_i := \sum_{j=i}^{n} k_j$. Thus,

$$\{\dim C(J) : J \text{ is } n \times n \text{ Jordan matrix with eigenvalues all the same}\} = \mathcal{U}_n.$$

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For a general Jordan matrix J, assume that J has m different eigenvalues, $\lambda_1, \dots, \lambda_m$, and suppose that the multiplicity of λ_i is n_i. Then $n = \sum_{i=1}^m n_i$. According to Claim 2.3, $\{\dim C(J) \mid J \text{ is } n \times n \text{ Jordan matrix}\} = \{\sum_{i=1}^m n_i \mid n_i \in \mathcal{U}_{n_i}\} := \mathcal{U}_n$.

For a general matrix A, there are a Jordan matrix J and an invertible matrix P such that $A = PJP^{-1}$. Then $C(A) = \{PBP^{-1} : B \in C(J)\}$. Therefore, $\dim C(A) = \dim C(J)$ and then

$$\{\dim C(A) : A \in M_{n\times n}(\mathbb{C})\} = \{\dim C(J) : J \text{ is } n \times n \text{ Jordan matrix}\} = \mathcal{U}_n.$$

This completes the proof. □

Proof of Theorem 1.6. We only prove the equality (1.3). Let

$$\mathcal{U}_{n,k} = \Big\{\sum_{i=1}^{k} n_i^2 \;\Big|\; \sum_{i=1}^{k} n_i = n, \text{ where } n_1, n_2, \dots, n_k \in \mathbb{N}^+\Big\}, \quad k = 1, 2, \dots, n;\ n \in \mathbb{N}^+.$$

Then $\mathcal{U}_n = \bigcup_{k=1}^n \mathcal{U}_{n,k}$ and $\mathcal{U}_{n,1} = \{n^2\}$. If $k \ge 2$, there is a number $j \in \{1, 2, \dots, k\}$ such that $n_j \le \lfloor n/2 \rfloor$. Without loss of generality, we may assume that $n_1 \in \{1, 2, \dots, \lfloor n/2 \rfloor\}$. Therefore, $n - n_1 = \sum_{i=2}^k n_i$ and then $\mathcal{U}_n \setminus \mathcal{U}_{n,1} = \bigcup_{n_1=1}^{\lfloor n/2\rfloor}(\{n_1^2\} + \mathcal{U}_{n-n_1})$. Accordingly, $\mathcal{U}_n = \bigcup_{i=1}^{\lfloor n/2\rfloor}(\{i^2\} + \mathcal{U}_{n-i}) \cup \{n^2\}$. □

Proof of Theorem 1.7. First, for any $n \in \mathbb{N}^+$, C_n can be written as

$$C_n = \bigsqcup_{k=1}^{n} C_{n,k}, \tag{2.1}$$

where

$$C_{n,k} = \Big\{\sum_{i=1}^{l} \binom{n_i}{2} \;\Big|\; \sum_{i=1}^{l} n_i = n, \text{ where } 1 \le n_1 \le n_2 \le \cdots \le n_l = k,\ l = 1, 2, \dots, n-k+1\Big\}$$

corresponds to the sum $\sum_{i=1}^l \binom{n_i}{2}$ over all partitions $(n_1, \dots, n_l)$ of n with $\max_{1\le i\le l} n_i = k$. From the elementary inequalities $\binom{k}{2} \ge \binom{k-1}{2} \ge \cdots \ge \binom{2}{2} \ge \binom{1}{2}$ and $\binom{k}{2} \ge \binom{k-1}{2} + \binom{1}{2} \ge \cdots \ge \binom{\lfloor k/2\rfloor}{2} + \binom{k-\lfloor k/2\rfloor}{2}$, it follows that

$$C_{n,k} \subset \Big[\binom{k}{2}, \lfloor n/k\rfloor\binom{k}{2} + \binom{n - \lfloor n/k\rfloor k}{2}\Big], \quad k = 1, 2, \dots, n. \tag{2.2}$$

Therefore, a straightforward calculation simplifies (2.2) to

$$C_{n,k} \subset \Big[0, 2\binom{\lfloor n/2\rfloor}{2}\Big], \quad k = 1, \dots, \lfloor n/2\rfloor \tag{2.3}$$

and

$$C_{n,k} \subset \Big[\binom{k}{2}, \binom{k}{2} + \binom{n-k}{2}\Big], \quad k = \lfloor n/2\rfloor + 1, \dots, n. \tag{2.4}$$

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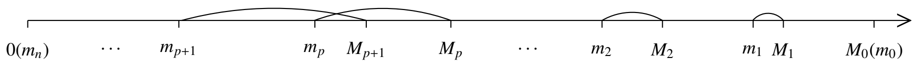

FIGURE 3. A figure used in the proof of Theorem 1.7.

For simplicity, we let $m_i = \binom{n-i}{2}$ and $M_i = \binom{i}{2} + \binom{n-i}{2}$, $i = 0, 1, \ldots, n$. Then, by (2.1), (2.3) and (2.4),

$$C_n = \bigsqcup_{i=1}^{n} C_{n,i} \subset \bigcup_{i=0}^{\lfloor n/2 \rfloor} [m_i, M_i] \cup [0, M_{\lfloor n/2 \rfloor}]. \tag{2.5}$$

Let p be the largest positive integer satisfying $M_p < m_{p-1} - 1$, that is, $\binom{p}{2} + \binom{n-p}{2} < \binom{n-p+1}{2} - 1$. It is easy to see that $p \approx \sqrt{2n}$. Figure 3 illustrates the relationship between M_i and m_i, $i \in \{0, 1, \ldots, n\}$. Obviously, $\lfloor n/2 \rfloor > p$ for sufficiently large n. So, without loss of generality, we may assume that $\lfloor n/2 \rfloor > p \approx \sqrt{2n}$. This, together with (2.5) and $M_p > M_{p+1} > \cdots > M_{\lfloor n/2 \rfloor}$, yields

$$C_n \subset \bigcup_{i=0}^{\lfloor n/2 \rfloor} [m_i, M_i] \cup [0, M_{\lfloor n/2 \rfloor}] \subset \bigcup_{i=0}^{p-1} [m_i, M_i] \cup [0, M_p]. \tag{2.6}$$

Let $\mathcal{A} = (\bigsqcup_{i=0}^{p-1} [m_i, M_i] \bigcup [0, M_p]) \bigcap \mathbb{N}$. Then (2.6) implies that $C_n \subset \mathcal{A}$. Hence, it follows from $E_n = \# C_n \leq \# \mathcal{A}$ and

$$\begin{aligned} \# \mathcal{A} &= \sum_{i=0}^{p-1} \left(\binom{i}{2} + 1 \right) + \binom{n-p}{2} + \binom{p}{2} + 1 = \binom{p+1}{3} + p + \binom{n-p}{2} + 1 \\ &= \frac{n^2}{2} - \frac{2}{3} \sqrt{2} n^{3/2} + \frac{n}{2} + o(n) \end{aligned}$$

that

$$E_n \leq \frac{n^2}{2} - \frac{2}{3} \sqrt{2} n^{3/2} + \frac{n}{2} + o(n). \tag{2.7}$$

According to Proposition 3.3 and the inequality (4) in [7], $G_n \geq \frac{1}{2} n^2 - c n^{3/2}$ for some $c > 0$, where $G_n + 1$ is the least positive integer that cannot be written in the form of $\sum_{i=1}^{k} \binom{n_i}{2}$ with $\sum_{i=1}^{k} n_i = n$. Thus,

$$[0, G_n] \cap \mathbb{N} \subset C_n. \tag{2.8}$$

From $M_p < m_{p-1} - 1$, we have $M_p + 1 \notin C_n$. Then the definition of G_n gives

$$[0, G_n] \subset [0, M_p]. \tag{2.9}$$

Let $\widetilde{\mathcal{A}} = (\bigsqcup_{i=0}^{p-1} [m_i, M_i] \bigcup \bigcup_{i=p}^{q} [m_i, M_i] \bigcup [0, G_n]) \bigcap \mathbb{N}$, where q is a given positive integer such that $M_q \leq G_n$. Then the definition of G_n gives

$$[0, M_q] \subset [0, G_n]. \tag{2.10}$$

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We should note that q can be chosen as $\lfloor 2c\sqrt{n}\rfloor$, since $M_{\lfloor 2c\sqrt{n}\rfloor} \le \frac{1}{2}n^2 - cn^{3/2} \le G_n$ for sufficiently large n. Hence, we may assume without loss of generality that $\lfloor n/2\rfloor > q \approx 2c\sqrt{n}$. From (2.9) and (2.10), we can obtain $[0, M_q] \subset [0, M_p]$, which yields $p \le q$. Therefore, $\bigcup_{i=p}^{q}[m_i, M_i] \subset [0, M_p]$. Combining this with (2.9) and the definitions of $\widetilde{\mathcal{A}}$ and $\mathcal{A}$, we derive $\widetilde{\mathcal{A}} \subset \mathcal{A}$.

Note that the definition of p and (2.10) yield

$$[0, M_p] = \bigcup_{i=p}^{q}[m_i, M_i]\bigcup[0, M_q] \subset \bigcup_{i=p}^{q}[m_i, M_i]\bigcup[0, G_n].$$

So, we immediately obtain $\mathcal{A} \subset \widetilde{\mathcal{A}}$ and thus

$$\mathcal{A} = \widetilde{\mathcal{A}} = \Big(\bigcup_{i=0}^{q}[m_i, M_i]\bigcup[0, G_n]\Big)\bigcap\mathbb{N}. \tag{2.11}$$

It follows from (2.11) and (2.8) that $\mathcal{A} \setminus C_n \subset \bigcup_{i=0}^{q}([m_i, M_i] \cap \mathbb{N} \setminus C_n)$. Combining this with

$$\begin{aligned}[m_i, M_i] \cap \mathbb{N} \setminus C_n &\subset [m_i, M_i] \cap \mathbb{N} \Big\backslash \Big(C_i + \Big\{\binom{n-i}{2}\Big\}\Big)\\ &= \Big(\Big[0, \binom{i}{2}\Big] \cap \mathbb{N} + \Big\{\binom{n-i}{2}\Big\}\Big) \Big\backslash \Big(C_i + \Big\{\binom{n-i}{2}\Big\}\Big) = \Big(\Big[0, \binom{i}{2}\Big] \cap \mathbb{N} \setminus C_i\Big) + \Big\{\binom{n-i}{2}\Big\}\end{aligned}$$

and

$$\#C_i \ge G_i \ge \frac{1}{2}i^2 - ci^{3/2} \ge \#\Big(\Big[0, \binom{i}{2}\Big] \cap \mathbb{N}\Big) - ci^{3/2},$$

we see that

$$\begin{aligned}\#(\mathcal{A} \setminus C_n) &\le \sum_{i=0}^{q}\#([m_i, M_i] \cap \mathbb{N} \setminus C_n) \le \sum_{i=0}^{q}\#\Big(\Big[0, \binom{i}{2}\Big] \cap \mathbb{N} \setminus C_i\Big)\\ &\le \sum_{i=0}^{q} ci^{3/2} \le \frac{2c}{5}(q+1)^{5/2} = \frac{1}{5}(2c)^{7/2}n^{5/4} + o(n^{5/4}).\end{aligned}$$

Therefore, $\#C_n = \#\mathcal{A} - \#(\mathcal{A} \setminus C_n) \ge \frac{1}{2}n^2 - \frac{2}{3}\sqrt{2}n^{3/2} + \frac{1}{2}n - Cn^{5/4} - o(n^{5/4})$, where $C := (2c)^{7/2}/5$. Accordingly,

$$E_n \ge \frac{1}{2}n^2 - \frac{2}{3}\sqrt{2}n^{3/2} - Cn^{5/4} - o(n^{5/4}). \tag{2.12}$$

So, we have proved (1.4) by (2.12) and (2.7). □

Appendix

Table A.1 gives some examples of $\mathcal{U}_n$ and $\mathcal{P}_n$, where $\mathcal{U}_n$ is obtained from its definition and $\mathcal{P}_n$ is obtained from Proposition 1.2.

The following Mathematica program is based on the recurrence formula (1.3) in Theorem 1.6.

https://doi.org/10.1017/S0004972716000575 Published online by Cambridge University Press

TABLE A.1. A table used in the Appendix.

n	$\mathcal{U}_n$	E_n	$\mathcal{P}_n$
1	1	1	2
2	2, 4	2	4, 3
3	3, 5, 9	3	7, 6, 4
4	4, 6, 8, 10, 16	5	11, 10, 9, 8, 5
5	5, 7, 9, 11, 13, 17, 25	7	16, 15, 14, 13, 12, 10, 6
6	6, 8, 10, 12, 14, 18, 20, 26, 36	9	22, 21, 20, 19, 18, 16, 15, 12, 7

```
U = {1}; A = {U}; m = 200;
For[n = 2, n <= m, n++,
    {U = {n^2}, For[i = 1, i <= n/2, i++,
      U = Union[A[[n - i]] + i^2, U]], A = Union[A, {U}]}];
Table[Length[A[[i]]], {i, 1, m}]
```

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https://doi.org/10.1017/S0004972716000575 Published online by Cambridge University Press