J. Austral. Math. Soc. (Series A) 47 (1989), 458–465

WALSH-FOURIER SERIES WITH COEFFICIENTS OF GENERALIZED BOUNDED VARIATION

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(Received 23 February 1988)

Communicated by W. Moran

Abstract

We extend in different ways the class of null sequences of real numbers that are of bounded variation and study the Walsh-Fourier series of integrable functions on the interval $[0, 1)$ with such coefficients. We prove almost everywhere convergence as well as convergence in the pseudometric of $L^r(0, 1)$ for $0 < r < 1$.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 42 C 10.
Keywords and phrases: Walsh-Paley system, coefficient sequence of bounded variation, pointwise convergence, convergence in $L^r(0, 1)$-metric, $0 < r < 1$, arithmetic mean, de la Vallée Poussin mean, Dirichlet kernel, Fejér kernel.

1. Introduction

We consider the Walsh orthonormal system $\{w_k(x): k = 0, 1, \dots\}$ defined on the interval $[0, 1)$ in the Paley enumeration (see, for example, [1, page 60]). We will study the Walsh-Fourier series

$$(1.1) \qquad \sum_{k=0}^{\infty} a_k w_k(x), \qquad a_k = \int_0^1 f(x) w_k(x)\, dx,$$

of an integrable function $f \in L^1(0, 1)$. In this paper, the integrals and the term "almost everywhere" (in abbreviation a.e.) are meant in the Lebesgue sense.

This research was completed while the author was a visiting professor at the University of Tennessee, Knoxville, during the academic year 1987/1988.

2. Main results

We denote by

$$s_n(f,x)=\sum_{k=0}^{n} a_k w_k(x) \qquad (n=0,1,\dots)$$

the partial sums of series (1.1). Furthermore, we write

$$\Delta a_k = a_k - a_{k+1}, \qquad \Delta^2 a_k = \Delta a_k - \Delta a_{k+1} \qquad (k=0,1,\dots)$$

for the first and second differences, and set

$$\lambda_n = [\lambda n] \qquad (n=0,1,\dots) \tag{2.1}$$

where λ is a fixed real number, $\lambda > 1$, and $[\cdot]$ means the integral part.

THEOREM 1. *If $f \in L^1(0,1)$ and*

$$\lim_{\lambda\to 1+0} \limsup_{n\to\infty} \frac{1}{\lambda_n - n + 1} \sum_{k=n}^{\lambda_n} (\lambda_n - k + 1)|\Delta^m a_k| = 0 \tag{2.2}$$

for $m = 1$ or 2, then

$$\lim_{n\to\infty} s_n(f,x) = f(x) \quad a.e. \tag{2.3}$$

THEOREM 2. *If $f \in L^1(0,1)$ and condition (2.2) is satisfied for $m=1$ or 2, then*

$$\lim_{n\to\infty} \int_0^1 |s_n(f,x) - f(x)|^r \, dx = 0 \quad \text{for } 0 < r < 1/m. \tag{2.4}$$

Clearly, if condition (2.2) is satisfied for $m = 1$, then it is automatically satisfied for $m = 2$, but the converse implication fails in general.

We draw two corollaries of Theorems 1 and 2.

COROLLARY 1. *If $f \in L^1(0,1)$ and*

$$\lim_{\lambda\to 1+0} \limsup_{n\to\infty} \sum_{k=n}^{\lambda_n} |\Delta^m a_k| = 0 \tag{2.5}$$

for $m = 1$ or 2, then we have conclusions (2.3) and (2.4).

EXAMPLE. Let $\lambda = \lambda^{(j)} = 1 + 2^{-j}$ for $j = 1, 2, \dots$ and consider the sequence $\{a_k\}$ defined as follows:

$$\Delta^m a_k = \begin{cases} \frac{(-1)^j}{j(2^{2j}+2^j-k+1)} & \text{if } 2^{2j} \le k \le 2^{2j} + 2^j \text{ for some } j = 1, 2, \dots; \\ 0 & \text{otherwise.} \end{cases}$$

https://doi.org/10.1017/S144678870003319X Published online by Cambridge University Press

It is not hard to check that $\{a_k\}$ is a null sequence and condition (2.2) is satisfied, but condition (2.5) is not. This example shows that Theorem 1 is more general than Corollary 1.

Corollary 1 applies to many particular cases. We refer the reader to [2] where seven main cases and even further subcases are listed and discussed in details. We present here one more special case of (2.5) which is not contained in [2].

COROLLARY 2. *If $f \in L^1(0,1)$ and the finite limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} k|\Delta^m a_k| = L \tag{2.6}$$

exists for $m = 1$ or 2, then we have conclusions (2.3) *and* (2.4).

We recall that a sequence $\{a_k\}$ is said to be of bounded variation if

$$\sum_{k=0}^{\infty} |\Delta a_k| < \infty.$$

Obviously, if $\{a_k\}$ is of bounded variation, then (2.6) is satisfied with $m = 1$ (and a fortiori with $m = 2$) and $L = 0$. Thus, each of the conditions (2.2), (2.5) and (2.6) for either $m = 1$ or $m = 2$ can be considered a generalization of the notion of bounded variation.

We note that the counterpart of Corollary 1 for trigonometric Fourier series was proved by Chen [2], while that of Corollary 2 was proved by Stanojevic [5].

3. Proofs

We denote by

$$\sigma_n(f,x) = \frac{1}{n+1} \sum_{j=0}^{n} s_j(f,x) \qquad (n = 0, 1, \dots)$$

the first arithmetic means of series (1.1). It is well-known (see [3] and [4], respectively) that if $f \in L^1(0,1)$, then

$$\lim_{n\to\infty} \sigma_n(f,x) = f(x) \quad \text{a.e.} \tag{3.1}$$

and

$$\lim_{n\to\infty} \int_0^1 |\sigma_n(f,x) - f(x)|\, dx = 0. \tag{3.2}$$

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Next, we consider the so-called generalized de la Vallée Poussin means defined by

$$\tau_n(f,\lambda,x) = \frac{1}{\lambda_n - n + 1}\sum_{j=n}^{\lambda_n} s_j(f,x)$$

where $\lambda > 1$ and λ_n is given by (2.1).

Using the representation

$$\tau_n(f,\lambda,x) = \frac{\lambda_n + 1}{\lambda_n - n + 1}\sigma_{\lambda_n}(f,x) - \frac{n}{\lambda_n - n + 1}\sigma_{n-1}(f,x),$$

we have from (3.1) and (3.2) that for any fixed $\lambda > 1$,

$$\lim_{n\to\infty} \tau_n(f,\lambda,x) = f(x) \quad \text{a.e.} \tag{3.3}$$

and

$$\lim_{n\to\infty} \int_0^1 |\tau_n(f,\lambda,x) - f(x)|\,dx = 0. \tag{3.4}$$

REMARK 1. Actually, we have (3.3) at each point x, at which (3.1) is satisifed, Furthermore, if the convergence of $\sigma_n(f,x)$ is uniform on a certain set E, then the convergence of $\tau_n(f,\lambda,x)$ is also uniform on E for fixed λ.

PROOF OF THEOREMS 1 AND 2 FOR $m = 1$. By definition,

$$\tau_n(f,\lambda,x) - s_n(f,x) = \frac{1}{\lambda_n - n + 1}\sum_{j=n+1}^{\lambda_n}\sum_{k=n+1}^{j} a_k w_k(x). \tag{3.5}$$

For each $j \geq n+2$, a summation by parts yields

$$\sum_{k=n+1}^{j} a_k w_k(x) = -a_{n+1}D_n(x) + \sum_{k=n+1}^{j-1} D_k(x)\Delta a_k + a_j D_j(x) \tag{3.6}$$

where

$$D_n(x) = \sum_{k=0}^{n} w_k(x) \qquad (n = 0, 1, \ldots)$$

is the Dirichlet kernel for the Walsh system. It is well known (see, for example, [3]) that

$$|D_n(x)| < 2/x \qquad (n = 0, 1, \ldots; 0 < x < 1). \tag{3.7}$$

From this, (3.5) and (3.6), a simple computation gives that for $0 < x < 1$,

$$\begin{aligned} &|\tau_n(f,\lambda,x) - s_n(f,x)| \\ &\leq \frac{2}{(\lambda_n - n + 1)x}\sum_{j=n+1}^{\lambda_n}\left(|a_{n+1}| + \sum_{k=n+1}^{j-1}|\Delta a_k| + |a_j|\right) \\ &= \frac{o(1)}{x} + \frac{2}{(\lambda_n - n + 1)x}\sum_{k=n+1}^{\lambda_n - 1}(\lambda_n - k)|\Delta a_k|, \end{aligned} \tag{3.8}$$

where $o(1)$ does not depend on x. Here we used the fact that $f \in L^1(0,1)$ implies that

$$\lim_{n\to\infty} a_n = 0. \tag{3.9}$$

By (2.2) and (3.8), for every $0 < x < 1$,

$$\lim_{\lambda\to 1+0} \limsup_{n\to\infty} |\tau_n(f,\lambda,x) - s_n(f,x)| = 0 \tag{3.10}$$

and for every $0 < r < 1$,

$$\lim_{\lambda\to 1+0} \limsup_{n\to\infty} \int_0^1 |\tau_n(f,\lambda,x) - s_n(f,x)|^r \, dx = 0. \tag{3.11}$$

Combining (3.3) and (3.10) yields (2.3), while combining (3.4) and (3.11) yields (2.4) in the case $m = 1$.

REMARK 2. It is easy to see that the convergence in (3.10) is uniform on any interval $[\delta, 1)$ with $0 < \delta < 1$.

PROOF OF THEOREMS 1 AND 2 FOR $m = 2$. We perform one more summation by parts on the right-hand side of (3.6), which results in the following:

$$\begin{aligned}\sum_{k=n+1}^{j} a_k w_k(x) &= -a_{n+1}D_n(x) - (n+1)F_n(x)\Delta a_{n+1} \\ &\quad + \sum_{k=n+1}^{j-2} (k+1)F_k(x)\Delta^2 a_k + jF_{j-1}(x)\Delta a_{j-1} + a_j D_j(x),\end{aligned} \tag{3.12}$$

where

$$F_n(x) = \frac{1}{n+1}\sum_{j=0}^{n} D_j(x) \qquad (n = 0, 1, \dots)$$

is the Fejér kernel for the Walsh system.

According to Fine [3], for all positive integers n and m, and for all x, except possibly for a dyadic rational x,

$$(n+1)|F_n(x)| < \frac{4}{x(x-2^{-m})} + \frac{4}{x^2} = C(x) \quad \text{if } 2^{-m} < x < 2^{-m+1}. \tag{3.13}$$

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It follows from (3.5), (3.7), (3.9), (3.12) and (3.13) that for all $0 < x < 1$, except perhaps the dyadic rationals,

$$(3.14)\quad |\tau_n(f,\lambda,x) - s_n(f,x)| \le \frac{1}{\lambda_n - n + 1} \sum_{j=n+1}^{\lambda_n} \left\{ \frac{2}{x}(|a_{n+1}| + |a_j|) + C(x)\left(|\Delta a_{n+1}| + \sum_{k=n+1}^{j-2} |\Delta^2 a_k| + |\Delta a_{j-1}| \right) \right\}$$
$$= \left(\frac{1}{x} + C(x)\right) o(1) + \frac{C(x)}{\lambda_n - n + 1} \sum_{k=n+1}^{\lambda_n - 2} (\lambda_n - k - 1)|\Delta^2 a_k|,$$

where $o(1)$ does not depend on x.

This and (2.2) imply that for all x, except the dyadic rationals, we have (3.10).

Let $0 < r < 1/2$. By (3.14),

$$\int_0^1 |\tau_n(f,\lambda,x) - s_n(f,x)|^r \, dx = o(1) + \left\{ o(1) + \frac{1}{\lambda_n - n + 1} \sum_{k=n+1}^{\lambda_n - 2} (\lambda_n - k - 1)|\Delta^2 a_k| \right\} \int_0^1 C^r(x)\, dx.$$

By (3.13),

$$\int_0^1 C^r(x)\, dx \le \sum_{m=1}^{\infty} \int_{2^{-m}}^{2^{-m+1}} \frac{4^r}{x^r(x - 2^{-m})^r}\, dx + \int_0^1 \frac{4^r}{x^{2r}}\, dx \le \sum_{m=1}^{\infty} \frac{4^r}{1-r} 2^{m(2r-1)} + \frac{4^r}{1-2r} < \infty.$$

Putting the last two estimates together gives (3.11) for $0 < r < 1/2$.

Finally, combining (3.3) and (3.10) yields (2.3), while combining (3.4) and (3.11) yields (2.4) in the case $m = 2$.

PROOF OF COROLLARY 2. It is enough to show that condition (2.6) implies (2.5). Clearly,

$$\sum_{k=n}^{\lambda_n} |\Delta^m a_k| \le \frac{1}{n} \sum_{k=n}^{\lambda_n} k|\Delta^m a_k| \le \frac{\lambda_n}{n} \frac{1}{\lambda_n} \sum_{k=1}^{\lambda_n} k|\Delta^m a_k| - \frac{n-1}{n} \frac{1}{n-1} \sum_{k=1}^{n-1} k|\Delta^m a_k|.$$

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Given any $\varepsilon > 0$, by (2.6) we have

$$\sum_{k=n}^{\lambda_n} |\Delta^m a_k| \leq \frac{\lambda_n}{n}(L+\varepsilon) - \frac{n-1}{n}(L-\varepsilon) = \frac{\lambda_n - n + 1}{n} L + \frac{\lambda_n + n - 1}{n}\varepsilon \tag{3.15}$$

provided n is large enough. Thus, it follows from (2.1) and (3.15) that

$$\limsup_{n\to\infty} \sum_{k=n}^{\lambda_n} |\Delta^m a_k| \leq (\lambda - 1)L + (\lambda + 1)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, hence we get (2.5).

4. Concluding remarks

It turns out from the proofs of Theorems 1 and 2 that we can also deduce (2.3) and (2.4) when the "lim sup" in (2.2) equals zero for a specific value of $\lambda > 1$. Here we formulate only the case $\lambda = 2$.

THEOREM 3. *If $f \in L^1(0,1)$ and*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=n}^{2n} (2n - k + 1)|\Delta^m a_k| = 0 \tag{4.1}$$

for $m = 1$ or 2, then we have conclusions (2.3) *and* (2.4).

Condition (4.1) is also a generalization of the notion of bounded variation.

Another by-product of the proof of Theorem 1 relates to continuous functions.

THEOREM 4. *If $f \in C[0,1)$ and condition* (2.2) *is satisfied for $m = 1$, then for every $0 < x < 1$,*

$$\lim_{n\to\infty} s_n(f, x) = f(x), \tag{4.2}$$

and this convergence is uniform on each interval $[\delta, 1)$ with $0 < \delta < 1$.

Relation (4.2) is an immediate consequence of Remarks 1 and 2 in Section 3 and the following well-known result (see [3]): if $f \in C[0,1)$, then for every x

$$\lim_{n\to\infty} \sigma_n(f, x) = f(x)$$

and this convergence is uniform on the whole interval [0, 1).

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We note that the counterpart of Theorem 4 for trigonometric Fourier series was proved by Chen [2].

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https://doi.org/10.1017/S144678870003319X Published online by Cambridge University Press