17
Bell Inequalities

17.1 Introduction

From the dawn of the quantum age in 1900, there has been an ongoing debate, often fierce and hostile, between the supporters of the classical world view and those of the quantum world view. This debate eventually led to remarkable experiments, the convincing and repeated results of which are claimed by quantum theorists to support the predictions of quantum mechanics (QM) and not those of classical mechanics (CM). However, old paradigms tend to linger on the shelves of science long past their sell-by dates and there are extant schools of theorists that strive to find loopholes in the above-mentioned experiments. That is a legitimate activity up to a point, but the divisive nature of the debate requires commentary, which is the subject matter of this chapter.

QM was discovered only by advanced technology, so it stands to reason that any test of QM will require even more advanced technology. In the previous chapter we stressed the differences between active and passive transformations. The experiments discussed here rely on such technically difficult active transformations that the Hidden Variables (HV) theorists (for that is really what they are) opposing QM are frequently able to think of objections, known as *loopholes*, to the empirical protocols employed by the experimentalists. Some of these loopholes are reasonable but many are not. Closing those loopholes convincingly is an ongoing important activity in experimental quantum physics.

A significant feature of the experiments discussed in this chapter is that they go beyond a certain point of complexity in terms of the number of classical and quantum degrees of freedom involved. Historically, before that point had been reached, classical interpretations of quantum wave functions appeared viable and perhaps even attractive, such as that proposed by Bohm (Bohm, 1952). We shall refer to the collective of such interpretations as the *hidden variables* (HV) paradigm and the point in question as the *Heisenberg point*.

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It seems obvious with hindsight that the Heisenberg point was reached almost immediately after Schrödinger introduced his wave mechanics formulation of QM in 1926 (Schrödinger, 1926). Certainly, the wave function $\Psi(\boldsymbol{x}, t)$ for a single-particle system under observation (SUO) can be visualized as an objective wave of sorts in physical space. By the term *physical space*, we mean the three-dimensional space $\mathbb{P}^3(\Lambda)$ of extension, position, and distance, associated with a real laboratory Λ,[1] a concept that would surely have been understandable to Aristotle, Galileo, Newton, Hamilton, and Lagrange. But the Schrödinger wave function $\Psi(\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^N; t)$ for an SUO consisting of N particles is actually a time-dependent complex function over a real $3N$-dimensional space $\mathcal{C}^{3N}$, known as *configuration space*, which is quite different conceptually from $\mathbb{P}^3(\Lambda)$.

Humans are remarkably stable creatures, both physically and mentally. Good health is measured in years and mental outlook is measured in decades. The phenomenon of *persistence*, which we have attributed as the origin of objectivization, seems to us responsible for the classical world view. We see objects apparently unchanged over significant stretches of time, and we come to believe that those objects have real identities. In fact, we are generally strongly conditioned mentally to think in such classical terms. We imagine ourselves as sitting in physical space $\mathbb{P}^3(\Lambda)$ and we try to relate all experience to it. We may refer to this as *mental persistence*, or equivalently, *classical conditioning*.

Absolute physical space and absolute time (Newton, 1687)[2] form the conceptual foundations on which Newtonian mechanics was built. We shall refer to this mathematical model as *space-time*, noting the hyphen between "space" and "time." This hyphen is important: it marks the recognition that observers operate in their physical space with a *process* view of time rather than a *manifold* or Block Universe perspective. Indeed, QM seems to us empirically meaningful only in the space-time perspective, simply because probability, as we know it, makes no sense otherwise.[3]

Not only does the space-time model give a remarkably good account of many physical phenomena such as planetary orbits, but it conforms excellently to our inherited classical conditioning: we think in terms of objects moving around physical space with the passage of absolute time.

The advent of relativity did little to change this in practice, for the basic reason that the speed of light is so great in relative terms that the space-time model is a good one for all practical purposes. Indeed, its replacement, the Block

[1] We follow here Schwinger's statement, quoted in Chapter 24, that space and time are contextually defined by apparatus.

[2] We refer the reader to the *Principia* for Newton's defining comments on what he meant by "absolute space" and "absolute time".

[3] There will be theorists who interpret QM and probability in terms of abstract mathematical structures over Block Universe manifolds, with operator norms, C^* algebras, and such like. Those approaches to QM equate empirical physics with mathematical physics and generally neglect observers and apparatus, thereby usually having a generalized propositional classification of one.

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Universe manifold $\mathbb{M}^4$ (four-dimensional Minkowski spacetime), came into the orbit of theorists' attention only relatively recently, in 1908 (Minkowski, 1908). It is too soon for human conditioning to have evolved (if it ever will) to the point where the world around us is interpreted naturally according to the $\mathbb{M}^4$ spacetime paradigm rather than the Newtonian space-time paradigm.

When Schrödinger introduced his wave function in 1926, therefore, classical conditioning ensured that attempts would be made to understand his theory in classical mechanical terms. Indeed, Schrödinger himself originally interpreted his wave function in realist terms, but soon realized that the configuration space argument required a revision in that interpretation.

Such attempts have persisted: there remain groups of theorists who have the agendas of either refuting relativity (Dingle, 1967) or refuting QM (Bohm, 1952), or refuting both (Kracklauer, 2002). A theme common to these agendas is contextual incompleteness, with little or no attention being paid to the details of relative internal context (discussed in Section 2.12). A significant point to make here is that none of these theorists has reported experiments that *they* have actually done, so *nullius in verba*[4] and *Hitchens' razor*[5] can legitimately be applied here.

Fortunately, the matter goes beyond talking-shop physics because of the remarkable contribution of the theorist J. S. Bell. In 1964, Bell discussed an HV model of a two spin-half SUO such as a two-electron state (Bell, 1964). He based his model on a number of reasonable assumptions about the nature of classical reality and discovered the possibility of testing those assumptions. The significance of this is that Bell opened the door to empirical tests of quantum principles, because these give predictions different from those based on classical principles.

To understand his ideas, we need to review some concepts.

Parameters

Parameters are used in the description of apparatus and observers, and they are to be found in relative internal context and relative external context. Every classical model, including that of Bell, relies on parameters, such as the masses of the particles being observed, orientation of apparatus relative to the laboratory, and so on. Parameters may be physical constants determined by previous experiments, such as particle masses, or they may be classical degrees of freedom under the control of the observer, such as orientation angles of a Stern–Gerlach (SG) main magnetization field.

Parameters should not be confused with dynamical variables, which are theoretical constructs related to states of SUOs. Parameters are generally expressed in terms of classical real numbers, associated with agreed systems of units. Some parameters are specific, meaning that they are assumed to be exact to within

[4] Take no one's word for it.

[5] That which is asserted without proof can be dismissed without proof.

measurement errors, or statistical in nature. An example of a specific parameter is the mass of the free electron in vacuo, $9.1938291\ldots \times 10^{-31}$ kg. An example of a statistical parameter would be the temperature of the laboratory in which an experiment was being carried out.

Variables

Variables have to do with states of SUOs. As the word suggests, variables change over the course of an experimental run. The point of doing experiments is to determine to what extent observers understand those changes.

Model Limits

A good model will make predictions about the range over which the variables can go, for a fixed set Θ of parameters. For example, Newtonian mechanics and Newton's law of gravitation predicts how far the Earth could go from the Sun in its annual orbit, given the known parameters, such as the mass of the Earth and of the Sun, and the current position and velocity of the Earth relative to the Sun.

Bell showed, in a CM model of two spin-half particles based on "reasonable" classical assumptions such as locality, that there was a limit Λ to a certain empirically measurable function F of the parameters Θ,[6] given in the form of an inequality $F(\Theta) \leqslant \Lambda$. Such inequalities are now universally referred to as Bell inequalities (Bell, 1988). They are in focus in this chapter, because they provide an empirical test of CM predictions versus those of QM.

In the following sections we shall first discuss the SG experiment from a classical perspective. Then we shall show how a classical Bell-type inequality can be derived. Then we shall show how standard QM predicts the possibility of a nonclassical violation of this inequality. Then we shall give the quantized detector network (QDN) account of the violation.

17.2 The Stern–Gerlach Experiment

Along with the double-slit experiment, the SG experiment serves as a standard test of QM principles. In this section we discuss the latter experiment from the perspective of HV theorists applying standard CM principles.

The stage diagram Figure 17.1 shows the SG architecture. Observer Ted operates a preparation device T that directs a beam of particles[7] 1_0 toward an SG module $S^{\boldsymbol{a}}$ containing a strong inhomogeneous magnetic field aligned principally

[6] By *empirically measurable*, we mean fixing the parameters and then performing an experiment to calculate a value for F.

[7] We emphasize again that such descriptions express a classical interpretation of what happens in the laboratory: Ted pushes buttons in preparation device T and Alice looks at signals on the detector screen. Neither Ted nor Alice see "particles" in the way spectators observe baseballs or cricket balls.

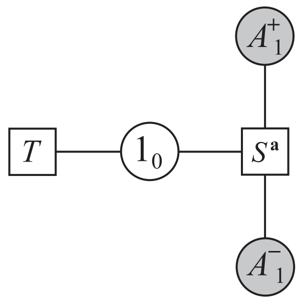

Figure 17.1. Stage diagram of the SG experiment.

along a fixed direction $\boldsymbol{a}$. This direction is the significant parameter in this experiment. Interestingly, while the classical explanation of what is going on requires a knowledge of the magnetic field in module $S^{\boldsymbol{a}}$, that turns out not to be directly relevant to the outcome of the experiment; what is important here is the forest, not the trees.

It was observed by Stern and Gerlach (Gerlach and Stern, 1922a,b) that on a detecting screen on the opposite side of $S^{\boldsymbol{a}}$ to the source T, signals were observed in two principal areas, or spots labeled A_1^+ and A_1^- in Figure 17.1. These spots will henceforth be assumed to be disjoint, that is, not overlapping, because experience suggests that this can always be arranged to a good approximation in real experiments.

The standard CM interpretation of these empirical observations is that particles from T have entered the magnetic field in $S^{\boldsymbol{a}}$ and that, by virtue of interaction with that field, some of them have been deflected into region A_1^+ while the rest have been deflected into region A_1^-. The standard CM theory is based on the following assumptions.

Extended Charged Particles

Each particle passing through $S^{\boldsymbol{a}}$ has nontrivial electromagnetic structure, meaning that it is not a point but an extended system of electric charge density that is swirling around in such a way as to create a time-dependent magnetic dipole moment, denoted $\boldsymbol{\mu}$.

Electrodynamic Forces

As it passes through the SG module $S^{\boldsymbol{a}}$, each particle's instantaneous magnetic dipole moment $\boldsymbol{\mu}$ interacts with the inhomogeneous magnetic field $\boldsymbol{B}$ in $S^{\boldsymbol{a}}$, an interaction modeled by a term proportional to $\boldsymbol{\mu} \cdot \boldsymbol{B}$ in the classical Hamiltonian. This generates an additional contribution to the standard Lorentz force $q\boldsymbol{v} \times \boldsymbol{B}$ on that particle, and it is this additional force that is interpreted classically as responsible for what Stern and Gerlach observed. Here q is the electric charge of the particle, $\boldsymbol{v}$ is its instantaneous velocity, and $\boldsymbol{B}$ is the effective magnetic field in which the particle is moving.

The fact that Stern and Gerlach observed two regions, labeled by us A_1^+ and A_1^- in Figure 17.1, is indisputable. Therefore, a CM calculation should account for that splitting. We assume that that can be done. There are good examples of such bifurcations in CM, such as comets either falling into elliptic captured orbits around the sun or entering the solar system once and then leaving forever on hyperbolic orbits. Another example is from the statistics of single car accidents on icy roads: cars will veer off a road either to the left or to the right.

Lacking more detailed information, particularly about any hidden variables that could contribute additional forces in the SG experiment, we simply assume that if we could complete such calculations, they would show the necessary bifurcation into either A_1^+ or A_1^-.[8]

Deterministic Outcomes

The classical electromagnetic forces guiding each particle through module $S^{\boldsymbol{a}}$ are deterministic, meaning the following. Suppose that at the start of a run, one of the particles, #1, in 1_0 has an initial position $\boldsymbol{r}_0$ and initial velocity $\boldsymbol{v}_0$. Subsequently, it enters $S^{\boldsymbol{a}}$ and ends up somewhere on the detecting screen. Now a typical beam will consist of a vast number of particles. Suppose another one of the particles, #2, started off in 1_0 at exactly the same initial position, the same initial velocity, and with the same internal degrees of freedom (HVs), as #1, although at some other time. Then it would subsequently follow exactly the same spatial path and would end up in exactly the same spatial position on the screen as #1.

Consider now a beam of many particles from 1_0 passing through the same apparatus. There will be a spread of initial positions and initial momenta, with most particle velocities approximately along the same common direction toward $S^{\boldsymbol{a}}$. There may also be some additional hidden variables, such as those involved with the internal charge structure of the particles. Suppose there are N particles in such a beam. Then the ith particle starts in 1_0 with a set of initial variables denoted $\boldsymbol{\theta}^i$. This includes any hidden variables.

Now according to the deterministic principle outlined above, that particle will certainly end up in A_1^+ or else certainly in A_1^-, assuming perfect transmission, that is, with no outside interference and with completely efficient detection. It is important to note that for given $\boldsymbol{\theta}^i$, which of the two outcomes occurs is not the question; what matters is the asserted fact that one of them will be definitely forced to occur by the deterministic nature of the mechanics. Moreover, this is not a random outcome: which of the two sites it will be is predetermined by θ^i.

[8] The assumption is made that cases where the particle would end up in the middle of the detecting screen between A_1^+ and A_1^- constitute a tiny proportion of the whole and can be neglected. There must be, for instance, relatively few, if any, comets that are on genuinely parabolic trajectories.

It is also important to note that the question of whether we could calculate that outcome is irrelevant to the discussion.[9]

The classical deterministic paradigm outlined above leads naturally to the following assertion. Consider the set $\Theta \equiv \left\{\boldsymbol{\theta}^1, \boldsymbol{\theta}^2, \ldots, \boldsymbol{\theta}^N\right\}$ of all the initial HVs associated with a beam in a given run. Then this set can be regarded as the union $\Theta = \mathcal{A}^+ \cup \mathcal{A}^-$ of two disjoint subsets $\mathcal{A}^+$ and $\mathcal{A}^-$, where $\mathcal{A}^+$ is the subset of HVs that send a beam particle into A_1^+ and $\mathcal{A}^-$ is the subset of HVs that send a beam particle into A_1^-.

Classical Counterfactuality

So far, nothing controversial has been written. To understand the next step, we need to make a small diversion. There is a piece of logic (or metaphysics, if you prefer) that is at the heart of the problem in "understanding" QM. It is the essential point on which Einstein, Podolsky, and Rosen pitched their famous argument against standard QM (Einstein et al., 1935). It goes by the name of *counterfactuality*. A basic definition is the following.

Definition 17.1 If $\boldsymbol{P}$ and $\boldsymbol{Q}$ are two propositions (that is, statements) and $\boldsymbol{P}$ is known to be false, then the statement $\boldsymbol{P}$ implies $\boldsymbol{Q}$, written $\boldsymbol{P} \Rightarrow \boldsymbol{Q}$, is a counterfactual (statement) if it is logically true, despite the fact that $\boldsymbol{P}$ is false.

Example 17.2 Let $\boldsymbol{P}$ be the proposition **My computer is broken today** and $\boldsymbol{Q}$ is the proposition **I cannot do any typing today**. Then it is true that $\boldsymbol{P} \Rightarrow \boldsymbol{Q}$. But actually, at the time of writing this, $\boldsymbol{P}$ is false: my computer is working and I have just typed out these words.

There are serious questions about the above definition, particularly in view of the fact that we cannot avoid employing counterfactual reasoning in physics. The most obvious problem is that Definition 17.1 is contextually incomplete: there is no reference in it to any observer for whom the "truth" concept is valid, nor is there any statement of a method by which a "truth" value could be established. This is not hair-splitting: it matters in physics. That is why we have qualified the word *truth* in that definition with the adjective *logical*. Logic is not physics.

When it comes to physics, CM adopts a modified form of counterfactuality, referred to here as *the principle of classical counterfactuality*, also known

[9] It is remarkable that QM does not even attempt any such calculation, because not only is it regarded as a vacuous enterprise, but there is also no mechanism in QM to deal with individual outcomes. QM is after all a theory about the statistics of observation. HV theorists, on the other hand, do not consider it contradictory to assert (1) that there are variables they cannot know anything about, but (2) if they did know about them, they could predict everything about their behaviour, in principle.

as *counterfactual definiteness*. In brief, this principle states that counterfactual statements can be empirically meaningful in physics, that it is meaningful to assume SUOs exist and have properties that are independent of actual observation.

Classical counterfactuality is used by observers in all experiments, even those that are described by QM. This is because state preparation is based on prior established context: how do we *know* that a beam of particles is entering an SG apparatus? Such knowledge arises only because we believe in the constancy, internal consistency, and reliability of the laws of physics that we have established over centuries, and because we have previously prepared such a beam many times and tested it for its properties, a process called *calibration*.

It is just a fact of life that when it comes to actually using such a beam in a complete run in an experiment, we can no longer test it. If we were to interrupt the beam before it entered the apparatus, in order to check that the beam is what we think it is, then the rest of the run could not subsequently take place. We literally cannot have our cake and eat it.

This is perhaps the only place in science where metaphysics is critical. It is a self-evidently vacuous assertion to say that we cannot complete an uninterrupted run if we constantly interrupt it to check on the state of the apparatus. It is a quasi-religious belief throughout all of science that calibration allows us to make certain assumptions without the need to check them: *I have pressed this button and I am confident that a beam of electrons is now on its way into an SG module.*

It is on such a basis that when each run of the real experiment starts, we do not check the beam any more but rely on classical counterfactuality: that what we are doing in the beam preparation stage means what we think it means.

Classical counterfactuality is an article of faith in the laws of persistence and consistency, and in a universe that does not play tricks on us by arbitrarily changing its laws. It is implicit in the protocol of *every* experiment. The point about Bell inequality experiments is that they show that on the quantum level, classical counterfactuality is a false principle. That surely is enough to worry anyone, because it means we do not really "understand" physical reality: it is not completely described by a classical world view, only most of the time.

The relevance of classical counterfactuality to the SG experiment is as follows. Suppose that the same, identical (at the HV level) beam had been sent into the given SG apparatus but with one crucial difference: the direction of the magnetic field had previously been altered by the observer and was now along some different direction $\boldsymbol{b}$ and not $\boldsymbol{a}$. According to classical counterfactuality, it is legitimate to discuss both scenarios "simultaneously." Of course, that is fanciful, for it is not possible to go back and alter the past, as far as we know.

Classical counterfactuality and the principle of CM, however, allow us to *imagine* the possibility of a different past. We are permitted to decompose Θ as the union $\Theta = \mathcal{B}^+ \cup \mathcal{B}^-$, where $\mathcal{B}^+$ is the subset of HVs that would have sent a beam particle into B_1^+ and $\mathcal{B}^-$ is the subset of HVs that would have sent a

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beam particle into B_1^-, **if** the magnetic field axis had been $\boldsymbol{b}$ and not $\boldsymbol{a}$. Here B_1^+ and B_1^- are the two regions on the detecting screen observed with module $S^{\boldsymbol{b}}$.

Classical counterfactuality allows us to go further and to make the decomposition

$$\Theta = (\mathcal{A}^+ \cap \mathcal{B}^+) \cup (\mathcal{A}^+ \cap \mathcal{B}^-) \cup (\mathcal{A}^- \cap \mathcal{B}^+) \cup (\mathcal{A}^- \cap \mathcal{B}^-). \tag{17.1}$$

This is a proposition rather different from the assertion $\Theta = \mathcal{A}^+ \cup \mathcal{A}^- = \mathcal{B}^+ \cup \mathcal{B}^-$: we could at least perform each experiment $S^{\boldsymbol{a}}$ and $S^{\boldsymbol{b}}$ separately, at different times, with magnetization direction well defined each time, whereas none of the four terms on the right-hand side of (17.1) could be tested directly. For instance, the first term, $\mathcal{A}^+ \cap \mathcal{B}^+$, is the set of HV that would send a beam particle into A_1^+ **if** the magnetization axis were along direction $\boldsymbol{a}$, and would send the same beam particle into B_1^+ **if** the magnetization axis were along direction $\boldsymbol{b}$ instead of $\boldsymbol{a}$.

In the real world it is physically not possible to have both directions $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *in the same run.*

That is not considered a problem by theorists who accept the CM paradigm, but is a point of view inconsistent with Wheeler's participatory principle, stated in Chapter 1. It is on such points of interpretation and natural philosophy that rest the irreconcilable differences between CM and QM. These *very* different, incompatible visions of physical reality are what this debate is all about.

Counterfactual Probabilities

HV theorists address the problem of the meaning of the counterfactual intersection $\mathcal{A}^+ \cap \mathcal{B}^+$ by the following argument. Consider a very large number $N^{\boldsymbol{a}}$ of particles, prepared by Ted in a standard way, passed through $S^{\boldsymbol{a}}$. By counting the number $N(A_1^+)$ that land in A_1^+, we can estimate the probability $P_{CM}(\mathcal{A}^+) \simeq N(A_1^+)/N^{\boldsymbol{a}}$ that a particle would land in A_1^+ and not in A_1^-. Likewise, if the axis were $\boldsymbol{b}$, then we can estimate the probability $P_{CM}(\mathcal{B}^+) \simeq N(B_1^+)/N^{\boldsymbol{b}}$ that a particle would land in B_1^+ and not in B_1^-.

Note that we are now discussing a probability measure on the set Θ of hidden variables associated with a beam prepared in a standard way by Ted.

According to CM, the rules of classical probability apply, so the counterfactual probability $P_{CM}(\mathcal{A}^+ \cap \mathcal{B}^+)$ that a particle would land in A_1^+ *if* the magnetization axis were $\boldsymbol{a}$, but would have landed in B_1^+ *if* the magnetization axis were $\boldsymbol{b}$, is given by the rule

$$P_{CM}(\mathcal{A}^+ \cap \mathcal{B}^+) = P_{CM}(\mathcal{A}^+) P_{CM}(\mathcal{B}^+). \tag{17.2}$$

HV theorists simply cannot escape this assertion, unless they are prepared to introduce novel possibilities such as nonlocality, contextual effects, and such like, and these generally make their line of argument unappealing.

The problem is, we cannot do any experiments to measure $P_{CM}(\mathcal{A}^+ \cap \mathcal{B}^+)$ directly, or any of the other counterfactual probabilities directly, because as

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stated, only one magnetization axis can be set up at a time. We shall call this the *simultaneity problem*.

17.3 Circumventing the Simultaneity Problem

An ingenious way of getting around the simultaneity problem is to use two electrons or photons at a time, as follows.

Spin-Zero Two-Electron States

In both CM and QM, electrons are imagined as having an internal degree of freedom called *spin*, associated with angular momentum. There is great evidence for the empirical validity of this idea.

CM and QM treat spin differently, however: the former puts no restriction on the direction of an electron's spin axis or the magnitude of the electron's angular momentum, whereas the latter requires external context (that is, the parameters associated with the apparatus being used) and predicts the quantization of angular momentum in units of $\hbar$, the reduced Planck's constant.[10] Despite their fundamental differences, however, both CM and QM theorists are happy to discuss a two-electron state with total angular momentum of zero.

This means the following in both paradigms: if a spin-zero state of two electrons is prepared and subsequently one of the electrons is passed through $S^{\boldsymbol{a}}$ and observed to land in region A_1^+, then if the other electron was passed through another, identical[11] apparatus $\overline{S}^{\boldsymbol{a}}$, then that second electron would land in region $\overline{A}_1^-$ and not in $\overline{A}_1^+$. Likewise, for every electron that landed in A_1^-, its partner electron would land in $\overline{A}_1^+$.

There is a useful fact about electrons that we can exploit: electrons repel each other. If we fired a beam of two-electron states along a given direction, we should expect that beam to spread out due to this repulsion. This would then allow us to make observations on separate electrons.

We are in position now to circumvent the simultaneity problem as follows.

Alice, Bob, and Ted

Imagine now three experimentalists, Alice, Bob, and Ted, with apparatus shown schematically in Figure 17.2(a). We will call this the *enhanced Stern–Gerlach* (eSG) experiment. In eSG, Bob has an identical copy $\overline{S}^{\boldsymbol{a}}$ of Alice's SG module $S^{\boldsymbol{a}}$, except Bob's module is displaced in the laboratory so as not to overlap Alice's module in their common physical space $\mathbb{P}^3(\Lambda)$.

By stage Σ_0, Ted has prepared a beam of two-electron, spin-zero states. By stage Σ_1, the beam has split by electric repulsion into subbeams 1_1 and 2_1.

[10] The angular momentum of an electron state is generally discussed in terms of $\hbar/2$.

[11] Empirically identical, apart from being spatially displaced, so not identical in the sense of Leibniz.

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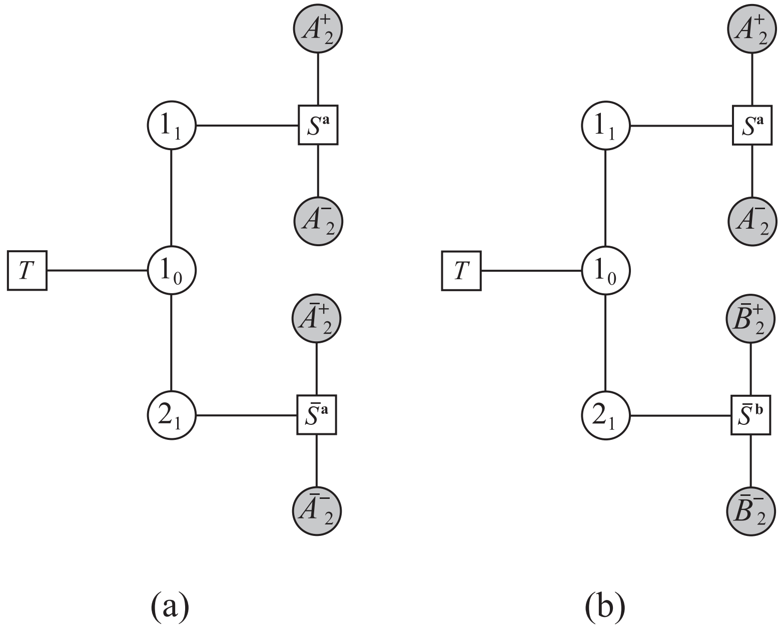

Figure 17.2. The enhanced SG experiment on two-electron, spinless states.

Subbeam 1_1 then enters Alice's SG module $S^{\boldsymbol{a}}$, while subbeam 2_1 enters Bob's module $\overline{S}^{\boldsymbol{a}}$. By conservation of angular momentum (which holds in both CM and QM), whenever Bob observes an electron in region $\overline{A}_2^+$ he can be sure that Alice has observed her electron in region A_2^-, and so on.

It is important to appreciate that this statement is not about statistics; it is about what happens in each individual run. The conservation of angular momentum (as well as that of electric charge, energy, linear momentum, and other classically motivated conserved quantities) takes place at the emergent, process time level of observation. It is an extraordinary and deep fact that in QM, there are classical concepts that can be relied on. The neutrino was discovered precisely because the conservation of energy applies at the individual-run level in beta decay, as proposed by Pauli in 1930, not at the statistical level, as proposed by Bohr.[12]

Now we come to the circumvention of simultaneity problem. The trick here is that Bob need not set the main magnetic field axis in his module to be in the same direction $\boldsymbol{a}$ as that set by Alice in her module $S^{\boldsymbol{a}}$. Suppose Bob sets his

[12] At the time, beta decay experiments could not account for a discrepancy between the energy going into a beta decay process and that going out. Pauli proposed that there was an unobserved particle, now known as the neutrino and subsequently detected by Cowan, Reines, and collaborators in 1956 (Cowan et al., 1956), carrying off the energy discrepancy.

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field axis to some different direction $\boldsymbol{b}$ (we will now refer to his module as $\overline{S}^{\boldsymbol{b}}$ rather than $\overline{S}^{\boldsymbol{a}}$). The relevant stage diagram is now Figure 17.2(b). Then we can give an operational definition of $\Pr_{CM}(\mathcal{A}^+ \cap \mathcal{B}^+)$ and the other counterfactual probabilities as follows.

Consider a large number N of runs in the eSG experiment. Suppose Alice and Bob count their respective outcomes, recording the time of each. Afterward, they come together and compare data sets and times. Then for every outcome in which Alice found her electron in A_2^+ *and* Bob found his electron in $\overline{B}_2^-$, in the same run, then that is counted as originating from the subset $\mathcal{A}^+ \cap \mathcal{B}^+$, as if all that information came from the original DS experiment, with a single electron. Essentially, the second electron observed by Bob is like a ghost version of the electron observed by Alice, and both are observed in the same run. Of course, care is taken to take the opposite spins into account. Given the total count, an estimate for the counterfactual probability $\Pr_{CM}(\mathcal{A}^+ \cap \mathcal{B}^+)$ follows immediately.

With this procedure, it is clear that total probability is conserved, that is, we have

$$\Pr_{CM}(\mathcal{A}^+ \cap \mathcal{B}^+) + \Pr_{CM}(\mathcal{A}^+ \cap \mathcal{B}^-) + \Pr_{CM}(\mathcal{A}^- \cap \mathcal{B}^+) + \Pr_{CM}(\mathcal{A}^- \cap \mathcal{B}^-) = 1. \tag{17.3}$$

Bob & Carol & Ted & Alice

To derive the appropriate Bell inequality, the analysis requires the above discussion to be extended to a third direction, $\boldsymbol{c}$, with observer Carol, giving the decomposition $\Theta = \mathcal{C}^+ \cap \mathcal{C}^-$, where $\mathcal{C}^+$ is that subset of Θ that would send an electron into $\mathcal{C}^+$, and similarly for $\mathcal{C}^-$. Classical counterfactuality also entitles us to make the decomposition

$$\begin{aligned}\Theta = {} & (\mathcal{A}^+ \cap \mathcal{B}^+ \cap \mathcal{C}^+) \cup (\mathcal{A}^+ \cap \mathcal{B}^+ \cap \mathcal{C}^-) \cup (\mathcal{A}^+ \cap \mathcal{B}^- \cap \mathcal{C}^+) \cup \\ & (\mathcal{A}^+ \cap \mathcal{B}^- \cap \mathcal{C}^-) \cup (\mathcal{A}^- \cap \mathcal{B}^+ \cap \mathcal{C}^+) \cup (\mathcal{A}^- \cap \mathcal{B}^+ \cap \mathcal{C}^-) \cup \\ & (\mathcal{A}^- \cap \mathcal{B}^- \cap \mathcal{C}^+) \cup (\mathcal{A}^- \cap \mathcal{B}^- \cap \mathcal{C}^-).\end{aligned} \tag{17.4}$$

A Bell Inequality

We are now in position to construct a Bell inequality for this experiment. The first step is to direct the discussion from the set Θ of HV (which by definition we have no knowledge about), to outcome probabilities, for which we have empirical data. Suppose we passed a beam with a large number N of electrons through our SG device, and repeated that run a large number of times. By counting electron impacts on the detector screen, we would then determine outcome frequencies, and finally we would be in position to discuss probabilities.

Given a probability measure P over Θ, then for any subsets A, B of Θ, we have the rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{17.5}$$

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With this rule and what we know about the subsets involved, we can deduce the following relations:

$$P_{CM}(\mathcal{A}^+ \cap \mathcal{B}^-) = P_{CM}(\mathcal{A}^+ \cap \mathcal{B}^- \cap \mathcal{C}^+) + P_{CM}(\mathcal{A}^+ \cap \mathcal{B}^- \cap \mathcal{C}^-), \tag{17.6}$$

$$P_{CM}(\mathcal{B}^+ \cap \mathcal{C}^-) = P_{CM}(\mathcal{A}^+ \cap \mathcal{B}^+ \cap \mathcal{C}^-) + P_{CM}(\mathcal{A}^- \cap \mathcal{B}^+ \cap \mathcal{C}^-), \tag{17.7}$$

$$P_{CM}(\mathcal{A}^+ \cap \mathcal{C}^-) = P_{CM}(\mathcal{A}^+ \cap \mathcal{B}^+ \cap \mathcal{C}^-) + P_{CM}(\mathcal{A}^+ \cap \mathcal{B}^- \cap \mathcal{C}^-). \tag{17.8}$$

Every term in these equations is a probability, so is nonnegative. Adding (17.6) to (17.7) and subtracting (17.8), we exploit this nonnegativity directly to find the Bell inequality

$$\boxed{P_{CM}(\mathcal{A}^+ \cap \mathcal{B}^-) + P_{CM}(\mathcal{B}^+ \cap \mathcal{C}^-) \geqslant P_{CM}(\mathcal{A}^+ \cap \mathcal{C}^-).} \tag{17.9}$$

This and similar inequalities are the focus of interest in numerous experiments.

Exercise 17.3 Use (17.5) and other relevant information to prove (17.6), (17.7), and (17.8). Hence prove (17.9).

It has to be pointed out that there are more serious questions about this inequality than those we raised about $P_{CM}(\mathcal{A}^+ \cap \mathcal{B}^+)$. We used two-spin states to circumvent that latter problem, but that approach cannot deal with *three* simultaneous magnetization axes. Essentially, we have to perform separate subexperiments to determine the empirical values $P_{EMP}(\mathcal{A}^+ \cap \mathcal{B}^-)$, $P_{EMP}(\mathcal{B}^+ \cap \mathcal{C}^-)$, and $P_{EMP}(\mathcal{A}^+ \cap \mathcal{C}^-)$ separately, ensuring that standardization across all three subexperiments makes the test of the inequality (17.9) beyond reasonable doubt.

We note in passing that (17.9) looks like the triangle inequality $d(a,b) + d(b,c) \geqslant d(a,c)$ defined for a metric space, where $d(a,b)$ is the "distance" between elements a and b of the metric space.

17.4 The Standard Quantum Calculation

We give now a brief account of the standard QM calculation used to test the inequality (17.9). We will use the nonrelativistic Pauli electron theory, but taking account only of the internal spin degree of freedom of each particle, assumed to be an electron. This is modeled by a qubit Hilbert space $\mathcal{Q}$, no different in mathematical structure from the qubits used to construct QDN quantum registers.

Calibration of Single Electron States

With reference to Figure 17.1, consider an uncalibrated beam of electrons sent by Ted into an SG calibration module $S^{\boldsymbol{k}}$ where $\boldsymbol{k} = (0,0,1)$ in standard Cartesian coordinates defined previously by Ted. As discussed above, this beam will split into two subbeams, denoted K_1^+ and K_1^-. An electron state entering K_1^+ will be

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denoted $|+\boldsymbol{k}\rangle$, while one entering K_1^+ will be denoted $|-\boldsymbol{k}\rangle$. These two states form a *calibration basis* for $\mathcal{Q}$ and satisfy the orthonormality conditions

$$\langle +\boldsymbol{k}|+\boldsymbol{k}\rangle = \langle -\boldsymbol{k}|-\boldsymbol{k}\rangle = 1, \quad \langle +\boldsymbol{k}|-\boldsymbol{k}\rangle = 0. \tag{17.10}$$

In contrast to QDN, standard QM tends to work with a fixed Hilbert space over any given run. Neither of these approaches is incorrect: they encode the same information differently. We will use the calibration basis as a reference to describe all state vectors and other bases in the following discussion.

Having calibrated his module T, Ted now uses it to create a beam of normalized particle states, $|\Psi\rangle$, at stage Σ_0, given by

$$|\Psi\rangle = u|+\boldsymbol{k}\rangle + v|-\boldsymbol{k}\rangle, \tag{17.11}$$

where $|u|^2+|v|^2=1$.[13]

With reference to Figure 17.1, now suppose a beam of particles represented by such a state is subsequently sent through SG module $S^{\boldsymbol{a}}$, where the main magnetic field direction $\boldsymbol{a}$ is given by $\boldsymbol{a} = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$, where θ and ψ are standard spherical polar coordinates relative to the standard Cartesians referred to hitherto. Any beam passing through $S^{\boldsymbol{a}}$ will in turn be split into two components, denoted A_1^+ and A_1^-, as discussed above. In standard QM, each of these components will be associated with orthogonal, normalized quantum outcome states $|+\boldsymbol{a}\rangle$ and $|-\boldsymbol{a}\rangle$, respectively, and these can be used to form an orthonormal preferred basis for $\mathcal{Q}$, associated with $S^{\boldsymbol{a}}$.

Standard quantum theory gives the following relations between the calibration basis $\{|+\boldsymbol{k}\rangle, |-\boldsymbol{k}\rangle\}$ and the preferred basis $\{|+\boldsymbol{a}\rangle, |-\boldsymbol{a}\rangle\}$ (the outcome states), up to inessential arbitrary overall phase factors:

$$\begin{aligned} |+\boldsymbol{k}\rangle &= \frac{\sin\theta}{\sqrt{2-2\cos\theta}}|+\boldsymbol{a}\rangle + \frac{\sin\theta}{\sqrt{2+2\cos\theta}}|-\boldsymbol{a}\rangle, \\ |-\boldsymbol{k}\rangle &= \frac{(1-\cos\theta)}{\sqrt{2-2\cos\theta}}|+\boldsymbol{a}\rangle - \frac{(1+\cos\theta)}{\sqrt{2+2\cos\theta}}|-\boldsymbol{a}\rangle. \end{aligned} \tag{17.12}$$

From this we can find the amplitude $\mathcal{A}(+\boldsymbol{a}|+\boldsymbol{k}) \equiv \langle +\boldsymbol{a}|+\boldsymbol{k}\rangle$ for outcome state $|+\boldsymbol{a}\rangle$ given initial state $|+\boldsymbol{k}\rangle$, and so on. Hence we can find the conditional probabilities. We find, for example,

$$\Pr(+\boldsymbol{a}|+\boldsymbol{k}) \equiv |\mathcal{A}(+\boldsymbol{a}|+\boldsymbol{k})|^2 = \cos^2(\tfrac{1}{2}\theta). \tag{17.13}$$

Two-Spin States

Disregarding all factors inessential to the present discussion, a normalized two half-spin state $|\Phi\rangle$ of zero total spin zero is given by

$$|\Phi\rangle = \tfrac{1}{\sqrt{2}}|+\boldsymbol{k}^a\rangle \otimes |-\boldsymbol{k}^b\rangle - \tfrac{1}{\sqrt{2}}|-\boldsymbol{k}^a\rangle \otimes |+\boldsymbol{k}^b\rangle, \tag{17.14}$$

[13] The reader will appreciate by now how difficult it is to describe such a process without using suggestive, misleading language.

where the superscripts label the two spin-half particles.[14] We now imagine that particle a is sent through Alice's module $S^{\boldsymbol{a}}$ and particle b is sent through Bob's module $S^{\boldsymbol{b}}$.

Using (17.12), we may write

$$\begin{aligned} |+\boldsymbol{k}^a\rangle &= \alpha^a|+\boldsymbol{a}\rangle + \beta^a|-\boldsymbol{a}\rangle, \quad |+\boldsymbol{k}^b\rangle = \alpha^b|+\boldsymbol{b}\rangle + \beta^b|-\boldsymbol{b}\rangle, \\ |-\boldsymbol{k}^a\rangle &= \gamma^a|+\boldsymbol{a}\rangle + \delta^a|-\boldsymbol{a}\rangle, \quad |-\boldsymbol{k}^b\rangle = \gamma^b|+\boldsymbol{b}\rangle + \delta^b|-\boldsymbol{b}\rangle, \end{aligned} \tag{17.15}$$

where again, superscripts label particles and

$$\alpha^a \equiv \frac{\sin\theta^a}{\sqrt{2-2\cos\theta^a}}, \quad \alpha^b \equiv \frac{\sin\theta^b}{\sqrt{2-2\cos\theta^b}}, \tag{17.16}$$

and so on. Note that here $\{|+\boldsymbol{a}\rangle, |-\boldsymbol{a}\rangle\}$ is a preferred basis for $\mathcal{Q}^a$, the qubit associated with Alice's electron, and $\{|+\boldsymbol{b}\rangle, |-\boldsymbol{b}\rangle\}$ is a preferred basis for $\mathcal{Q}^b$, the qubit associated with Bob's electron.

Given (17.15), we readily find

$$|\Phi\rangle = \tfrac{1}{\sqrt{2}} \begin{matrix} \{(\alpha^a\gamma^b - \gamma^a\alpha^b)|+\boldsymbol{a}\rangle\otimes|+\boldsymbol{b}\rangle + (\alpha^a\delta^b - \gamma^a\beta^b)|+\boldsymbol{a}\rangle\otimes|-\boldsymbol{b}\rangle \\ +(\beta^a\gamma^b - \delta^a\alpha^b)|-\boldsymbol{a}\rangle\otimes|+\boldsymbol{b}\rangle + (\beta^a\delta^b - \delta^a\beta^b)|-\boldsymbol{a}\rangle\otimes|-\boldsymbol{b}\rangle\} \end{matrix} . \tag{17.17}$$

The Bell Inequality

Now recall that the focus of attention here is the classical Bell inequality (17.9). Considering the CM single particle counterfactual probability $P_{CM}(\mathcal{A}^+, \mathcal{B}^-)$, this translates into the QM outcome probability $P_{QM}(+\boldsymbol{a}, +\boldsymbol{b}|\Phi)$ in the extended SG experiment (involving both Alice and Bob). This means that the classical Bell inequality (17.9) is replaced by the assertion that

$$P_{QM}(+\boldsymbol{a}, +\boldsymbol{b}|\Phi) + P_{QM}(+\boldsymbol{b}, +\boldsymbol{c}|\Phi) - P_{QM}(+\boldsymbol{a}, +\boldsymbol{c}|\Phi) \geqslant 0. \tag{17.18}$$

The first term in this expression is just the squared modulus of the coefficient of the tensor product term $|+\boldsymbol{a}\rangle\otimes|+\boldsymbol{b}\rangle$ in (17.17). Hence we deduce

$$P_{QM}(+\boldsymbol{a}, +\boldsymbol{b}|\Phi) = \tfrac{1}{2}|\alpha^a\gamma^b - \gamma^a\alpha^b|^2. \tag{17.19}$$

We can simplify this expression by using rotational symmetry, orienting our Cartesian coordinates along the direction of vector $\boldsymbol{a}$. Then we find

$$P_{QM}(+\boldsymbol{a}, +\boldsymbol{b}|\Phi) = \tfrac{1}{2}\sin^2(\tfrac{1}{2}\theta^{ab}), \tag{17.20}$$

where θ^{ab} is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. A similar calculation for the other two terms in (17.15) gives

$$\sin^2(\tfrac{1}{2}\theta^{ab}) + \sin^2(\tfrac{1}{2}\theta^{bc}) - \sin^2(\tfrac{1}{2}\theta^{ac}) \geqslant 0. \tag{17.21}$$

[14] We can ignore the fact that electrons are identical and obey Fermi–Dirac statistics, because the spreading of the beam prior to the electrons entering either $SG(\boldsymbol{a})$ or else $SG(\boldsymbol{b})$ has introduced a form of classical labelling.

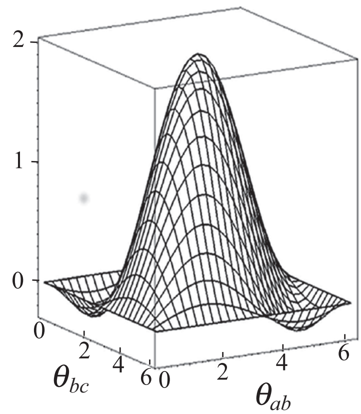

Figure 17.3. Plot of the function $f(\theta_{ab}, \theta_{bc})$ over a suitable domain.

The problem is that we can find vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ for which (17.21) is wrong. For example, take these vectors to lie in a plane, with $\theta^{ab} = \pi/3$, $\theta^{bc} = \pi/3$, $\theta^{ac} = \theta^{ab} + \theta^{bc} = 2\pi/3$. Then

$$\sin^2(\tfrac{1}{2}\tfrac{\pi}{3}) + \sin^2(\tfrac{1}{2}\tfrac{\pi}{3}) - \sin^2(\tfrac{1}{2}\tfrac{2\pi}{3}) = \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{3}{4} = -\tfrac{1}{4}. \tag{17.22}$$

In Figure 17.3 we show a plot of the function $f(\theta^{ab}, \theta^{bc}) \equiv \sin^2(\frac{1}{2}\theta^{ab}) + \sin^2(\frac{1}{2}\theta^{bc}) - \sin^2(\frac{1}{2}\theta^{ab} + \frac{1}{2}\theta^{bc})$ over a range of possibilities. There are two distinct regions where the function value is negative, while the HV calculation predicts that there should be no such regions.

There have been many experiments related to the one discussed here that have shown violations of Bell's inequalities, a frequently quoted one being that of Aspect and others using photons (Aspect et al., 1982).

There is now not much doubt among the majority of physicists that classical counterfactuality has been shown empirically to be a false principle in physics. It remains an excellent principle as far as relative external context (the wider Universe) is concerned: we can usually go to work and remain confident that our house will still be there when we get back in the evening.

There remains a relatively small group of committed HV theorists who continue to probe this issue and have come up with classically based possible explanations for the observed empirical violations of Bell's inequalities. Experimentalists continue to test these loopholes, and have reached the point where the HV classically motivated "explanations" seem more unpalatable than the quantum theory they are trying to circumvent.

17.5 The QDN Calculation

In this section we apply QDN to the eSG scenario discussed above. The relevant stage diagram is Figure 17.4.

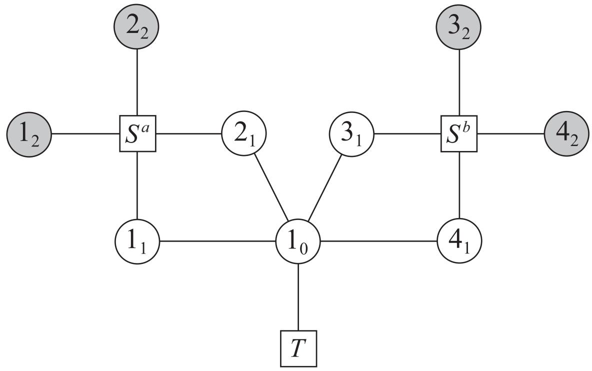

Figure 17.4. The QDN stage diagram of the enhanced SG experiment.

It was found that the total state at stage Σ_1, corresponding to the entangled state (17.14) needs to be described as a state in the tensor product of a four-dimensional internal spin space and a rank-four quantum register.

Stage Σ_0
The preparation switch stage Σ_0 creates a beam of spin-zero, two electron states, represented by the total state $|\Psi_0) \equiv |s_0^1\rangle \otimes \widehat{\mathbb{A}}_0^1 \mathbf{0}_0$. Here, $|s_0^1\rangle$ represents a normalized state in the one-dimensional Hilbert space of two-electron spin zero states.

Stage Σ_0 to Stage Σ_1
By the first stage, Σ_1, the beam has split into two entangled spin-half subbeams. This stage is *before* any of the subbeams enter their respective SG apparatus. Bitification requires each of these sub-beams to be associated with *two* signal qubits,[15] so we require a rank-four quantum register at that stage, as stated above.

The dynamics is given by the rule

$$U_{1,0}\left\{|s_0^1\rangle \otimes \widehat{\mathbb{A}}_0^1 \mathbf{0}_0\right\} = \frac{1}{\sqrt{2}} \left\{ \begin{array}{l} |+\boldsymbol{k}_1^a\rangle \otimes |-\boldsymbol{k}_1^b\rangle \otimes \widehat{\mathbb{A}}_1^1 \widehat{\mathbb{A}}_1^3 - \\ |-\boldsymbol{k}_1^a\rangle \otimes |+\boldsymbol{k}_1^b\rangle \otimes \widehat{\mathbb{A}}_1^2 \widehat{\mathbb{A}}_1^4 \end{array} \right\} \mathbf{0}_1. \tag{17.23}$$

Stage Σ_1 to Stage Σ_2
Using (17.15) and from Figure 17.4 we have

$$\begin{aligned} U_{2,1}\left\{|+\boldsymbol{k}_1^a\rangle \otimes |-\boldsymbol{k}_1^b\rangle \otimes \widehat{\mathbb{A}}_1^1 \widehat{\mathbb{A}}_1^3 \mathbf{0}_1\right\} &= \left\{\alpha^a|+\boldsymbol{a}_2\rangle \otimes \widehat{\mathbb{A}}_2^1 + \beta^a|-\boldsymbol{a}_2\rangle \otimes \widehat{\mathbb{A}}_2^2\right\} \\ &\quad \times \left\{\gamma^b|+\boldsymbol{b}_2\rangle \otimes \widehat{\mathbb{A}}_2^3 + \delta^b|-\boldsymbol{b}_2\rangle \otimes \widehat{\mathbb{A}}_2^4\right\} \mathbf{0}_2, \\ U_{2,1}\left\{|-\boldsymbol{k}_1^a\rangle \otimes |+\boldsymbol{k}_1^b\rangle \otimes \widehat{\mathbb{A}}_1^2 \widehat{\mathbb{A}}_1^4 \mathbf{0}_1\right\} &= \left\{\gamma^a|+\boldsymbol{a}_2\rangle \otimes \widehat{\mathbb{A}}_2^1 + \delta^a|-\boldsymbol{a}_2\rangle \otimes \widehat{\mathbb{A}}_2^2\right\} \\ &\quad \times \left\{\alpha^b|+\boldsymbol{b}_2\rangle \otimes \widehat{\mathbb{A}}_2^3 + \beta^b|-\boldsymbol{b}_2\rangle \otimes \widehat{\mathbb{A}}_2^4\right\} \mathbf{0}_2. \end{aligned} \tag{17.24}$$

[15] This is on account of the fact that entanglement can be detected, given the right apparatus.

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This is all that is needed to run our computer algebra program MAIN, which gives us much more information than we need here. Our interest originated in $P_{CM}(+\boldsymbol{a}, -\boldsymbol{b})$. Context tells us that we need to calculate the probability that detectors 1_2 and 3_2 are each in their respective signal state at stage Σ_2. Program MAIN gives us the answer

$$\Pr(\widehat{\mathbb{A}}_2^1 \widehat{\mathbb{A}}_2^3 \mathbf{0}_2 | \Psi_0) = \frac{1}{2} |\alpha^a \gamma^b - \gamma^a \alpha^b|^2, \tag{17.25}$$

which is precisely (17.19), the result of the standard QM calculation. The same argument applies to the other terms in the Bell inequality (17.18).

Our conclusion is that QDN gives the same results as standard QM, and hence the same prediction of violations of Bell inequalities.

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