BOUNDARY VALUE PROBLEMS OF SINGULAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

by CHI YEUNG LO

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1. Introduction. In a recent paper [6], this author has extended the method of the kernel function [1] to the boundary value problems of the generalized axially symmetric potentials

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{k}{y}\frac{\partial u}{\partial y}=0 \qquad (k>0,\ y>0).$$

This method can also be applied to a more general class of singular differential equations, namely

$$L[u] \equiv \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{2\mu}{x}\frac{\partial u}{\partial x}+\frac{2\nu}{y}\frac{\partial u}{\partial y}=0 \qquad (\mu,\nu>0,\ x>0,\ y>0), \tag{1.1}$$

or, equivalently,

$$L[u] \equiv \frac{\partial}{\partial x}\left(x^{2\mu}y^{2\nu}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(x^{2\mu}y^{2\nu}\frac{\partial u}{\partial y}\right)=0 \qquad (\mu,\nu>0,\ x>0,\ y>0). \tag{1.1$'$}$$

We shall derive in the sequel explicit formulas for the Dirichlet problems of (1.1) in the first quadrant of the x-y plane in terms of sufficiently smooth boundary data, and obtain an error-bound for their approximate solutions. We shall also indicate how the Neumann problem can be solved.

2. The kernel function. We introduce the following notations, where $P=(x,y)$ is a point in the x-y plane, and R is an arbitrary fixed positive constant.

$$\begin{aligned}
D &= \{P : x^2+y^2 < R^2,\ x>0,\ y>0\},\\
C_R &= \{P : x^2+y^2 = R^2,\ x \geqq 0,\ y \geqq 0\},\\
\Gamma_x &= \{P : 0 \leqq x < R,\ y=0\},\\
\Gamma_y &= \{P : x=0,\ 0 \leqq y < R\},\\
C &= C_R \cup \Gamma_x \cup \Gamma_y,
\end{aligned}$$

$s=$ the arc length on C, and $n=$ the exterior normal on C.

Then we have the following Green's formulas for two regular functions $u(x,y)$ and $v(x,y)$:

$$\iint_D vL[u]\,dx\,dy = -\iint_D x^{2\mu}y^{2\nu}[u_x v_x+u_y v_y]\,dx\,dy+\int_C x^{2\mu}y^{2\nu}v\frac{\partial u}{\partial n}\,ds \tag{2.1}$$

$$\iint_D (vL[u]-uL[v])\,dx\,dy = \int_C x^{2\mu}y^{2\nu}\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)ds. \tag{2.2}$$

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Let u be a solution of (1.1). Then, from (2.1), we have

$$E\{u, v\} \equiv \iint_D x^{2\mu}y^{2\nu}[u_x v_x + u_y v_y]\,dx\,dy = \int_C x^{2\mu}y^{2\nu}v\frac{\partial u}{\partial n}\,ds. \tag{2.3}$$

In particular, for $v = 1$,

$$\int_C x^{2\mu}y^{2\nu}\frac{\partial u}{\partial n}\,ds = 0. \tag{2.4}$$

Let $\mathscr{F}(D)$ be a class of functions satisfying (1.1) in D, such that

$$E\{u\} \equiv E\{u, u\} < \infty \tag{2.5}$$

and

$$\int_C x^{2\mu}y^{2\nu}u\,ds = 0. \tag{2.6}$$

Then $\|u\| = E\{u\}^{1/2}$ represents a Dirichlet norm or D-norm for the class $\mathscr{F}(D)$ and any nontrivial element in $\mathscr{F}(D)$ must be a non-constant function. It is known [4, 5] that a complete set of solutions of (1.1) regular at the origin, when expressed in polar coordinates, is given by

$$f_n(r, \theta) = r^{2n}P_n^{(\nu-1/2,\mu-1/2)}(1-2\sin^2\theta),$$

where n is a positive integer and $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree n. This set of functions is orthogonal with respect to the D-norm over the domain D and can be normalized to

$$u_n(r, \theta) = [c_{n,\nu-1/2,\mu-1/2}]^{-1/2}r^{2n}R^{-2n-\nu-\mu}P_n^{(\nu-1/2,\mu-1/2)}(\cos 2\theta), \tag{2.7}$$

where

$$c_{n,\alpha,\beta} = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n)\Gamma(n+\alpha+\beta+1)}. \tag{2.8}$$

The orthonormal property of $u_n(r, \theta)$ is deduced from the following formula for Jacobi polynomials [8, p. 68]:

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta[P_n^{(\alpha,\beta)}(x)]^2\,dx = 2^{\alpha+\beta+1}n^{-1}c_{n,\alpha,\beta}. \tag{2.9}$$

Let $P = (\rho, \phi)$ and $Q = (r, \theta)$ be two arbitrary points in D. Define

$$\begin{aligned} K(P, Q) = K(\rho, \phi;\ r, \theta) &= \sum_{n=1}^{\infty} u_n(\rho, \phi)u_n(r, \theta) \\ &= \sum_{n=1}^{\infty} [c_{n,\nu-1/2,\mu-1/2}]^{-1}r^{2n}\rho^{2n}R^{-4n-2\mu-2\nu} \\ &\quad \times P_n^{(\nu-1/2,\mu-1/2)}(\cos 2\theta)P_n^{(\nu-1/2,\mu-1/2)}(\cos 2\phi). \end{aligned} \tag{2.10}$$

Using a classical inequality for Jacobi polynomials [8, p. 168]

$$\max_{-1 \leqq x \leqq 1} |P_n^{(\alpha,\beta)}(x)| \sim n^q \quad \text{for} \quad q = \max(\alpha, \beta) \geqq -1/2$$

and the asymptotic expansion for the Gamma function, we see that the series (2.10) is dominated by the series

$$\sum_{n=1}^{\infty} Ar^{2n}\rho^{2n}R^{-4n}n^{2q-1} \quad \text{for} \quad q = \max(\nu - 1/2, \mu - 1/2), \tag{2.11}$$

where A is a constant independent of n. Hence the series (2.10) converges uniformly if either P and Q lie in any closed subdomain of D or Q is in C_R and P is in any closed subdomain of D. In addition,

$$u(P) = E\{K(P, Q), u(Q)\} \qquad (P \in D). \tag{2.12}$$

We shall call $K(P, Q)$ the kernel function of the class $\mathscr{F}(D)$ with respect to the metric E.

3. Dirichlet problems.

Definition. A function $f(x)$ defined on $-1 \leqq x \leqq 1$ is said to *belong to the class* L if

(A) $$\int_0^{\pi/2} \cos^{2\mu}\theta \sin^{2\nu}\theta f(\cos 2\theta)\, d\theta = 0,$$

and

(B) $f(x)$ can be expanded into a uniformly convergent series of Jacobi polynomials, i.e.,

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x), \tag{3.1}$$

where

$$a_n = [2^{\alpha+\beta+1}n^{-1}c_{n,\alpha,\beta}]^{-1} \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x) P_n^{(\alpha,\beta)}(x)\, dx. \tag{3.2}$$

Remarks.

1. Condition (A) implies that $a_0 = 0$ for any $f(x)$ in L.
2. Let $q = \max(\alpha, \beta) \geqq -1/2$ and let p be a positive integer greater than or equal to $2q+2$. Then $f(x)$ satisfies condition (B) if $f(x) \in C^p[-1, 1]$. (See [7, p. 301].)

We want to determine a function $u(P)$ in $\mathscr{F}(D)$ such that

$$\lim_{P \to Q} u(P) = f(Q), \tag{3.3}$$

where Q is a point in C_R and $f(x)$ is an element in the class L. This problem will be called the Dirichlet problem.

The representation formula for the Dirichlet problem can be obtained formally from (2.12) as

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$$u(P)=E\{K(P,Q),u(Q)\}=R^{2\mu+2\nu+1}\int_0^{\pi/2}\cos^{2\mu}\theta\sin^{2\nu}\theta F(\rho,\phi;r,\theta)f(\cos 2\theta)\,d\theta, \tag{3.4}$$

by putting $u(Q)=f(Q)=f(\cos 2\theta)$ and

$$F(P,Q)=F(\rho,\phi;R,\theta)=\frac{\partial K}{\partial r}_{|r=R}$$
$$=\sum_{n=1}^{\infty}2nR^{-2\mu-2\nu-1}[c_{n,\nu-1/2,\mu-1/2}]^{-1}\rho^{2n}R^{-2n}$$
$$\times P_n^{(\nu-1/2,\mu-1/2)}(\cos 2\theta)P_n^{(\nu-1/2,\mu-1/2)}(\cos 2\phi). \tag{3.5}$$

Using the classical inequality for Jacobi polynomials and the asymptotic expansion for the Gamma function, we can readily show that $F(P,Q)$ and its partial derivatives converge uniformly for P in any closed subdomain of D, and $u(P)$ represented by (3.4) is a solution of (1.1). To show (3.3), we note that, from our hypothesis,

$$f(\cos 2\theta)=\sum_{n=1}^{\infty}a_nP_n^{(\nu-1/2,\pi-1/2)}(\cos 2\theta), \tag{3.6}$$

where a_n is given by (3.2). For each $\theta=\theta_0$, $0\leqq\theta_0\leqq\pi/2$, the series (3.6) is a convergent series of constants. Then, by Abel's theorem, the series

$$\sum_{n=1}^{\infty}a_nt^nP_n^{(\nu-1/2,\mu-1/2)}(\cos 2\theta_0) \tag{3.7}$$

converges uniformly for $0\leqq t\leqq 1$. The condition (B) implies the uniform convergence of (3.7) for all t with $0\leqq t\leqq 1$ and θ with $0\leqq\theta\leqq\pi/2$. What we need to show is that $u(P)$ defined by (3.4) can be written as (3.7) with $t=\rho^2/R^2$.

Let $P\in D_0$ (a closed subdomain of D). Then

$$u(\rho,\phi)=R^{2\mu+2\nu+1}\int_0^{\pi/2}\sin^{2\nu}\theta\cos^{2\mu}\theta f(\cos 2\theta)F(\rho,\phi;r,\theta)\,d\theta$$
$$=\sum_{n=1}^{\infty}2n[c_{n,\nu-1/2,\mu-1/2}]^{-1}P_n^{(\nu-1/2,\mu-1/2)}(\cos 2\phi)$$
$$\times\int_0^{\pi/2}\cos^{2\mu}\theta\sin^{2\nu}\theta f(\cos 2\theta)P_n^{(\nu-1/2,\mu-1/2)}(\cos 2\theta)\,d\theta.$$

As $\rho\to R$, we have (3.3). The interchange of integration and summation is valid because of uniform convergence of (3.5) in D_0.

We summarize our result as

THEOREM 1. *Let $f(x)$ belong to the class L. Then there exists a solution of* (1.1) *given by* (3.4) *and* (3.5) *such that $u(R,\phi)=f(\cos 2\phi)$.*

The series (3.5) can be put into closed form by means of the formula for the generating function of Jacobi polynomials. Explicitly, we have

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$$F(P, Q) = F(\rho, \phi; R, \theta) = (t-1)R^{-2\mu-2\nu-1}(\sqrt{t})^{-\mu-\nu-1}.$$

$$\times \int_0^\pi \frac{d}{dk}\left[\frac{\cos(\nu-\mu)\omega}{\{k^2-(a^2+2ab\cos\chi+b^2)\}^{(\mu+\nu)/2}}\right.$$

$$\left.\times\left(\frac{k^2-(a+b\cos\chi)^2}{k^2-(b+a\cos\chi)^2}\right)^{(\nu-\mu)/2}\right]\sin^{\mu+\nu+1}\chi\,d\chi, \tag{3.8}$$

where ω is the acute angle (positive or negative) such that

$$\cot\omega = \frac{k\sin\chi\sqrt{k^2-(a^2+2ab\cos\chi+b^2)}}{k^2\cos\chi-(a+b\cos\chi)(b+a\cos\chi)}, \tag{3.9}$$

where $a = \sin\phi\sin\theta$, $b = \cos\phi\cos\theta$, $t = \rho^2/R^2$, $k = [t^{-1/2}+t^{1/2}]/2$. For details, we refer to the paper of G. N. Watson [9].

When we put $\mu = \nu = \sigma$, (3.8) is simplified to

$$F_\sigma(P,Q) = (t-1)R^{-4\sigma-1}\pi^{-1}t^{-\sigma-1/2}\int_0^\pi 2k\sigma[k^2-(a^2+2ab\cos\chi+b^2)]^{-\sigma-1}\sin^{2\sigma-1}\chi\,d\chi. \tag{3.10}$$

We observe that $F_\sigma(P, Q)$ is of constant sign whereas $F(P, Q)$ is not for arbitrary values of μ and ν. Hence, if we consider the particular case of equation (1.1) for which $\mu = \nu = \sigma$, i.e.,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+2\sigma\left(\frac{1}{x}\frac{\partial u}{\partial x}+\frac{1}{y}\frac{\partial u}{\partial y}\right) = 0 \qquad (\sigma > 0,\ x > 0,\ y > 0), \tag{3.11}$$

a stronger result on the Dirichlet problem can be obtained.

THEOREM 2. *Let $f(\cos 2\theta)$ be a continuous function defined on $0 \leqq \theta \leqq \pi/2$ and satisfying the condition* (A) *of the class L. Then there exists a solution of* (3.11) *given by*

$$u(\rho, \phi) = R^{4\sigma+1}\int_0^{\pi/2}\sin^{2\sigma}\theta\cos^{2\sigma}\theta F_\sigma(\rho, \phi; R, \theta)f(\cos 2\theta)\,d\theta \tag{3.12}$$

such that

$$\lim_{\rho\to R} u(\rho, \phi) = f(\cos 2\phi). \tag{3.13}$$

Proof. It is clear from Theorem 1 that $u(\rho, \phi)$ is a solution of (3.12). We need only to show (3.13). Firstly, we note that

$$1 = R^{4\sigma+1}\int_0^{\pi/2}\sin^{2\sigma}\theta\cos^{2\sigma}\theta F_\sigma(\rho, \phi; R, \theta)\,d\theta \tag{3.14}$$

on account of orthonormal property of the functions $u_n(r, \theta)$ defined by (2.7) and the uniform convergence of $F_\sigma(P, Q)$ for P in any closed subdomain of D.

Let $Q_0 = (R, \phi_0)$ be a fixed point in C_R, so that $0 \leqq \phi_0 \leqq \pi/2$. Let $\varepsilon > 0$ be given. Then

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there exists a $\delta > 0$ such that $|f(\cos 2\theta) - f(\cos 2\phi_0)| < \varepsilon/2$ for all θ in $S_1 = \{\theta : |\theta - \phi_0| < \delta\}$. On the other hand, for all θ in $S_2 = [0, \pi/2] - S_1$, the function $F_\sigma(P, Q) \to 0$ uniformly as $P \to Q_0$. Hence, for all P such that $|P - Q_0| < \delta_1 < \delta$, we have

$$R^{4\sigma+1} \int_{S_2} F_\sigma(\rho, \phi; R, \theta) \sin^{2\sigma}\theta \cos^{2\sigma}\theta \, d\theta < \varepsilon/4M, \tag{3.15}$$

where $M = \max_{0 \leqq \theta \leqq \pi/2} |f(\cos 2\theta)|$.

Thus we have

$$|u(\rho, \phi) - f(\cos 2\phi_0)| = \left| \int_0^{\pi/2} R^{4\sigma+1} \sin^{2\sigma}\theta \cos^{2\sigma}\theta F_\sigma(\rho, \phi; R, \theta)[f(\cos 2\theta) - f(\cos 2\phi_0)] \, d\theta \right|.$$

On splitting the integral into two parts S_1 and S_2, and applying (3.14) and (3.15), we have $|u(\rho, \phi) - f(\cos 2\phi_0)| < \varepsilon$ for all P such that $|P - Q_0| < \delta_1$.

We may obtain approximations to the solution of the Dirichlet problem by taking a finite number of terms in the series (3.5). Let

$$F_N(\rho, \phi; R, \theta) = \sum_{n=1}^{N} 2nR^{-2\mu-2\nu-1}[c_{n,\nu-1/2,\mu-1/2}]^{-1} \times \rho^{2n} R^{-2n} P_n^{(\nu-1/2,\mu-1/2)}(\cos 2\theta) P_n^{(\nu-1/2,\mu-1/2)}(\cos 2\phi). \tag{3.16}$$

Since

$$\max_{-1 \leqq x \leqq 1} |P_n^{(\nu-1/2,\mu-1/2)}(x)| = \Gamma(n+q+1)/\Gamma(n+1)\Gamma(q+1)$$

for $q = \max(\nu - 1/2, \mu - 1/2) \geqq 0$, we have

$$\begin{aligned} &|F(\rho, \phi; R, \theta) - F_N(\rho, \phi; R, \theta)| \\ &\leqq \sum_{n=N+1}^{\infty} 2R^{-2\mu-2\nu-1} \frac{(2n+\mu+\nu)\Gamma(n+1)\Gamma(n+\mu+\nu)}{\Gamma(n+\mu+1/2)\Gamma(n+\nu+1/2)} \left[\frac{\Gamma(n+q+1)}{\Gamma(n+1)\Gamma(q+1)} \right]^2 (\rho/R)^{2n} \\ &\leqq 2R^{-2\mu-2\nu-1} K v^{N+1}, \end{aligned} \tag{3.17}$$

where $\rho^2/R^2 \leqq v < 1$, and K is the sum of the convergent series

$$\sum_{p=0}^{\infty} \frac{(2p+2N+\mu+\nu+1)\Gamma(p+N+2)\Gamma(p+N+\mu+\nu+1)[\Gamma(N+p+q+2)]^2}{\Gamma(p+N+\mu+3/2)\Gamma(p+N+\nu+3/2)\Gamma(p+N+2)[\Gamma(q+1)]^2} (\rho^2/R^2)^p.$$

Thus we have the following theorem.

THEOREM 3. *Let $f(\cos 2\theta)$ satisfy the hypothesis of Theorem 1, and $\max_{0 \leqq \theta \leqq \pi/2} |f(\cos 2\theta)| = M$. Then the error in using the approximating kernel $F_N(P, Q)$ in Theorem 1 is bounded by $KM\pi v^{N+1}$ for those points P in the closed subdomain $D_0 = \{(\rho, \phi) : \rho^2/R^2 \leqq v, 0 \leqq \delta \leqq \phi \leqq \pi/2 - \delta\}$.*

REMARK. Bergman and Herriot [2] obtained a numerical solution of boundary value problem for the equation $u_{xx}+u_{yy}-C(x,y)u=0$, $C>0$. Their method can also be applied to our case by considering the kernel $F_N(\rho,\phi;R,\theta)$ in the representation formula.

4. Neumann problems. The Neumann problem is to determine a function in $\mathscr{F}(D)$ such that its normal derivative assumes a given function on the boundary C_R. The representation formula for the solution of the Neumann problem is also given by (2.12),

$$u(\rho,\phi)=u(P)=E\{K(P,Q),u(Q)\}=E\{u(Q),K(P,Q)\}$$
$$=R^{2\mu+2\nu+1}\int_0^{\pi/2}\cos^{2\mu}\theta\sin^{2\nu}\theta K(\rho,\phi;R,\theta)f(\cos 2\theta)\,d\theta, \tag{4.1}$$

where

$$K(\rho,\phi;R,\theta)=\sum_{n=1}^{\infty}[c_{n,\nu-1/2,\mu-1/2}]^{-1}\rho^{2n}R^{-2n-2\mu-2\nu}\cdot$$
$$\cdot P_n^{(\nu-1/2,\mu-1/2)}(\cos 2\phi)P_n^{(\nu-1/2,\mu-1/2)}(\cos 2\theta). \tag{4.2}$$

We shall state the main results here and omit all the details since the approach and arguments are essentially the same as in the Dirichlet problem.

THEOREM 4. *Let $f(x)$ belong to the class L. Then there exists a solution of* (1.1) *given by* (4.1) *and* (4.2), *such that*

$$\lim_{\rho\to R}\frac{\partial u(\rho,\phi)}{\partial\rho}=f(\cos 2\phi).$$

THEOREM 5. *Let $f(\cos 2\theta)$ be a continuous function defined on $0\leqq\theta\leqq\pi/2$ and satisfying the condition* (A) *of the class L. Then there exists a solution of* (3.11) *given by*

$$u(\rho,\phi)=R^{4\sigma+1}\int_0^{\pi/2}\cos^{2\sigma}\theta\sin^{2\sigma}\theta K_\sigma(\rho,\phi;R,\theta)f(\cos 2\theta)\,d\theta, \tag{4.3}$$

such that

$$\lim_{\rho\to R}\frac{\partial u(\rho,\phi)}{\partial\rho}=f(\cos 2\phi),$$

where

$$K_\sigma(\rho,\phi;R,\theta)=\sum_{n=1}^{\infty}\frac{(n+\sigma)\Gamma(n)\Gamma(n+2\sigma)}{[\Gamma(n+\sigma+1/2)]^2}\rho^{2n}R^{-2n-2\mu-2\nu}C_n^\sigma(\cos 2\phi)C_n^\sigma(\cos 2\theta)$$

and $C_n^\sigma(x)$ is the Gegenbauer polynomial of degree n.

As in the Dirichlet problem, we can get an approximate solution by taking a finite number of terms in $K(P,Q)$. However, we shall not go into details for the estimates of its error-bound.

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MICHIGAN STATE UNIVERSITY