Compositio Mathematica **134:** 269–282, 2002.

Linear Systems of Cubics Singular at General Points of Projective Space

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(Received: 29 March 2001; accepted in final form: 25 July 2001)

Abstract. We present an elementary proof that given a general collection of d points in $\mathbb{P}^n$ the linear system of cubics singular on each point has the expected codimension except when $n = 4$ and $d = 7$. In that case the cubic is unique. This, together with previous work of the author, gives a proof of the Alexander–Hirschowitz interpolation theorem.

Mathematics Subject Classifications (2000). Primary 13D40, secondary 14F17.

Key words. Hilbert function, infinitesimal interpolation, singular points, special linear system.

1. Introduction

Let $\mathcal{K}$ be an infinite field and $\mathbb{P}^n = \mathbb{P}^n_{\mathcal{K}}$.
We prove the following result of J. Alexander and A. Hirschowitz:

THEOREM 1 ([AH3]). *Let Φ be a general collection of d points in $\mathbb{P}^n$. Then the codimension in $H^0(\mathcal{O}_{\mathbb{P}^n}(3))$ of the space of sections singular on Φ is equal to $\min\left((n+1)d, \binom{n+3}{3}\right)$ unless $n = 4$ and $d = 7$.*

This completes the proof begun in [C3] of the Alexander–Hirschowitz interpolation theorem comprised by [H], [A], [AH1], [AH2], [AH3]. The general theorem states that the codimension in $H^0(\mathcal{O}_{\mathbb{P}^n}(m))$ of the space of sections singular on a general collection of d points is as expected, with four exceptions (namely, $(n, m, d) = (2, 4, 5), (3, 4, 9), (4, 3, 7)$, and $(4, 4, 14)$).

The overall theorem has the immediate consequence of determining when a polynomial of given degree together with its partial derivatives may be interpolated to a general collection of points in affine space. Less obvious is Lasker's discovery ([L]) of the connection to the Waring problem for general linear forms. That problem asks: when is a general degree m form in $n+1$ variables expressible as a sum of mth powers of linear forms? More recently, R. Lazarsfeld observed and A. Iarrobino ([I]) made explicit the equivalence of this problem to that of Alexander and Hirschowitz, in appropriate characteristic. For example, if char $\mathcal{K} \neq 3$, a general cubic form in $\mathcal{K}[X_0, \ldots, X_n]$ may be written as a sum of d cubes of linear forms if $(n+1)d \geqslant (n+3)(n+2)(n+1)/6$ except when $n=4$ and $d=7$. The Waring problem in turn connects ([P], [T]) to the study of the variety of secant $(d-1)$-planes to the Veronese variety $v_m(\mathbb{P}^n) \subset |\mathcal{O}_{\mathbb{P}^n}(m)|$. (The reader should consult [G] and [IK] for very readable and detailed descriptions of these phenomena together with ample bibliographies on the activity in this area over the last few centuries.)

We present here an elementary proof of Theorem 1. The weighty techniques of blowing up and degeneration given in [AH3] are avoided. Instead we develop a direct method extendible to the study of linear systems describing higher order vanishing on a general collection of points, particularly in the (difficult) situations of low degrees.

Among the cases of the Alexander–Hirschowitz interpolation theorem, that of degree 3 is the most subtle. Each of the lines between pairs of points lies in the base locus of the linear system of cubics singular on a set of points. Hence the standard method (as described below) of specialising to a hyperplane has the complication that a line must meet a hyperplane. A key feature of the technique employed here is that the obstruction given by such lines is turned into an advantage.

This shows promise towards generalisation to the study of higher order singularities on a general collection of points. Specifically, Alexander and Hirschowitz (see [AH5]) show that the dimension of a linear system describing forms of given degree vanishing to specified orders at most k on a collection of points of $\mathbb{P}^n$ may be obtained inductively using the méthode d'Horace différentielle in sufficiently large degree (unspecified, and dependent on n and k). But in degree at most $2k-1$ the method is hindered, again, by linear constraints. Results verifying instances of a conjecture of R. Fröberg and A. Iarrobino in [I] in the case of degree $k+1$ are obtained in [C2], as here, by specialising to a codimension k plane (and bypassing blow-ups!). In [C4], we proceed on the case of multiplicity $k=3$ in the lower degrees by a method analogous to that presented here, thereby finding the codimension in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$ of sections vanishing to specified orders at most 3 on a general collection of points in degrees $m \geqslant 7$. Sharper results may be achieved, as here, by taking advantage of linear obstructions.

Let us start by rewording the problem in terms of schemes:

DEFINITION 1. Let $p \in \mathbb{P}^n$. The *double point* at p in $\mathbb{P}^n$ is the scheme given by the square of the ideal (sheaf) of p.

A $\delta - dot$ is a subscheme of degree δ of a *double point*. (Hence a double point in $\mathbb{P}^n$ is an $(n+1)$-dot.)

If $\Phi \subset \mathbb{P}^n$ we denote by Φ^2 the union of the double points supported on Φ.

Hence, a form vanishes on Φ^2 exactly if it is singular along Φ.

To prove that a general collection of d points of $\mathbb{P}^n$ verifies the theorem, we start by choosing a codimension 2 plane $\mathbb{P}^{n-2} \subset \mathbb{P}^n$ and construct a collection X of d

double points for which the base locus of cubics through X contains $\mathbb{P}^{n-2}$. First, a maximal number α of double points of X are specialised onto $\mathbb{P}^{n-2}$ so that $X \cap \mathbb{P}^{n-2}$ imposes independent conditions on $|\mathcal{O}_{\mathbb{P}^{n-2}}(3)|$. Of the remaining $d - \alpha$ points we may further specialise by forming pairs so that the line between each pair *does* meet $\mathbb{P}^{n-2}$. This may be done (by induction together with numerics) so that $\mathbb{P}^{n-2}$ is in the base locus of the system of cubics passing through X.

Hence the theorem is proved by a general statement in Lemma 2 regarding schemes $X \subset \mathbb{P}^n$ consisting of double points, some of which lie on $\mathbb{P}^{n-2}$ and some of which occur in, say, γ pairs so that the line between each pair meets $\mathbb{P}^{n-2}$. The lemma gives criteria under which the union of such a scheme X with $\mathbb{P}^{n-2}$ imposes the expected number $\deg X - \deg(X \cap \mathbb{P}^{n-2}) - \gamma$ of conditions on the linear system of cubics of $\mathbb{P}^n$ passing through $\mathbb{P}^{n-2}$.

The proof of Lemma 2 proceeds by induction on dimension, using the standard Castelnuovo exact sequence (1). We choose a hyperplane H along with its own codimension 2 plane $H \cap \mathbb{P}^{n-2}$. A scheme X is constructed so that all but one (say $\{p\}^2$) of the double points that are not on $\mathbb{P}^{n-2}$ are specialised onto$H$ and all but a certain number $\alpha_0 \leqslant n-1$ of the double points on $\mathbb{P}^{n-2}$ are further specialised onto $\mathbb{P}^{n-2} \cap H$. So by induction $(X \cup \mathbb{P}^{n-2}) \cap H$ imposes conditions on cubics of H as desired. In the base locus of cubics through X lies the union of α_0 lines between the point p and $\mathbb{P}^{n-2}$. These meet H in a further α_0 points, whose union with $X \cap H$ may be viewed as general points on a hyperplane of H containing $\mathbb{P}^{n-2} \cap H$. Hence their union with $(X \cup \mathbb{P}^{n-2}) \cap H$ is easily seen (with appropriate choice of numbers), not to lie on a cubic of H. So each of the α_0 double points on $\mathbb{P}^{n-2} - (\mathbb{P}^{n-2} \cap H)$ is 'split in half'. (Whereas, specialising all of the doubles onto H would cause parity problems. Indeed, we do profit from the initial cases' having odd values.) The result required from lower degree is then simply that the union of one double point with $\mathbb{P}^{n-2}$, α_0 double points on $\mathbb{P}^{n-2}$, and $n - \alpha_0$ general points does not lie on a quadric.

Let us review the basic techniques used here and in [AH3].

DEFINITION/NOTATION 1. Given a hyperplane $H \subset \mathbb{P}^n$, the *residual* of X with respect to H is the scheme $\tilde{X}$ described by the ideal sheaf $\mathcal{I}_{\tilde{X}} = \mathcal{I}_X : \mathcal{O}_{\mathbb{P}^n}(-H)$.

Then the Castelnuovo exact sequence is given by

$$0 \to \mathcal{I}_{\tilde{X}}(-1) \to \mathcal{I}_X \to \mathcal{I}_{X \cap H, H} \to 0. \tag{1}$$

Generally, to construct a scheme $X \subset \mathbb{P}^n$ of double points imposing the expected number of conditions in a given degree m, one may proceed by induction on n and m as follows: Specialise points of X onto a hyperplane H, and apply the exact sequence (1). (Here $\tilde{X}$ is the union of the simple points on H with the remaining double points of X.) Then by induction on dimension, $H^i(H, \mathcal{I}_{X \cap H, H}(m))$ should vanish for $i = 0$ or 1 and by induction on degree $H^j(\mathbb{P}^n, \mathcal{I}_{\tilde{X}}(m-1))$ should vanish for $j = 0$ or 1. If one can arrange the specialisation so that $i = j$ then X is the desired scheme. If not (which happens often, strictly as a matter of numerics) one applies a more subtle variation on the exact sequence estimate such as that of Horace différentielle.

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A difficulty in applying such methods to cubics is that a set X of at least 2 double points does not impose deg X conditions on quadrics (because, for example, a quadric singular at one point and vanishing at another must vanish on the line joining them). However (as in [A], [C3], and [AH4]) standard Horace may be used to find a set $X \subset \mathbb{P}^n$ of double points that does impose the expected number *provided* that deg X is not too close to $\binom{n+3}{3}$, and this is enough to carry the induction to degree 4. Specifically, in this case if X is supported on a set Φ then cubics vanishing on X vanish on the union, Sec Φ, of lines joining points of Φ. Hence by applying standard Castelnuovo to schemes of the form $(\mathrm{Sec}\, \Gamma \cap \mathbb{P}^n) \cup \Sigma^2$, a weak result on cubics is obtained by specialisation and induction. However, the Horace differential technique does not apply well to cubics (witness the case of dimension 6 below!).

In [AH3] the complete degree 3 result is proved by blowing up. The relevant scheme X is constructed by specialising points onto a codimension c plane, $c > 1$ so that the union of lines between pairs of the remaining points does not meet this plane. Further the codimension $c = 3$ or 5 is chosen depending on n (in fact, on $n (\mathrm{mod}\, 3)$) so that just enough double points may be specialised to the plane so that it is in the base locus of cubics through X. Of course $\mathbb{P}^{n-c}$ isn't a divisor, which is taken to warrant blowing it up. Then the main effort of the proof goes toward the specialisation to the exceptional divisor $E \simeq \mathbb{P}^{n-c} \times \mathbb{P}^{c-1}$ on the blowup, where Horace différentielle is applied.

By contrast we do not use Horace differential nor Horace blowup methods. Although the devout blower-up may view the argument as using the blowup $\mathbb{B}^n$ of $\mathbb{P}^n$ with respect to a codimension 2 plane, a key difference (in blowup parlance) is that we work only with a divisor $\mathbb{B}^{n-1}$ rather than the exceptional one. Hence we avoid starting from scratch on the exceptional divisor $\mathbb{P}^{n-2} \times \mathbb{P}^1$. We can imagine that the exceptional divisor analysis would become onerous in the study of points of higher order.

2. Main Argument

To verify Theorem 1 in $\mathbb{P}^n$ we shall form a collection X of double points that yields a $\mathbb{P}^{n-2}$ in its cubic base locus. This is done by specialising points onto $\mathbb{P}^{n-2}$ and pairs of points onto hyperplanes containing $\mathbb{P}^{n-2}$. Then the problem is reduced to studying cubics through $X \cup \mathbb{P}^{n-2}$.

Lemma 2 gives a statement on cubics through $X \cup \mathbb{P}^{n-2}$ where X is any general union of double points, some on $\mathbb{P}^{n-2}$ and some in pairs whose line meets $\mathbb{P}^{n-2}$. (In particular, X itself need not inflict $\mathbb{P}^{n-2}$ in the base locus of cubics.)

Let us categorise such schemes X.

DEFINITION 2. Fix $\mathbb{P}^{n-2} \subset \mathbb{P}^n$. Given nonnegative integers $\mu, \gamma, \alpha, \delta$ so that $\mu \geqslant 2\gamma$, $\delta \leqslant n+1$ we define a $(\mu, \gamma, \alpha, \delta)$-subscheme of $\mathbb{P}^n$ with respect to $\mathbb{P}^{n-2}$ as a union of:

- $\mu - 2\gamma$ double points
- γ pairs of double points for which the line between each pair meets $\mathbb{P}^{n-2}$.
- α double points lying on a $\mathbb{P}^{n-2}$, and
- a δ-dot.

Notice that a $(\mu, \gamma, \alpha, \delta)$-scheme X imposes at most

$$\deg X - \deg(X \cap \mathbb{P}^{n-2}) - \gamma \leqslant (n+1)\mu - \gamma + 2\alpha + \delta$$

conditions on the linear system of cubics of $\mathbb{P}^n$ through $\mathbb{P}^{n-2}$.

Let us abbreviate the description of conditions imposed by subschemes of $\mathbb{P}^n$ with respect to a codimension 2 plane:

DEFINITION 3. Take $\mathbb{P}^{n-2} \subset \mathbb{P}^n$. We will say that a $(\mu, \gamma, \alpha, \delta)$-scheme X is 3-*agreeable* with respect to $\mathbb{P}^{n-2}$ if X imposes $\min((n+1)\mu - \gamma + 2\alpha + \delta, (n+1)^2)$ conditions on the linear system of cubics of $\mathbb{P}^n$ through $\mathbb{P}^{n-2}$. For any subscheme $Z \subset \mathbb{P}^n$ we will say that Z is m-*maximal* with respect to $\mathbb{P}^{n-2}$ if $Z \cup \mathbb{P}^{n-2}$ does not lie on an m-ic.

We obtain the main tool in the proof of Theorem 1:

LEMMA 2. *Choose* $\mathbb{P}^{n-2} \subset \mathbb{P}^n$. *Let* $\mu, \gamma, \alpha, \delta$ *be nonnegative integers satisfying:*

- $\mu \geqslant \min(3, n+1)$,
- *if* $\gamma > 0$ *then* $\mu \geqslant \min(5, n+1)$ *and* $\mu \geqslant 2\gamma$, *and*
- $0 \leqslant \delta \leqslant n+1$.

Let $X \subset \mathbb{P}^n$ *be a generic* $(\mu, \gamma, \alpha, \delta)$-*scheme with respect to* $\mathbb{P}^{n-2}$. *Then* X *is* 3-*agreeable*.

Proof. Observe that the result holds when $n = 1$ (or, for that matter, $n = 0$). Now assume by induction that it holds in dimension $n - 1$.

Choose a hyperplane $H \subset \mathbb{P}^n$.

Let us start with a few hypotheses to simplify the numerics.

We may assume that $\mu \leqslant n+2$ (and $\mu \leqslant n+1$ if $\gamma = 0$) since if equality holds then

$$(n+1)\mu - \gamma \geqslant (n+1)^2.$$

Next, we observe that any case having $\delta \geqslant 2$ may be deduced from cases in which $\delta \leqslant 1$: choose a $(\mu, \gamma, \alpha, 0)$-scheme X along with a point p so that each of X, $X \cup \{p\}$, and $X \cup \{p\}^2$ are 3-agreeable (according to the cases $(\mu, \gamma, \alpha, 0), (\mu, \gamma, \alpha, 1)$, and $(\mu + 1, \gamma, \alpha, 0)$). It follows that there is a δ-dot supported at p whose union with X is 3-agreeable. Hence we assume that $\delta \leqslant 1$.

Further, we may assume that

$$(n+1)\mu - \gamma + 2\alpha + \delta \geqslant (n+1)^2.$$

To see this, given $\mu, \gamma, \alpha, \delta$, if

$$(n+1)\mu - \gamma + 2\alpha + \delta < (n+1)^2$$

we may solve

$$(n+1)\mu - \gamma + 2\alpha' + \delta' = (n+1)^2$$

with $\delta' \leqslant 1$. If a $(\mu, \gamma, \alpha', \delta')$-scheme is 3-agreeable, so is a $(\mu, \gamma, \alpha, 0)$-scheme (since it is a subscheme) and hence so is a $(\mu, \gamma, \alpha, \delta)$-scheme (choosing a general point if $\delta = 1$).

We divide into 3 cases:

- $n \geqslant 3$, $\mu = 3, \gamma = 0$;
- $n \geqslant 5$, $\mu = 5$, $\gamma > 0$; and
- the general case.

General case: Assume that $\mu \geqslant \min(4, n+1)$ and if $\gamma > 0$ then $\mu \geqslant \min(6, n+1)$.

We start by selecting a subscheme Z of $\mathbb{P}^n$ supported on H according to the following numerical specifications:

$$\gamma_0 = \begin{cases} 1, & \text{if } \gamma > 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\alpha_0 = \begin{cases} n+1+\gamma_0-\mu, & \text{if } \alpha > 0, \\ 0, & \text{if } \alpha > 0, \end{cases}$$

(note that $0 \leqslant \alpha_0 \leqslant \alpha$)

$$\delta_0 = \begin{cases} 1, & \text{if } \gamma = 1, \quad \alpha = 0, \quad \text{and} \quad \mu = n+1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$(\mu_1, \gamma_1, \alpha_1, \delta_1) = (\mu - 1, \gamma - \gamma_0, \alpha - \alpha_0, \delta - \delta_0)$$

so that

$$n\mu_1 - \gamma_1 + 2\alpha_1 + \delta_1 + \alpha_0 \geqslant n^2 \tag{2}$$

and

$$\mu_1 + \alpha_0 + \delta_0 - \gamma_0 = n. \tag{3}$$

Let Z be a $(\mu_1, \gamma_1, \alpha_1, \delta_1)$-subscheme of $\mathbb{P}^n$ supported on H. Then $\mu_1 = \mu - 1$ satisfies $\mu_1 \geqslant \min(3, n)$ and if $\gamma_1 > 0$ then $\mu_1 \geqslant \min(5, n)$. Hence, by induction we may choose Z so that $Z \cap H$ is 3-agreeable with respect to $\mathbb{P}^{n-3} = \mathbb{P}^{n-2} \cap H$.

Now we take $p \in \mathbb{P}^n - H$. If $\gamma = 0$ we choose a general such point. Otherwise (since $\mu_1 > 2\gamma_1$) we may take one of the unpaired points, say q, of Z and choose p on a general line between q and $\mathbb{P}^{n-2}$.

Consider the union Φ of α_0 points on $\mathbb{P}^{n-2}$, to be chosen, and $X = \{p\}^2 \cup \Phi^2 \cup Z$. Then X is a $(\mu, \gamma, \alpha, \delta)$-scheme. We wish to find Φ so that X is 3-maximal.

Let $J(\Phi)$ be the union of the α_0 lines joining each point of Φ to p. Then any cubic passing through $\{p\}^2 \cup \Phi^2$ must vanish on $J(\Phi)$, hence it is equivalent to see that $X \cup J(\Phi)$ is 3-maximal.

First we show that Φ may be chosen so that $(X \cup J(\Phi)) \cap H$ is 3-maximal with respect to $\mathbb{P}^{n-2} \cap H$.

Let us abbreviate $Z_1 = Z \cap H$, $\mathbb{P}^{n-3} = \mathbb{P}^{n-2} \cap H$.

Let $K = \operatorname{span}(p, \mathbb{P}^{n-2}) \cap H$. (Note that $K = \operatorname{span}(q, \mathbb{P}^{n-3})$ if $\gamma > 0$.)

We start by observing that $(X \cup J(\Phi)) \cap H$ is 3-maximal for general Φ provided that $Z_1 \cup K$ does not lie on a cubic.

To see this, imagine that $Z_1 \cup K$ does not lie on a cubic. According to (2) along with the induction hypothesis, Z_1 imposes at least $n^2 - \alpha_0$ conditions on the system of cubics through $\mathbb{P}^{n-3}$. It follows that there is a collection $\Psi \subset K$ of α_0 points so that $Z_1 \cup \Psi \cup \mathbb{P}^{n-3}$ is not on a cubic. Hence, if $\Phi \subset \mathbb{P}^{n-2} - \mathbb{P}^{n-3}$ is given by the union of α_0 points on $\mathbb{P}^{n-2}$, each given by the intersection of $\mathbb{P}^{n-2}$ with a line between p and a point of Ψ we have $J(\Phi) \cap H = \Psi$. So $(X \cup J(\Phi) \cup \mathbb{P}^{n-2}) \cap H$ does not lie on a cubic hypersurface of H.

Now we show that $Z_1 \cup K$ does not lie on a cubic of H; equivalently, that the residual $\tilde{Z}_1$ of Z_1 with respect to K does not lie on a quadric of H. To see this, we divide into the two cases: $\gamma = 0$ or $\gamma > 0$.

If $\gamma = 0$ then $\tilde{Z}_1$ consists of μ_1 double points, a δ_1-dot, and α_1 points on $\mathbb{P}^{n-3}$. We have

$$n\mu_1 + 2\alpha_1 + \delta_1 + (n - \mu_1) \geqslant n^2$$

so by Lemma 6 (substituting $(\mu_1, \alpha_1, \delta_1)$ for (μ, α, ϵ)) we have

$$n\mu_1 \binom{\mu_1}{2} + \alpha_1 + \delta_1 \geqslant \binom{n+1}{2}.$$

Hence, by Lemma 3, $\tilde{Z}_1$ does not lie on a quadric.

Now suppose $\gamma > 0$. Since $q \in K$, $\tilde{Z}_1$ consists of $\mu_1 - 1$ double points, a δ_1-dot, α_1 points on $\mathbb{P}^{n-3}$, and the (reduced) point q on K. We have:

$$n\mu_1 + 2\alpha_1 + \delta_1 + \alpha_0 - \gamma_1 \geqslant n^2, \text{ so}$$
$$n\mu_1 + 2\alpha_1 + \delta_1 + 1 + (n - \mu_1) \geqslant n^2$$

Then by substituting $(\mu_1, \alpha_1, \delta_1 + 1)$ in lemma 6 we obtain

$$n(\mu_1 - 1) - \binom{\mu_1 - 1}{2} + \alpha_1 + \delta_1 + 1 \geqslant \binom{n+1}{2}.$$

Again by lemma 3, $\tilde{Z}_1$ does not lie on a quadric.

Residual to H we have the union $\tilde{X}$ of the double point at p, $\mu - 1$ simple points on H α_0 double points on $\mathbb{P}^{n-2}$, and δ_0 general points. (If $\gamma > 0$ note that any quadric

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through the double point at p and $\mathbb{P}^{n-2}$ must vanish at q, so that q 'disappears' from the residual calculation.) By Lemma 4 $\tilde{X}$ is 2-maximal (with appropriate choice of Φ).

Hence X is 3-maximal.

Case $\mu = 3, \gamma = 0$: We have seen this case for $n = 2$, hence we may take $n \geqslant 3$. We may take α as (the integer!) $\alpha = (n+1)(n-2)/2$, so that $3(n+1) + 2\alpha = (n+1)^2$. Set $\alpha_0 = n-1$, $\alpha_1 = \alpha - \alpha_0 \geqslant 0$. Let X be the union of: three double points on H, α_1 double points on $\mathbb{P}^{n-3} = \mathbb{P}^{n-2} \cap H$, α_0 general double points on $\mathbb{P}^{n-2}$. Then $X \cap H$ is 3-maximal by induction, and $\tilde{X}$ is 2-maximal by Lemma 5.

Case $\mu = 5, \gamma > 0$: The case $n = 4$ is already complete so we may assume $n \geqslant 5$. Take $\alpha_0 = n-2 \leqslant \alpha$, $\alpha_1 = \alpha - \alpha_0$. Let X be the union of five double points on H, δ points on H, α_1 double points on $\mathbb{P}^{n-2} \cap H$, α_0 general double points of $\mathbb{P}^{n-2}$. Then $X \cap H$ is 3-maximal by induction and $\tilde{X}$ is 2-maximal by Lemma 5.

It remains to complete the lemmas on quadrics and the calculation used here. □

LEMMA 3. *If* $d(n+1) - \binom{d}{2} + s \geqslant \binom{n+2}{2}$ *then a general union of d double points and s simple points does not lie on a quadric of $\mathbb{P}^n$. If $d \geqslant c$ we may take the s points to lie on a codimension c plane.*

Proof. If $n = 0$ or $d = 0$ there is nothing to prove. Otherwise, specialize $d-1$ double points along with the s simple points to a hyperplane, apply induction, and observe that a hyperplane does not contain a double point of $\mathbb{P}^n$. □

LEMMA 4. *Choose $\mathbb{P}^{n-2} \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$. Suppose $\alpha + \beta = n$, $\beta \geqslant 1$. Then the generic union of $\mathbb{P}^{n-2}$ with α double points on $\mathbb{P}^{n-2}$, β points on $\mathbb{P}^{n-1}$, and a double point does not lie on a quadric.*

Proof. Let $\Phi \subset \mathbb{P}^{n-2}$ and $\Sigma \subset \mathbb{P}^{n-1}$ be collections of α, β points so that $\mathbb{P}^{n-1}$ is spanned by $\Sigma \cup \Phi$. Let $p \in \mathbb{P}^n - \mathbb{P}^{n-1}$.

Then for $X = \Phi^2 \cup \Sigma \cup \{p\}^2$ the residual $\tilde{X}$ with respect to $\mathbb{P}^{n-1}$ contains $\{p\}^2$ which does not lie on a hyperplane. Hence it suffices to show that $\Sigma \cup \Phi \cup \mathbb{P}^{n-2}$ is not on a quadric of $\mathbb{P}^{n-1}$, i.e. that $\Sigma \cup \Phi$ is not on a hyperplane of $\mathbb{P}^{n-1}$. But this is just how Φ and Σ were chosen. □

LEMMA 5. *Choose $\mathbb{P}^{n-2} \subset \mathbb{P}^n$. Let α, β, γ be nonnegative integers satisfying $2\alpha + \beta + 2\gamma = 2n+1$. Assume that either $\beta = 3$ and $\gamma = 0$ or $\beta + 2\gamma = 5$. Let Z be the generic union of α double points of $\mathbb{P}^n$ lying on $\mathbb{P}^{n-2}$ and $\beta + 2\gamma$ points, 2γ of these points lying (in pairs) on the union of γ lines that meet $\mathbb{P}^{n-2}$. Then $Z \cup \mathbb{P}^{n-2}$ does not lie on a quadric.*

Proof. The case of $\beta = 3$, $\alpha = n-1$ is straightforward.

Now suppose that $\beta + 2\gamma = 5$. We may assume that $\gamma = 2$. Observe that any quadric passing through $\mathbb{P}^{n-2}$ and two general points of a line meeting $\mathbb{P}^{n-2}$ must go through that line. Further, a quadric through $\mathbb{P}^{n-2}$ and two general lines that meet a

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point $q \in \mathbb{P}^{n-2}$ vanishes on $\{q\}^2$ and two general points, one on each line. We reduce then to the case $\beta = 3, \alpha = n - 1$. $\square$

LEMMA 6. *Let n, μ, α, ϵ be nonnegative integers. Suppose that*

$$n\mu + n - \mu + 2\alpha + \epsilon \geqslant n^2.$$

If $\mu = n$ or $2 \leqslant \mu \leqslant n - 1$ then

$$n\mu - \binom{\mu}{2} + \alpha + \epsilon \geqslant \binom{n+1}{2}.$$

Further, suppose $\epsilon \geqslant 1$. If $\mu = n$ or $4 \leqslant \mu \leqslant n - 1$ then

$$n(\mu - 1) - \binom{\mu - 1}{2} + \alpha + \epsilon \geqslant \binom{n+1}{2}.$$

Proof. Fix n. Fix $\epsilon \geqslant 0$ and let

$$P(t) = nt - t(t-1)/2 + (n^2 - nt - n + t + \epsilon)/2 - \binom{n+1}{2}.$$

We show that $P(t) \geqslant 0$ if $t = n$ or $2 \leqslant t \leqslant n - 1$. Observe that $P(t) = P(n + 2 - t)$, and, as a function of t, $P(t)$ is concave down. Hence, it suffices to check the endpoints $P(n) = P(2) = \epsilon/2 \geqslant 0$.

Now suppose $\epsilon \geqslant 1$. Call

$$Q(t) = n(t-1) - (t-1)(t-2)/2 + (n^2 - nt - n + t + \epsilon)/2 - \binom{n+1}{2}$$

so that

$$\lceil Q(\mu) \rceil \leqslant n(\mu - 1) - \binom{\mu - 1}{2} + \alpha + \epsilon - \binom{n+1}{2}$$

(since the right-hand side is an integer!). Analogous to the function $P(t)$, $Q(t) = Q(n + 4 - t)$, Q is concave down, and now we simply check: $Q(n) = Q(4) = -1 + \epsilon/2 \geqslant -1/2$. Thus if $\mu = n$ or $4 \leqslant \mu \leqslant n - 1$ we have $\lceil Q(\mu) \rceil \geqslant \lceil -1/2 \rceil = 0$.

This completes the proof of Lemma 2. $\square$

3. Proof of Theorem

For each n write

$$\binom{n+3}{3} = (n+1)a_n + b_n, 0 \leqslant b_n \leqslant n.$$

It is easy to see that

$$b_n = \begin{cases} \frac{n+1}{3}, & \text{if } 3|n+1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$a_n - a_{n-2} = \left\lfloor \frac{2n+4}{3} \right\rfloor.$$

We show that the general union of a_n double points and a b_n-dot in $\mathbb{P}^n$ imposes the expected number $\binom{n+3}{3}$ of conditions on cubics except when $n = 4$. (Since any collection of double points has a subscheme or is a subscheme of such a scheme it follows that for any d there is a collection of d double points imposing min $\left((n+1)d, \binom{n+3}{3}\right)$ conditions on cubics.)

For $n = 2$, $a_2 = 3$, $b_2 = 1$, the result is immediate from applying Lemma 2 to a (3,0,0,0)-scheme with respect to a point.

Induction Step. Now suppose that $n \geqslant 3$. Assume that the conclusion of Theorem 1 holds in dimension $n-2$ (so $n \neq 6$, say).

We construct a union X of a_n double points and a b_n-dot that does not lie on a cubic. First, we may specialise a_{n-2} of the double points to a $\mathbb{P}^{n-2}$, so that by induction $\mathbb{P}^{n-2}$ lies in the base locus of cubics through X unless $b_{n-2} \neq 0$. In that case, we shall make the further specialisation: of the remaining $a_n - a_{n-2}$ points of X, since $a_n - a_{n-2} > 2b_{n-2}$, we may take b_{n-2} pairs of points of $\mathbb{P}^n - \mathbb{P}^{n-2}$ so that the line spanned by each of the pairs meets $\mathbb{P}^{n-2}$. Now, with a general choice of pairs, $\mathbb{P}^{n-2}$ is in the base locus of cubics through X.

Thus, we take

$$(\mu, \gamma, \alpha, \delta) = (a_n - a_{n-2}, b_{n-2}, a_{n-2}, b_n)$$

for the application of Lemma 2.

We have

$$\mu = \left\lfloor \frac{2n+4}{3} \right\rfloor \geqslant 3, \quad \text{and} \quad \mu = \left\lfloor \frac{2n+4}{3} \right\rfloor > \frac{2(n-1)}{3} \geqslant 2\gamma.$$

Suppose $\gamma > 0$. If $n = 4$ then $\mu = 4$, so that Lemma 2 does not apply. However, for $n \neq 4$ (since $n \equiv 1 \pmod 3$) we have $n \geqslant 7$ so $\mu \geqslant 5$.

Then by Lemma 2 there is a $(\mu, \gamma, \alpha, \delta)$-scheme that is 3-maximal. We obtain, therefore, a collection of a_n double points and a b_n-dot whose union does not lie on a cubic.

Hence, the cases $n \leqslant 3$ and all odd values of n are complete, and for even $n \geqslant 4$ we are done once the cases $n = 4$ and $n = 6$ are established.

Dimension 4. Let us observe that the case of seven double points in $\mathbb{P}^4$ is exceptional: although $5 \times 7 = \binom{7}{3}$, given a (general) set of seven points in $\mathbb{P}^4$ there is a cubic that is singular on all 7. Since such a collection lies on a rational normal curve, described by the 2×2 minors of, say,

$$\begin{bmatrix} X_0 & X_1 & X_2 & X_3 \\ X_1 & X_2 & X_3 & X_4 \end{bmatrix},$$

then the determinant of

$$\begin{bmatrix} X_0 & X_1 & X_2 \\ X_1 & X_2 & X_3 \\ X_2 & X_3 & X_4 \end{bmatrix}$$

vanishes on the double curve and, hence is a cubic singular on the seven points.

However, the cubic singular on seven general points *is* unique. (One may see this from the argument of Lemma 2, but that is not the easiest way!) Take X to be seven double points, exactly five of which lie on $\mathbb{P}^3$. We may arrange (as we have just seen) that $X \cap \mathbb{P}^3$ does not lie on a cubic and (by Lemma 3) that $\tilde{X}$ lies on a unique quadric. So by Castelnuovo, the cubic through X is unique. In particular, there is no cubic singular on $d \geqslant 8$ general points of $\mathbb{P}^4$ and a general collection of $d \leqslant 6$ double points imposes $5d$ conditions on cubics (specialise $\min(5, d)$ onto a hyperplane).

(See [CH] or [C1] for alternative proofs.)

Dimension 6. We find 12 double points of $\mathbb{P}^6$ that do not lie on a cubic.

The idea is to specialise, as in higher degree cases, to a hyperplane, but deal with the consequences of the lines in the base locus.

An optimistic first step in producing such a scheme would be to take a set $\Sigma \subset \mathbb{P}^6 - \mathbb{P}^5$ of three (noncollinear) points, together with a set of $a_5 = 9$ points in $\mathbb{P}^5$. Then any cubic singular on Σ must vanish on Sec Σ, in particular, on the set $\Psi = \text{Sec}\,\Sigma \cap \mathbb{P}^5$ of three collinear points. (Of course, any set of three collinear points may be obtained in this manner.) Then, as we see in Lemma 7, nine double points of $\mathbb{P}^5$ may be chosen whose union with Ψ doesn't lie on a cubic of $\mathbb{P}^5$. The residual of the scheme (consisting of the three double points on Σ with nine points on $\mathbb{P}^5$) lies on a unique quadric of $\mathbb{P}^6$, 'due to' to the numerics of the situation, and indeed the configuration lies on a unique cubic.

A natural remedy would appear to be in studying in $\mathbb{P}^5$ the union of eight double points, the set Ψ, with a point $p \in \mathbb{P}^5$. This, again by Lemma 7, is 3-independent and (by numbers) yields a degree 2 scheme Λ supported at p in the base locus. Then 'normal' Horace différentielle would say that it suffices to verify that $\tilde{X} \cup \Lambda$ is 2-independent. Unfortunately, one may have little idea of how Λ depends on X; in particular, unlike in the usual Horace differential situation, Λ may well depend on the set Σ of points lying outside H.

Hence, we construct a subscheme as above in which the base locus scheme Λ at p may be identified; in fact, so that it does visibly depend on Σ. Indeed, the scheme $\tilde{X} \cup \Lambda$ turns out *not* to be 2-independent, so that Horace différentielle does not quite apply. Instead, a degeneration argument appears to save the day.

We abbreviate the accounting as follows:

DEFINITION/NOTATION 2. Given a scheme $X \subset \mathbb{P}^n$ we denote by $h_{\mathbb{P}^n}(X, m)$ the *Hilbert function* of X, namely,

$$h_{\mathbb{P}^n}(X, m) = \dim H^0(\mathcal{O}_{\mathbb{P}^n}(m) - \dim H^0(\mathcal{I}_X(m)),$$

that is, the number of conditions that X imposes on m-ics.

From the Castelnuovo exact sequence comes a basic estimate of the Hilbert function of a scheme $X \subset \mathbb{P}^n$:

$$h_{\mathbb{P}^n}(X, m) \geqslant h_{\mathbb{P}^n}(\tilde{X}, m-1) + h_H(X \cap H, m).$$

Let us start with the required results from dimension 5:

LEMMA 7. *Choose a line* $L \subset \mathbb{P}^5$. *There is a set* $\Psi \subset L$ *of three points, a collection* $\Phi \cup \{y\} \subset \mathbb{P}^5$ *of eight points, and a point* $x \in (y, L)$ *so that:*

- $h_{\mathbb{P}^5}(\Psi \cup \Phi^2 \cup \{x, y\}^2, 3) = 56$,
- $h_{\mathbb{P}^5}(\Psi \cup \Phi^2 \cup \{x\} \cup \{y\}^2, 3) = 52$, *and hence*
- *the base locus of cubics through* $\Psi \cup \Phi^2 \cup \{x\} \cup \{y\}^2$ *meets* $\{x\}^2$ *in precisely* $\{x\}^2 \cap \operatorname{span}(x, y)$.

Proof. Choose a general collection $\Phi \cup \{y\}$ of eight points in $\mathbb{P}^5$.

Suppose that L is any line, $\Psi \subset L$ is a set of three points, and $x \in \operatorname{span}(y, L)$. Assume that $\operatorname{span}(x, y) \cap \Psi = \emptyset$. Then any cubic through $\{x, y\}^2 \cup \Psi$ must vanish on L. Therefore we start by choosing L and $x \in \operatorname{span}(y, L)$ for which $L \cup \Phi^2 \cup \{x, y\}^2$ is not on a cubic.

Write $\Phi = \Phi_5 \cup \Phi_3$, where Φ_3 consists of 4 points. Take $\mathbb{P}^3 = \operatorname{span}(\Phi_3)$. Then by Lemma 2 we may choose $L \subset \mathbb{P}^3$ so that $\Phi_3^2 \cup L$ does not lie on a cubic of $\mathbb{P}^3$. Now we choose $x \in \operatorname{span}(y, L)$ and apply Lemma 2 with $\mu = 5, \gamma = 1, \alpha = 4$. Since $6 \times 5 - 1 + 2 \cdot 4 > 36$, there is no cubic through $\Phi^2 \cup \{x, y\}^2 \cup \mathbb{P}^3$ and, hence, none through $\Phi^2 \cup \{x, y\}^2 \cup \Psi$.

Next, consider $X = \Phi^2 \cup \{y\}^2 \cup \{x_0\} \cup \Psi$, where $\{x_0\} = \operatorname{span}(x, y) \cap L$. Again L and, hence, $\mathbb{P}^3$ are in the base locus of cubics through X. Applying Lemma 2 (with $\mu = 4, \gamma = 0, \alpha = 4$) we see that the scheme imposes $6 \times 4 + 2 \times 4 + 20 = 52$ conditions on cubics.

Hence we may find $x \in \operatorname{span}(x_0, y)$ satisfying the first two stated properties. These together imply that the intersection of $\{x\}^2$ with the base locus of cubics through $\Psi \cup \Phi^2 \cup \{x\} \cup \{y\}^2$ has degree 2. $\square$

Now choose a hyperplane $\mathbb{P}^5 \subset \mathbb{P}^6$. Suppose that $\Sigma \subset \mathbb{P}^6 - \mathbb{P}^5$ is a set of three (noncollinear) points. Let $\Psi = \operatorname{Sec} \Sigma \cap \mathbb{P}^5$. We may choose $\Phi \subset \mathbb{P}^5$ (seven points) and $x, y \in \mathbb{P}^5$ so that$x \in \operatorname{span}(y, \Sigma) \cap \mathbb{P}^5$. Then

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$$\begin{aligned}&h_{\mathbb{P}^6}(\Sigma^2 \cup \Phi^2 \cup \{x, y\}^2, 3)\\ &\quad = h_{\mathbb{P}^6}(\Sigma^2 \cup \mathrm{Sec}\Sigma \cup \Phi^2 \cup \{x, y\}^2, 3)\\ &\quad \geqslant h_{\mathbb{P}^6}(\Sigma^2 \cup \Phi \cup \{x, y\}, 2) + h_{\mathbb{P}^5}(\Psi \cup \Phi^2 \cup \{x, y\}^2, 3)\\ &\quad = 27 + 56 = 83.\end{aligned}$$

In particular, if $\Lambda \subset \{x, y\}^2$ is curvilinear then by Lemma 7

$$h_{\mathbb{P}^6}(\Sigma^2 \cup \Phi^2 \cup \Lambda, 3) = 70 + \deg \Lambda$$

except when $\Lambda = \{x, y\}^2 \cap \mathrm{span}\ (x, y)$.

Now choose points $p, q \in \mathbb{P}^6$ so that $p \in \mathrm{span}(q, \Sigma) = \mathbb{P}^3$ (and they are five general points of this $\mathbb{P}^3$). Call $M = \mathrm{span}(p, q)$. Then there is a collection $\Phi \subset \mathbb{P}^6$ consisting of seven points so that

$$h_{\mathbb{P}^6}(\Sigma^2 \cup \Phi^2 \cup \{p, q\}^2, 3) \geqslant 83.$$

Indeed, for Λ curvilinear with support on $\{p, q\}$ we have

$$h_{\mathbb{P}^6}(\Sigma^2 \cup \Phi^2 \cup \Lambda, 3) = 70 + \deg \Lambda$$

with the possible exception of the subscheme $\Lambda = \{p, q\}^2 \cap M$.

Therefore, by Lemma 4 of [C3] (e.g.), we are done once we produce a collection $\Gamma \subset \mathbb{P}^6$ of 10 points so that $\Sigma \subset \Gamma$ and

$$h_{\mathbb{P}^6}(\Gamma^2 \cup \Lambda, 3) = 74$$

for $\Lambda = \{p, q\}^2 \cap M$; or equivalently

$$h_{\mathbb{P}^6}(\Gamma^2 \cup M, 3) = 74.$$

Choose a set Γ_3 of four points in $\mathbb{P}^3 = \mathrm{span}(q, \Sigma)$ so that $\Sigma \subset \Gamma_3$. We may assume that $M \cup (\Gamma_3^2 \cap \mathbb{P}^3)$ is not on a cubic of $\mathbb{P}^3$ by Lemma 2 ($\mu = 4, \gamma = 0$).

Then choose a hyperplane H of $\mathbb{P}^6$ containing $\mathbb{P}^3$. Take a set $\Gamma_5 \subset H$ of four points and a point $r \in H$ for which

$$h_{\mathbb{P}^5}(\{r\} \cup \Gamma_5^2 \cup \Gamma_3^2 \cup \mathbb{P}^3, 3) = 54$$

(applying Lemma 2 with $\mu = 4, \gamma = 0, \alpha = 4, \delta = 1$).

Now choose a pair of points $\Gamma_6 \subset \mathbb{P}^6 - \mathbb{P}^5$ satisfying $\mathrm{Sec}\,\Gamma_6 \cap \mathbb{P}^5 = \{r\}$ and

$$h_{\mathbb{P}^6}(\Gamma_6^2 \cup \Gamma_5 \cup \Gamma_3, 2) = 20.$$

Hence for $\Gamma = \Gamma_6 \cup \Gamma_5 \cup \Gamma_3$ we have

$$\begin{aligned}&h_{\mathbb{P}^6}(\Gamma^2 \cup M, 3)\\ &\quad = h_{\mathbb{P}^6}(\Gamma_6^2 \cup \mathrm{Sec}(\Gamma_6) \cup \Gamma_5^2 \cup \Gamma_3^2 \cup M, 3)\\ &\quad \geqslant h_{\mathbb{P}^6}(\Gamma_6^2 \cup \Gamma_5 \cup \Gamma_3, 2) + h_{\mathbb{P}^5}(\{r\} \cup \Gamma_5^2 \cup \Gamma_3^2 \cup M, 3)\\ &\quad = 20 + 54 = 74.\end{aligned}$$

This completes the proof. □

https://doi.org/10.1023/A:1020905322129 Published online by Cambridge University Press

Acknowledgements

I am very grateful to J. Alexander for helpful discussions. Special thanks go to A. Hirschowitz.

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