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Rigidity of flat holonomies

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Abstract. We prove that the existence of one horosphere in the universal cover of a closed Riemannian manifold of dimension $n \geq 3$ with strongly $1/4$-pinched or relatively $1/2$-pinched sectional curvature, on which the stable holonomy along one horosphere coincides with the Riemannian parallel transport, implies that the manifold is homothetic to a real hyperbolic manifold.

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1. *Introduction*

Mostow's seminal rigidity theorem [**19**] asserts that the geometry of a closed hyperbolic manifold of dimension greater than two is determined by its fundamental group. Inspired by Mostow's theorem, we undertake a study of related, yet, more general themes. In this paper, we look at natural geometric submanifolds, the horospheres, and ask to what extent do these determine the geometry of the whole manifold. Precisely, we are concerned with the following general question.

Question 1.1. Does the geometry of the horospheres of a closed, negatively curved manifold of dimension greater than two determine the geometry of the whole manifold?

In general, there are very few answers to Question 1.1, and all of these relate the *extrinsic* geometry of the horospheres to the geometry of M. For instance, by combining [**5**, **10**] (see [**5**, Corollary 9.18]), one shows that if all the horospheres have constant mean curvature, then the underlying manifold is locally symmetric (of negative curvature). Let us recall that the mean curvature of a hypersurface is related to the derivative of its volume element in the normal direction to the hypersurface, and hence the mean curvature is an extrinsic quantity. In this paper, our main hypothesis is to relax the assumption on the sectional curvature in Mostow's theorem and allow it to be strictly quarter negatively curved pinched. In this case, constant mean curvature of the horospheres only occurs for real hyperbolic manifolds (up to homothety). In contrast, we would like to emphasize that we only consider the intrinsic properties of the induced metric on the horospheres.

Before stating our main theorem, let us recall a few important features of the manifolds under consideration and results that are related to our work in this paper. Let M denote an $(n+1)$-dimensional, closed, Riemannian manifold endowed with a metric of negative sectional curvature, $n \geq 2$. It follows from the Cartan–Hadamard theorem that $\tilde{M}$, the universal cover of M, is diffeomorphic to $\mathbb{R}^{n+1}$. Let $\tilde{M}$ be endowed with the pull-back Riemannian metric from M, under the natural projection $\pi : \tilde{M} \to M$. The geometric boundary $\partial\tilde{M}$ of $\tilde{M}$ is the set of equivalence classes of geodesic rays in $\tilde{M}$, where two geodesic rays are equivalent if they remain at a bounded Hausdorff distance. We recall that, in our context, it is homeomorphic to $\mathbb{S}^n$.

Given a point, $x_0 \in \tilde{M}$, and a unit tangent vector, $\tilde{v} \in T_{x_0}\tilde{M}$, we let $c_{\tilde{v}}$ denote the unique geodesic ray determined by $c_{\tilde{v}}(0) = x_0$ and $\dot{c}_{\tilde{v}}(0) = \tilde{v}$. It is well known that the map, $\tilde{v} \in T_{x_0}\tilde{M} \mapsto [c_{\tilde{v}}] \in \partial\tilde{M}$, defines a homeomorphism between the unit sphere in $T_{x_0}\tilde{M}$ and $\partial\tilde{M}$. Given a point $\xi = [c_{\tilde{v}}] \in \partial\tilde{M}$, the Busemann function $B_\xi(\cdot)$ is then defined for all $\xi \in \partial\tilde{M}$ and for all $x \in \tilde{M}$ by $B_\xi(x) = \lim_{t\to\infty}(d(x, c_{\tilde{v}}(t)) - d(x_0, c_{\tilde{v}}(t)))$.

Since M is a closed negatively curved manifold, for each $\xi \in \partial\tilde{M}$, it is known that the Busemann function $B_\xi(\cdot)$ is C^∞-smooth. Furthermore, for any $t \in \mathbb{R}$, the level set

$$H_\xi(t) = \{x \in \tilde{M};\ B_\xi(x) = t\}$$

is a smooth submanifold of $\tilde{M}$ which is diffeomorphic to $\mathbb{R}^n$ and which is called a *horosphere* centered at ξ. The sublevel set

$$HB_\xi(t) = \{x \in \tilde{M};\ B_\xi(x) \leq t\}$$

is called a *horoball*. It follows that horospheres inherit a complete Riemannian metric induced by the restriction of the metric of $\tilde{M}$. For instance, if (M, g) is a real hyperbolic manifold, every horosphere of $\tilde{M}$ is flat and therefore isometric to the Euclidean space $\mathbb{R}^n$.

So far, we defined horospheres as special submanifolds in $\tilde{M}$. However, a dynamical perspective turns out to be important in the proof of the main theorem. Let $\tilde{p} : T^1\tilde{M} \to \tilde{M}$ and $p : T^1M \to M$ denote the natural projections. The geodesic flow $\tilde{g}_t$ on $T^1\tilde{M}$ is known to be an Anosov flow, that is, the tangent bundle $TT^1\tilde{M}$ admits a decomposition as $TT^1\tilde{M} = \mathbb{R}X \oplus \tilde{E}^{ss} \oplus \tilde{E}^{su}$, where X is the vector field generating the geodesic flow and $\tilde{E}^{ss}$, $\tilde{E}^{su}$ are the strong stable and strong unstable distributions, respectively. These distributions are known to be integrable, invariant under the differential $d\tilde{g}_t$ of the geodesic flow, and to give rise to two transverse foliations of $T^1\tilde{M}$, $\tilde{W}^{ss}$ and $\tilde{W}^{su}$, the strong stable and strong unstable foliations, respectively, whose leaves are smooth submanifolds. A classical property of these foliations is that, in general, they are transversally Hölder with exponent less than one, and when the sectional curvature, denoted by K, is strictly 1/4-pinched (that is, $-4 < K \leq -1$), they are transversally C^1 (see [16, p. 226]), but we do not use such a regularity.

A link between the two points of view on horospheres is the following. For $\tilde{v} \in T^1\tilde{M}$, the strong stable leaf $\tilde{W}^{ss}(\tilde{v})$ through $\tilde{v}$ is defined to be the set of unit vectors $\tilde{w} \in T^1\tilde{M}$ which are normal to the horosphere $H_\xi(t)$ and pointing inward of the horoball $HB_\xi(t)$ in the direction of $\xi = c_{\tilde{v}}(+\infty)$, with $t = B_\xi(\tilde{p}(\tilde{v}))$ so that $H_\xi(t) = \tilde{p}(W^{ss}(\tilde{v}))$.

With this notation in place, let us now describe our main theorem and the foundational work we build upon. In §3, we will recall the construction of the stable holonomy. The notion of stable holonomy goes back to the work of Bonatti, Gómez-Mont, and Viana [6], and has been extensively studied by various authors, Viana [21] (also in the non-uniformly hyperbolic setting), Avila and Viana [4], Avila, Santamaria, and Viana [3], and Kalinin and Sadovskaya [17], in the context of partially hyperbolic systems. In our setting, it is a family, for each horosphere, of isomorphisms between the tangent spaces at any two points of it. Given $\xi \in \partial\tilde{M}$ and x, y a pair of points on a horosphere H_ξ centered at ξ, we will informally denote $\Pi^\xi(x, y)$ the isomorphism between the tangent spaces to this horosphere at x and y, and the stable holonomy will be the collection of all these isomorphisms $\Pi^\xi(x, y)$. The stable holonomy was originally constructed as a family, for each strong stable leaf of the geodesic flow, of isomorphisms $\mathcal{H}(\tilde{v}, \tilde{w})$ between the tangent spaces to this leaf at any pair of points $\tilde{v}, \tilde{w}$ and not on the horospheres as we will present here. However, the two constructions are equivalent since there is the conjugation $\mathcal{H}(\tilde{v}, \tilde{w}) = D\tilde{p}(\tilde{w})^{-1} \circ \Pi^\xi(x, y) \circ D\tilde{p}(\tilde{v})$, where $\tilde{p}\tilde{v} = x$, $\tilde{p}\tilde{w} = y$ and $c_{\tilde{v}}(+\infty) = c_{\tilde{w}}(+\infty) = \xi$ (see more in Appendix A). This construction, which holds in the general setting of linear cocycles over partially hyperbolic diffeomorphisms, requires the 'fiber bunched' condition, as in [17] (more on this in §3). In the context of the geodesic flow of a negatively curved closed manifold, the fiber bunched condition is a consequence of a pinching condition on the sectional curvature. We will consider two kinds of pinching. The strong 1/4-pinching of the curvature means that for every $x \in M$, the sectional curvature $K(x)$ satisfies

$$-4 < K(x) \leq -1. \tag{1.1}$$

Given $a > 0$, the curvature of M is said to be relatively a-pinched if there exists a strictly negative function $C : M \to \mathbb{R}_{<0}$ such that for every $x \in M$, the sectional curvature satisfies

$$C(x) \leq K(x) < aC(x). \tag{1.2}$$

In general, none of these two pinching conditions imply the other. To the best of our knowledge, a stable holonomy cannot be defined without *some* pinching condition on the curvature. In what follows, we will describe the construction of the stable holonomy on the horospheres under the strong 1/4-pinching or the relative 1/2-conditions. However, every horosphere $H_\xi(s)$ carries the Riemannian metric induced by the one of $\tilde{M}$. In particular, for every pair of sufficiently close points $x, y \in H_\xi(s)$, there is a unique minimizing geodesic of $H_\xi(s)$ joining them. Hence, we may consider the parallel transport associated to the Levi-Civita connection of the induced metric on $H_\xi(s)$, denoted by $P_s^\xi(x, y)$, between the tangent spaces to $H_\xi(s)$ at these points x and y. As mentioned before, in the case of $K \equiv -1$, the induced Riemannian metric on horospheres is flat and the stable holonomy $\Pi_s^\xi(x, y)$ and the parallel transport $P_s^\xi(x, y)$ coincide for every pair of points x and y

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on $H_\xi(s)$. Our main result is that the *converse* is true among strongly 1/4-pinched or 1/2-relatively pinched negatively curved manifolds.

THEOREM 1.2. (Main Theorem) *Let M be a closed, Riemannian manifold of dimension $n \geq 3$, endowed with a strongly 1/4-pinched or 1/2-relatively pinched negatively curved sectional curvature. Assume that there exists $\xi \in \partial \tilde{M}$ and $s \in \mathbb{R}$ such that for every pair of points $x, y \in H_\xi(s)$ joined by a unique minimizing geodesic, the stable holonomy $\Pi_s^\xi(x, y)$ is identical to the parallel transport $P_s^\xi(x, y)$. Then, (M, g) is homothetic to a real hyperbolic manifold.*

As mentioned before, the restriction on the sectional curvature ensures the existence of the stable holonomy. For Theorem 1.2 to hold, it is indeed sufficient to make the assumption for a *single* horosphere in $\tilde{M}$ since in Proposition 2.1, we show that it implies that *all* horospheres satisfy it. Since it is known that pinched curvature implies pinching of Lyapunov exponents, we could hope, as suggested by the anonymous referee, that a pinching of Lyapunov exponents might be sufficient. This point is left for further study.

In the case that $\dim M = 2$, Theorem 1.2 may still be true. However, our proof in the case of $\dim M \geq 3$ does not apply since it relies on Theorems 1.3 and 1.5 which both require an assumption on the dimension, see more details below.

Essential to the proof of our main theorem is the following deep characterization of closed, real hyperbolic manifolds stated by Butler [8]. This result is related to the way the geometry of horospheres evolves under the action of the geodesic flow. Butler showed, in what might be called now as *Lyapunov rigidity*, that the equality of the modulus of the eigenvalues of $dg_t|E^{ss}(v)$ along *every* periodic geodesic has an important geometric consequence. Let us recall his theorem.

THEOREM 1.3. [8, Theorem 1.1] *Let M be a closed, negatively curved manifold of dimension $n \geq 3$. For a periodic orbit $g_t(v)$ of the geodesic flow on T^1M with period $l(v)$, let $\xi_1(v), \ldots, \xi_n(v)$ be the complex eigenvalues of $Dg_{l(v)}(v)|E^{ss}(v)$, counted with multiplicities. Assume that $|\xi_1(v)| = \cdots = |\xi_n(v)|$ holds for each periodic orbit $g_t(v)$, then M is homothetic to a compact quotient of the real hyperbolic space.*

In this theorem, the assumption on $\dim M \geq 3$ is indeed necessary. Indeed, let us consider a closed surface M with a 1/4-pinched negative sectional curvature Riemannian metric g. The metric g can be chosen to be, for example, a small perturbation of an hyperbolic metric. In this case, the horospheres in $\tilde{M}$ endowed with their induced metric are complete Riemannian lines and the assumption on the eigenvalues of $Dg_{l(v)}(v)|E^{ss}(v)$ along periodic orbits $g_t(v)$ does not provide any useful information; indeed, there is a single eigenvalue and the action of Dg_t on E^{ss} is therefore trivially conformal.

Theorem 1.2 is a consequence of Theorem 1.3, Proposition 2.1, and the following result.

THEOREM 1.4. *Under the assumptions of Theorem 1.2, let $c_{\tilde{v}}(t)$ project to a periodic geodesic $c_v(t)$ of period $l(v)$ in M and let $\xi = c_{\tilde{v}}(+\infty)$. Then, the complex eigenvalues of $Dg_{l(v)}(v)|E^{ss}(v)$ satisfy $|\xi_1(v)| = \cdots = |\xi_n(v)|$.*

Let us now briefly describe the proof of Theorem 1.4. First note that the closeness of the manifold of M is a necessary assumption as one can verify on the examples given

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by the *Heintze groups*. Recall that a Heintze group is a solvable group $G_A := \mathbb{R} \ltimes_A \mathbb{R}^n$, where A is an $n \times n$ real matrix and $\mathbb{R}$ acts on $\mathbb{R}^n$ by $x \to e^{tA}x$. In the case where the real parts of the eigenvalues of A have the same sign, Heintze [14] showed the existence of left invariant metrics on G_A with negative sectional curvature. In this case, horospheres centered at a particular point on ∂G_A and endowed with the induced metric are flat (see §2 and, in particular, equation (2.8)). Notice that for a Heintze group, the existence of one 'flat' horosphere does not imply that all horospheres are flat. Indeed, crucial in the proof of Proposition 2.1 is the fact that the metric under consideration comes from a closed Riemannian manifold while a Heintze group does not have any cocompact quotient unless it is the hyperbolic space. If A is a multiple of the identity matrix, G_A is then homothetic to the real hyperbolic space; furthermore, it was proved by Heintze in [13] that the Heintze groups G_A have no cocompact lattice unless they are homothetic to the hyperbolic space. Moreover, X. Xie obtained a necessary condition for G_A to be quasi-isometric to a finitely generated group. His result is also essential for the proof of our main theorem. Before stating it, recall that, given an $n \times n$-matrix A, the 'real part Jordan form' of A is obtained from the Jordan form of A by replacing each diagonal entry with its real part and reordering to make it canonical.

THEOREM 1.5. [22, Corollary 1.6] *Let A be an $n \times n$ real matrix whose eigenvalues all have positive real parts. If G_A is quasi-isometric to a finitely generated group, then the real part Jordan form of A is a multiple of the identity matrix.*

The main idea of the proof of Theorem 1.4 is therefore to show that for each periodic orbit $g_t(v)$ of the geodesic flow of T^1M of period $l(v)$, $\tilde{M}$ is quasi-isometric to a Heintze group G_A, where A is a matrix whose eigenvalues *all* have positive real parts and such that $e^{l(v)A}$ is conjugate to $Dg_{l(v)}(v)|E^{ss}(v)$. By assumption, M is a closed manifold endowed with a negatively curved metric. It is well known that $\tilde{M}$ is quasi-isometric to the fundamental group of M which is, in particular, finitely generated. Hence, G_A turns out to be quasi-isometric to a finitely generated group. It now follows from the above-mentioned theorem of Xie that the real part of the eigenvalues of A coincide and therefore, the eigenvalues of $Dg_{l(v)}(v)|E^{ss}(v)$ have the same modulus.

Therefore, we are left with proving that $\tilde{M}$ is quasi-isometric to a Heintze group G_A. This is done as follows. Let us fix a geodesic in $\tilde{M}$ with an endpoint $\xi \in \partial\tilde{M}$. The set of stable horospheres $H_\xi(t)$ centered at ξ and the set of geodesics asymptotic to ξ define two orthogonal foliations of $\tilde{M}$. These foliations determine horospherical coordinates $\mathbb{R} \times H_\xi(0) = \mathbb{R} \times \mathbb{R}^n$ on $\tilde{M}$. In these coordinates, the metric of $\tilde{M}$ decomposes at every point $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ as an orthogonal sum

$$\tilde{g} = dt^2 + h_t, \tag{1.3}$$

where dt^2 is the standard metric on $\mathbb{R}$ and h_t is a one-parameter family of flat metrics on $H_\xi(0) = \mathbb{R}^n$. However, a Heintze group G_A is, by definition, also diffeomorphic to $\mathbb{R} \times \mathbb{R}^n$ with a metric, written similarly at every point $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, as the orthogonal sum

$$g_A := dt^2 + \langle e^{tA}\cdot, e^{tA}\cdot\rangle, \tag{1.4}$$

where $\langle e^{tA}\cdot, e^{tA}\cdot\rangle$ is a one-parameter family of flat metrics on $\mathbb{R}^n$, with $\langle\cdot,\cdot\rangle$ being the standard scalar product on $\mathbb{R}^n$. It is worth recalling that the family of flat metrics $\langle e^{tA}\cdot, e^{tA}\cdot\rangle$ on the $\mathbb{R}^n$ factor have the same Levi-Civita connection. This implies that the geodesic flow $(s, y) \to (s+t, y)$ acting on $G_A \approx \mathbb{R}\times\mathbb{R}^n$ commutes with the parallel transport along the horospheres $\{s\}\times\mathbb{R}^n$.

Turning back to $\tilde{M} \approx \mathbb{R}\times\mathbb{R}^n$ with its horospherical coordinates associated to $\xi = c_{\tilde{v}}(+\infty)$, where $c_{\tilde{v}}$ projects to a closed geodesic c_v of period $l(v)$ in M, we will prove that $\tilde{M}$ is quasi-isometric to G_A, for A defined by

$$e^{l(v)A} = D\tilde{p} \circ (D(\gamma \circ \tilde{g}_{l(v)}(\tilde{v})|E^{ss}(\tilde{v})) \circ D\tilde{p}^{-1} \tag{1.5}$$

and where γ is the element of the fundamental group of M such that $D\gamma(\tilde{g}_{l(\tilde{v})}(\tilde{v})) = \tilde{v}$, by proving that $h_{l(v)k} = \langle e^{kA}\cdot, e^{kA}\cdot\rangle$ for all positive integer k.

The proof of this equality reduces to a consequence of our assumptions that the parallel transport along the horospheres commutes with the flow $(s, y) \to (s+t, y)$ acting on $\tilde{M} \approx \mathbb{R}\times\mathbb{R}^n$. Indeed, it follows form this commutation that the computation of $h_{l(v)k}(l(v)k, y)(X, X)$ for any tangent vector X to $\mathbb{R}^n$ at the point $(l(v)k, y)$ does not depend on the point $y \in \mathbb{R}^n$. Thus, it will be computed at the point $(l(v)k, y_0)$, where y_0 is chosen so that $(0, y_0)$ are the coordinates of the point $x_0 \in \tilde{M}$ lying on the intersection of the geodesic $c_{\tilde{v}}$ with the horosphere $H_\xi(0) = \mathbb{R}^n$; the relation $h_{l(v)k}(l(v)k, y_0)(X, X) = \langle e^{kA}X, e^{kA}X\rangle$ is then easily derived from the fact that the flow $(s, y) \to (s+t, y)$ is the projection by $\tilde{p}$ on $\tilde{M}$ of the geodesic flow.

Let us conclude this quick description by briefly describing how the commutation of the parallel transport along the horospheres with the geodesic flow is derived. To this end, we adapt the construction in [**3**] and [**17**], which amounts to using the geodesic flow to construct a transportation along horospheres, which is called the stable holonomy. By construction, it is invariant by the geodesic flow. It turns out that to make this construction work, we need the strict 1/4-pinching curvature assumption or the relatively 1/2-pinched sectional curvature, which in turn corresponds to the notion of a bunched dynamical system appearing in [**3**, **17**].

The organization of the paper is as follows. In §2, we show that the assumption of the main theorem on one horosphere implies that it is satisfied on all of them using the properties of the stable foliation of T^1M and the density of each leaf. We also describe the geometry of the Heintze groups in the same section. In §3, we describe the construction of our version of the *stable holonomy*, adapted from [**3**] and [**17**]. We also prove that for this new transportation, if it coincides with the parallel transport for the induced metric on *one* horosphere, then it is also the case for *all* horospheres. Finally, in §4, this new tool allows us to prove that $\tilde{M}$ is quasi-isometric to the hyperbolic space, and that the derivative of the flow on the stable manifolds has complex eigenvalues which all have the same modulus. This concludes the proof of Theorem 1.4, and therefore of Theorem 1.2. In Appendix A, we show that the strong 1/4-pinching assumption in equation (1.2) implies the bunching of the stable cocycle of the geodesic flow defined in [**17**]. We will also prove that the stable holonomy defined is this work on horospheres is actually conjugate to the stable holonomy defined on the strong stable leaves of the geodesic flow.

2. *Geometry of horospheres and the Heintze groups*
In this section, we first prove Proposition 2.1 below which, among others, we prove several continuity properties of horospheres and assert that if one of them is flat, then *all* horospheres are flat. We then describe the main family of examples showing that the closeness assumption in Theorem 1.2 is necessary. These examples, consisting of simply connected Lie groups endowed with negatively curved left invariant metrics (see [14]), are due to E. Heintze and are called 'Heintze groups'. At the end of this section, we provide a proof of the fact that for every $\xi \in \partial\tilde{M}$, the Busemann function $B(\cdot, \xi)$ is smooth.

2.1. *Geometry of horospheres.* Let us start by recalling a few facts about the dynamical approach describing horospheres. We first note that the strong stable and unstable distributions $\tilde{E}^{ss}$, $\tilde{E}^{su}$ and their associated foliations $\tilde{W}^{ss}$, $\tilde{W}^{su}$ are invariant under the action of the fundamental group of M, and hence they all project onto their natural counterparts denoted by E^{ss}, E^{su}, W^{ss}, and W^{su} in TT^1M and T^1M, respectively. An important consequence of the closeness of M is that each leaf of the strong stable or unstable foliations W^{ss} and W^{su} is dense in T^1M (see [1]). An application of the dynamical interpretation is described in the proposition below and will be important in what follows. Given a unit tangent vector $\tilde{v} \in T^1_z\tilde{M}$, we will denote by $H_{\tilde{v}}$ the horosphere centered at the point $c_{\tilde{v}}(+\infty) \in \partial\tilde{M}$ and passing through the base point z of $\tilde{v}$. Observe that $H_{\tilde{v}} = H_\xi(s)$, where $\xi = c_{\tilde{v}}(+\infty)$ and $s = B_\xi(z)$. This notation will make the formulation of the next proposition easier. If $x, y \in H_{\tilde{v}}$ are two points such that there exists a unique geodesic of $H_{\tilde{v}}$ joining x and y, we write $P_{H_{\tilde{v}}}(x, y) : T_xH_{\tilde{v}} \to T_yH_{\tilde{v}}$ as the parallel transport along the geodesic path between x and y. We will denote by $d_{H_{\tilde{v}}}$ the distance on $H_{\tilde{v}}$. Recall that the parallel transport is measured with respect to the induced Riemannian metric on $H_{\tilde{v}}$.

PROPOSITION 2.1. *Let M be a closed $(n+1)$-dimensional Riemannian manifold with negative sectional curvature, then the following hold.*

(1) *Let $(\tilde{v}_k)_k$ be a sequence in $T^1\tilde{M}$ such that $\lim_k \tilde{v}_k = \tilde{v}$. Then, $H_{\tilde{v}_k} C^\infty$-converge to $H_{\tilde{v}}$ on compact subsets.*
(2) *It is equivalent that one or every horosphere in $\tilde{M}$ is flat.*
(3) *There exists a positive constant $\rho > 0$ such that the injectivity radius of each horosphere is bounded below by ρ.*
(4) *Let $(\tilde{v}_k)_k \in T^1_{x_k}\tilde{M}$ such that $\lim_k \tilde{v}_k = \tilde{v} \in T^1_x\tilde{M}$ (notice that $\lim_k x_k = x$). Let $X_k \in T_{x_k}H_{\tilde{v}_k}$ and $y_k \in H_{\tilde{v}_k}$ such that $\lim_k y_k = y \in H_{\tilde{v}}$, $\lim_k X_k = X \in T_xH_{\tilde{v}}$, and, if $d_{H_{\tilde{v}}}(x, y) < \rho$, then $\lim_k P_{H_{\tilde{v}_k}}(x_k, y_k)(X_k) = P_{H_{\tilde{v}}}(x, y)(X)$.*

Proof. Let us prove the first part of the proposition. Suppose that the sequence $(\tilde{v}_k)_k$ is converging to $\tilde{v}$ in $T^1\tilde{M}$. The set of unit vectors $\tilde{w}$ normal to $H_{\tilde{v}}$ such that $[c_{\tilde{w}}] = [c_{\tilde{v}}] \in \partial\tilde{M}$ is the strong stable leaf $\tilde{W}^{ss}(\tilde{v})$. Recall that the projection $\tilde{p} : T^1\tilde{M} \to \tilde{M}$ maps the strong stable leaf $\tilde{W}^{ss}(\tilde{v})$ diffeomorphically onto $H_{\tilde{v}} = \tilde{p}(\tilde{W}^{ss}(\tilde{v}))$. Similarly, for each k, the horosphere $H_{\tilde{v}_k}$ is the projection of a strong stable leaf $\tilde{W}^{ss}(\tilde{v}_k)$, $H_{\tilde{v}_k} = \tilde{p}(\tilde{W}^{ss}(\tilde{v}_k))$. Let v_k and v denote the projection under $d\pi : T^1\tilde{M} \to T^1M$ of $\tilde{v}_k$ and $\tilde{v}$, where $\pi : \tilde{M} \to M$ is the projection. Let us consider a chart $U \subset T^1M$ of the strong stable foliation W^{ss} containing v and let $Q = U \cap W^{ss}(v)$ be the plaque of the foliation W^{ss} through v. Since U

is a chart of the foliation W^{ss}, for k large enough, $U \cap W^{ss}(v_k) \neq \emptyset$ and the plaques $Q_k := U \cap W^{ss}(v_k)$ Hausdorff converge to Q. Consequently, for the lift $\tilde{Q} \subset T^1\tilde{M}$ of Q containing $\tilde{v}$, the set $\tilde{p}(\tilde{Q}) \subset H_{\tilde{v}}$ is the Hausdorff limit of the sequence of sets $\tilde{p}(\tilde{Q}_k) \subset H_{\tilde{v}_k}$, where $\tilde{Q}_k$ are lifts of Q_k containing $\tilde{v}_k$. We will show that for all $r \geq 0$, $\tilde{p}(\tilde{Q})$ is the limit in the C^r-topology, $r \geq 0$, of $\tilde{p}(\tilde{Q}_k)$, which will conclude the first part of the proposition.

Let us choose a chart U small enough so that $\tilde{Q}_k$ and $\tilde{Q}$ project diffeomorphically onto Q_k and Q. Similarly, we can assume that the projection $p : T^1M \to M$ also maps diffeomorphically Q_k and Q into M. Finally, if U is small enough, we have that $p(Q_k)$ and $p(Q)$ are isometrically covered by $\tilde{p}(\tilde{Q}_k)$ and $\tilde{p}(\tilde{Q})$, respectively. We can therefore work equivalently with $p(Q_k)$ and $p(Q)$ instead of $\tilde{p}(\tilde{Q}_k)$ and $\tilde{p}(\tilde{Q})$. Note that for any $t_0 > 0$, the strong stable foliation W^{ss} of the geodesic flow g_t coincides with the strong stable foliation of the diffeomorphism g_{t_0}, which we will denote by f. The time t_0 will be fixed later on.

We will now apply [20, Theorem IV.1, Appendix IV, p. 79] to the diffeomorphism f of T^1M, the decomposition of $TT^1M = E_1 \oplus E_2$ with $E_1 := \mathbb{R}X \oplus E^{su}$ and $E_2 := E^{ss}$. Moreover, since the geodesic flow on T^1M is an Anosov flow, we can choose t_0 so that the following hold:

$$\|Df(v)\| \leq \lambda\|v\| \tag{2.1}$$

for every $v \in E_2\backslash\{0\}$ and

$$\|Df(v)\| \geq \mu\|v\| \tag{2.2}$$

for every $v \in E_1\backslash\{0\}$, with the parameters $\mu = 1$ and $\lambda = e^{-1}$. Notice that in equations (2.1) and (2.2), the norm is the Riemannian metric on T^1M. The theorem mentioned above can now be applied while asserting that the set of plaques Q of the leaves of the strong stable foliation W^{ss} of f is locally a continuous family of C^r-embeddings into T^1M, for any $r \geq 0$, of the unit disk D^n in $\mathbb{R}^n$. More precisely, for $\varepsilon > 0$, let us define

$$W^{ss}_\epsilon(v) = \{u \in T^1M \,|\, d(f^n(v), f^n(u)) \leq \epsilon,\ \text{for all } n \geq 0,\ \text{and } d(f^n(v), f^n(u)) \underset{n\to+\infty}{\longrightarrow} 0\}. \tag{2.3}$$

Let $\mathcal{E}^r(D^n, T^1M)$ denote the space of C^r embeddings of D^n into T^1M, endowed with the C^r topology, where D^n is the unit disk in $\mathbf{R}^n$. Since f is C^r, for any $r \geq 0$, the assertions of the theorem are that for every $v \in T^1M$, we can choose a neighborhood V of v such that there exists a continuous map

$$\Theta : V \to \mathcal{E}^r(D^n, T^1M)\,, \tag{2.4}$$

such that $\Theta(w)(0) = w$ and $\Theta(w)(D^n) = W^{ss}_\epsilon(w)$ for all $w \in V$. We deduce that the sequence of maps $\Theta(v_k) : D^n \to W^{ss}_\epsilon(v_k)$ converges to the map $\Theta(v) : D^n \to W^{ss}_\epsilon(v)$. We may also choose $V \subset U$ and $\epsilon > 0$ small enough so that p maps $W^{ss}_\epsilon(v_k)$ diffeomorphically into Q_k for k large enough and, similarly, p maps $W^{ss}_\epsilon(v)$ diffeomorphically into Q. We may also assume that Q_k and Q lift diffeomorphically to $\tilde{Q}_k \subset T^1\tilde{M}$ and $\tilde{Q} \subset T^1\tilde{M}$. We then deduce that the sequence of diffeomorphism

$$\alpha_k := \pi^{-1} \circ p \circ \Theta(v_k) : D^n \to \tilde{p}(\tilde{Q}_k) \tag{2.5}$$

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converges to the diffeomorphism

$$\alpha := \pi^{-1} \circ p \circ \Theta(v) : D^n \to \tilde{p}(\tilde{Q}), \tag{2.6}$$

which proves the first part of the proposition.

Remark 2.2. Notice that in the above convergence, $\tilde{p}(\tilde{Q}_k) \subset H_{\xi_{\tilde{v}_k}}$ and $\tilde{p}(\tilde{Q}) \subset H_{\xi_{\tilde{v}}}$ contains balls of radius $\epsilon' := \epsilon'(\epsilon) > 0$ centered at $\tilde{p}(\tilde{v}_k)$ and $\tilde{p}(\tilde{v})$, respectively. The above convergence therefore holds on open sets of uniform size.

We now prove the second part of the proposition. Let us assume that $H_{\tilde{v}}$ is flat for the induced metric and consider $H_{\tilde{w}}$. Since M is a closed manifold, each leaf of the strong stable foliation W^{ss}, in particular, $W^{ss}(v)$, is dense in T^1M (see [1, Theorem 15]). Therefore, each plaque Q of $W^{ss}(w)$ contained in a chart $U \subset T^1M$ of the foliation is the Hausdorff limit of a sequence of plaques Q_l of $W^{ss}(v)$ in the same chart. Consequently, for the lift $\tilde{Q} \subset T^1\tilde{M}$ containing $\tilde{w}$, the set $\tilde{p}(\tilde{Q}) \subset H_{\tilde{w}}$ is the Hausdorff limit of a sequence of sets $\tilde{p}(\tilde{Q}_l) \subset H_{\tilde{v}}$, where $\tilde{Q}_l$ are lifts of Q_l.

Let Ψ be any transversal to W^{ss} passing through w (for example, Ψ could be a neighborhood of w in its weak unstable manifold), and let v_l be the intersection of Ψ with the plaque $Q_l \subset W^{ss}(v)$ which approximate Q, that is, $v_l \to w$ when $l \to +\infty$. Applying the first part of the proposition, the sequence $H_{\tilde{v}_l}$ locally converges in the C^r-topology to $H_{\tilde{w}}$. To be more precise, the metric

$$(\pi^{-1} \circ p \circ \Theta(w))^*(g)$$

is pulled back to D^n of the metric induced by the metric g of $\tilde{M}$ on $\pi^{-1}(p(\Theta(w)(D^n))) \subset H_{\tilde{w}}$ and, by the first part of the proposition, we deduce that

$$(\pi^{-1} \circ p \circ \Theta(w))^*(g) = \lim_{l\to\infty} (\pi^{-1} \circ p \circ \Theta(v_l))^*(g)$$

in the C^{r-1}-topology for every r. By tensoriality, the curvature of $(p \circ \Theta(w))^*(g)$ is pulled back of the intrinsic curvature of this projected horosphere (note that the curvature depends only on the differential of $p \circ \Theta$). Since all of these quantities depend continuously on w, it follows that $\tilde{p}(\tilde{Q})$ with the induced metric is flat, just as the $\tilde{p}(\tilde{Q}_l)$ are for all l.

This concludes the second part of the proposition.

The fourth part of the proposition follows along the same lines as above. Let $\tilde{v}_k \in T^1_{x_k}\tilde{M}$ and $\tilde{v} \in T^1_x\tilde{M}$ as in the statement. As above, we have convergence

$$(\pi^{-1} \circ p \circ \Theta(v))^*(g) = \lim_{k\to\infty} (\pi^{-1} \circ p \circ \Theta(v_k))^*(g)$$

in the C^{r-1}-topology for every r and therefore the Levi-Civita connection of $(\pi^{-1} \circ p \circ \Theta(v_k))^*(g)$ converges to the Levi-Civita of $(\pi^{-1} \circ p \circ \Theta(v))^*(g)$. In particular, for k large enough and $d_{H_{\tilde{v}_k}}(x_k, y_k) < \rho$, the unique geodesic between x_k and y_k converges to the unique geodesic joining x and y, and thus the corresponding parallel transport along these geodesics converges. This concludes the proof of the fourth part of the proposition.

Let us prove the third part of the proposition. We argue by contradiction assuming that there exists a sequence $\tilde{v}_k \in T^1_{x_k}\tilde{M}$ such that the injectivity radius $\mathrm{inj}_{H_{\tilde{v}_k}}(x_k)$ of $H_{\tilde{v}_k}$ at x_k tends to zero. By compactness of M, we may assume, after translation by elements

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of $\pi_1(M)$, that $\tilde{v}_k$ converges to $\tilde{v} \in T_x^1\tilde{M}$. As above, we have convergence of the metrics $(\pi^{-1} \circ p \circ \Theta(v))^*(g) = \lim_{l\to\infty}(\pi^{-1} \circ p \circ \Theta(v_k))^*(g)$ in the C^r-topology for every $r \geq 2$, and hence the injectivity radii $\mathrm{inj}_{H_{\tilde{v}_k}}(x_k)$ of $H_{\tilde{v}_k}$ at x_k converges to the injectivity radius $\mathrm{inj}_{H_{\tilde{v}}}(x)$ of $H_{\tilde{v}}$ at x. Since $\mathrm{inj}_{H_{\tilde{v}}}(x) > 0$, we get a contradiction, which concludes the proof of the third part of the proposition. □

2.2. *Heintze groups.* We now describe a family of examples illustrating that the compactness of M is a necessary assumption in Theorem 1.2. A Heintze group is a solvable group $G_A = \mathbb{R} \ltimes_A \mathbb{R}^n$, where A is an $n \times n$ matrix whose entries are real numbers. Such a group G_A is diffeomorphic to $\mathbb{R} \times \mathbb{R}^n$ with a group action given by $(s, y).(s', y') = (s + s', y + e^{sA}y')$. In what follows, we will use the coordinates given by the diffeomorphism $\psi : \mathbb{R} \times \mathbb{R}^n \to G_A$ defined by $\psi(s, y) := (s, e^{sA}y)$. When the real parts of the eigenvalues of A have the same sign, Heintze showed the existence of left invariant metrics on G_A with negative sectional curvature, see [**14**]. When the matrix A is a multiple of the identity, G_A, endowed with any left invariant metric is homothetic to the hyperbolic space. Furthermore, a Heintze group G_A contains no cocompact lattice unless it is homothetic to the hyperbolic space [**13**].

As an example, consider the $n \times n$ matrix A defined by

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \tag{2.7}$$

where $a_1 \leq a_2 \cdots \leq a_n < 0$. The left invariant metric g given at $(0, 0)$ by the standard Euclidean scalar product $dt^2 + |dy_1|^2 + \cdots + |dy_n|^2$ is written in the above coordinates $G_A = \mathbb{R} \times \mathbb{R}^n$ as

$$g = ds^2 + e^{2a_1 s}|dy_1|^2 + \cdots + e^{2a_n s}|dy_n|^2 \tag{2.8}$$

and gives G_A the structure of a Cartan–Hadamard manifold with pinched negative sectional curvature satisfying $-a_n^2 \leq K \leq -a_1^2$. In the above coordinates and for this metric, for every $y \in \mathbb{R}^n$, the curves $t \to (t, y)$ are geodesics, all being asymptotic to a point $\xi \in \partial G_A$ when $t \to +\infty$. For each $s \in \mathbb{R}$, the sets $\{(s, y)\ ,\ y \in \mathbb{R}^n\}$ are horospheres $H_\xi(s)$ centered at ξ. For each s, the horospheres $H_\xi(s)$ are clearly isometric to the Euclidean space $\mathbb{R}^n$. However, G_A is isometric to the real hyperbolic space if and only if $a_1 = a_2 = \cdots = a_n$ and it does not admit a compact quotient unless the a_i terms coincide, as proved in [**13**]. This exemplifies that having a family of Euclidean horospheres $H_\xi(t)$ centered at a given boundary point does not characterize the real hyperbolic space.

Also note that the flow φ_t defined in the above coordinates of G_A by

$$\varphi_t(s, y) := (s + t, y)$$

permutes the horospheres, mapping $H_\xi(s)$ on $H_\xi(s + t)$. Writing h_s as the metric induced by g on $H_\xi(s)$, we have

$$h_s := e^{2a_1 s}|dy_1|^2 + \cdots + e^{2a_n s}|dy_n|^2$$

and

$$\varphi_t^* \,(h_{s+t}) = e^{2a_1(s+t)}|dy_1|^2 + \cdots + e^{2a_n(s+t)}|dy_n|^2,$$

and hence the two metrics h_s and $\varphi_t^* \,(h_{s+t})$ are linearly equivalent and therefore they share the same Levi-Civita connecyion. The flow φ_t then preserves the Levi-Civita connecyions and thus commutes with the parallel transport of the induced metrics on the $H_\xi(s)$ terms.

2.3. *Busemann function.* Let $\tilde{M}$ be a Cartan Hadamard manifold endowed with pinched negative sectional curvature $-a^2 \leq K \leq -b^2 < 0$. The Busemann functions $B(\cdot, \xi)$ are C^2 for every $\xi \in \partial\tilde{M}$, [**15**, Proposition 3.1], and it is also known that they are C^∞ in the case that $\tilde{M}$ is the universal cover of a closed manifold.

For the sake of completeness, let us give here the proof of this fact. The geodesic flow $\tilde{g}_t$ on $\tilde{M}$ is generated by the smooth vector field $Z := d/dt_{|t=0}\tilde{g}_t$ on $T^1\tilde{M}$. For every $\xi \in \partial\tilde{M}$, the set defined by

$$\tilde{W}^s(\xi) = \{\tilde{v} \mid c_{\tilde{v}}(+\infty) = \xi\} \tag{2.9}$$

is a weak stable leaf of $\tilde{g}_t$, preserved by $\tilde{g}_t$. It is a smooth submanifold of $T^1\tilde{M}$ [**20**, Theorem IV.1] and the projection $\tilde{p}$ induces a diffeomorphism between $\tilde{W}_\xi$ and $\tilde{M}$. For every $\tilde{v} \in T^1\tilde{M}$, the vector $Z(\tilde{v}) := d/dt_{|t=0}(\tilde{g}_t(\tilde{v}))$ is tangent to the flow direction at $\tilde{v}$ and the following holds:

$$D\tilde{p}(\tilde{v})(Z(\tilde{v})) = \dot{c}_{\tilde{v}}(0) = -\nabla B(\tilde{p}(\tilde{v}), \xi). \tag{2.10}$$

Therefore, if we defined $\tilde{p}^{-1}(x) = \tilde{v} \in \tilde{W}_\xi$, we get that $\nabla B(x, \xi) = -D\tilde{p}(\tilde{p}^{-1}(x))(Z(\tilde{p}^{-1}(x))$ is a smooth vector field on $\tilde{M}$ and therefore $B(\cdot, \xi)$ is smooth.

This fact will be useful in §4 while constructing a quasi-isometry between $\tilde{M}$ and G_A using horospherical coordinates.

3. *Stable holonomies for horospheres in negatively curved manifolds*

A priori, the parallel transport associated to the induced metrics on horospheres does not commute with the action of the geodesic flow. In a sharp contrast, at the end of §2.2, we noticed that for Heintze groups, it does. In this section, we will describe another transport along horospheres, called the *stable holonomy*, which, by construction, commutes with the geodesic flow. A consequence of the equality of these *a priori* unrelated two parallel transports is that the Levi-Civita connections of the horospheres are flat and commute with the geodesic flow. We will see in §4 that when these two properties hold true on the family of horospheres $H_\xi(s)$, $s \in \mathbb{R}$, for $\xi \in \partial\tilde{M}$ fixed by some element $\gamma \in \pi_1(M)$, then $\tilde{M}$ is quasi-isometric to the Heintze group G_A, where A is the derivative of the Poincaré first return map along the periodic geodesic associated to γ.

We now describe the construction of the stable holonomy following [**3**, **17**]. It uses in a crucial way either the strong 1/4-pinching or the relative 1/2-pinching assumption on the curvature which corresponds to the 'fiber bunched' condition of [**17**] (see Appendix A). In fact, Propositions 3.5 and 3.12 are a consequence of [**17**, Proposition 4.2]. However, we will construct the stable holonomy in a way which is adjusted to our particular geometric setting and to make the paper self contained. We conclude this section with Proposition 3.13 and

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Corollary 3.14, stating that equality of the two transports on a single horosphere implies equality on all horospheres.

Throughout this section, we will work with the tangent bundle of horospheres in $\tilde{M}$ which, in turn, as a level set of Busemann functions, are smooth submanifolds of the universal cover of M. Keeping the notation from §1, let $g_t : T^1M \to T^1M$ denote the geodesic flow on M, i.e., the projection of $\tilde{g}_t$ under the map $D\pi : T^1\tilde{M} \to T^1M$. Let us choose a point $\xi \in \partial\tilde{M}$. It is a well-known feature of the negative curvature of $\tilde{M}$ that any point in $\tilde{M}$ lies on a unique geodesic ray ending at ξ. Hence, the canonical projection $\tilde{p} : T^1\tilde{M} \to \tilde{M}$ induces a diffeomorphism from the set of unit vectors that are pointing in the direction of ξ and $\tilde{M}$. This subset of unit tangent vectors will be denoted by $\tilde{W}^s(\xi)$, and is usually called the (weak) stable manifold and the induced diffeomorphism will be denoted by $\tilde{p}_\xi$.

With this identification, for every $t \in \mathbb{R}$ and for every $\xi \in \partial\tilde{M}$, the action of the geodesic flow on $\tilde{W}^s(\xi)$ provides us with a one-parameter group of diffeomorphism of $\tilde{M}$,

$$\varphi_{t,\xi} = \tilde{p}_\xi \circ \tilde{g}_t \circ \tilde{p}_\xi^{-1}. \tag{3.1}$$

For $\tilde{v}_0 \in T^1\tilde{M}$, let $\xi = c_{\tilde{v}_0}(+\infty)$ and assume that $\tilde{p}_\xi(\tilde{v}_0) = x_0$ with $c_{\tilde{v}_0}(0) = x_0$. By definition, $\tilde{p}_\xi$ maps $\tilde{W}^{ss}(\tilde{v}_0)$ diffeomorphically onto the unique horosphere centered at ξ which contains x_0. If we denote this horosphere by $H_\xi(0)$, then it also follows from the definitions that the derivative $D\tilde{p}_\xi(\tilde{v}_0)$ maps $\tilde{E}^{ss}(\tilde{v}_0)$ isomorphically onto $T_{x_0}H_\xi(0)$. Finally, we note that the family of horospheres centered at ξ can be parameterized by the time parameter, i.e., for $s \in \mathbb{R}$, the horosphere $H_\xi(s)$ will denote the unique horosphere in $\tilde{M}$, centered at ξ, which intersects the geodesic $c_{\tilde{v}_0}$ at time s. By the property of invariance of the strong stable foliation by the geodesic flow, it follows that the diffeomorphisms $\varphi_{t,\xi}$ permutes the set of horospheres centered at ξ, namely, $\varphi_{t,\xi}H_\xi(s) = H_\xi(s+t)$.

We now turn to the main construction of this section, see [3, 17]. The stable holonomy, which we describe below, provides a geodesic flow invariant way to identify tangent spaces at different points on any fixed horosphere. We fix $x_0 \in \tilde{M}$ and recall that the horospheres are defined by

$$H_\xi(s) = \{x \in \tilde{M} \mid B_\xi(x) = s\},$$

where the Buseman function B_ξ has been normalized such that $B_\xi(x_0) = 0$.

We start with the following definition (see [17, Definition 4.1] and Figure 1).

Definition 3.1. (Stable holonomy for horospheres) A stable holonomy is a family of maps $(x, y, \xi) \to \Pi_s^\xi(x, y)$, $s \in \mathbb{R}$, defined on the set of points (x, y, ξ) such that x, y belong to the horosphere $H_\xi(s)$, and such that the following properties hold:

(1) $\Pi_s^\xi(x, y)$ is a linear map from $T_xH_\xi(s)$ to $T_yH_\xi(s)$ for every $s \in \mathbb{R}$, $x, y \in H_\xi(s)$;

(2) $\Pi_s^\xi(x, x) = \mathrm{Id}$ and $\Pi_s^\xi(x, y) = \Pi_s^\xi(z, y) \circ \Pi_s^\xi(x, z)$ for every $s \in \mathbb{R}$, $x, y, z \in H_\xi(s)$;

(3) $\Pi_s^\xi(x, y) = D\varphi_{t,\xi}^{-1}(\varphi_{t,\xi}(y)) \circ \Pi_{s+t}^\xi(\varphi_{t,\xi}(x), \varphi_{t,\xi}(y)) \circ D\varphi_{t,\xi}(x)$ for all $t \in \mathbb{R}$, $s \in \mathbb{R}$;

(4) for every $\gamma \in \pi_1(M)$, $\Pi_{s+B_{\gamma\xi}(\gamma x_0)}^{\gamma\xi}(\gamma x, \gamma y) = D\gamma(y) \circ \Pi_s^\xi(x, t) \circ (D\gamma(x))^{-1}$;

in condition (3), $D\varphi_{t,\xi}(z)$ denotes the differential of $\varphi_{t,\xi}$ at the point z.

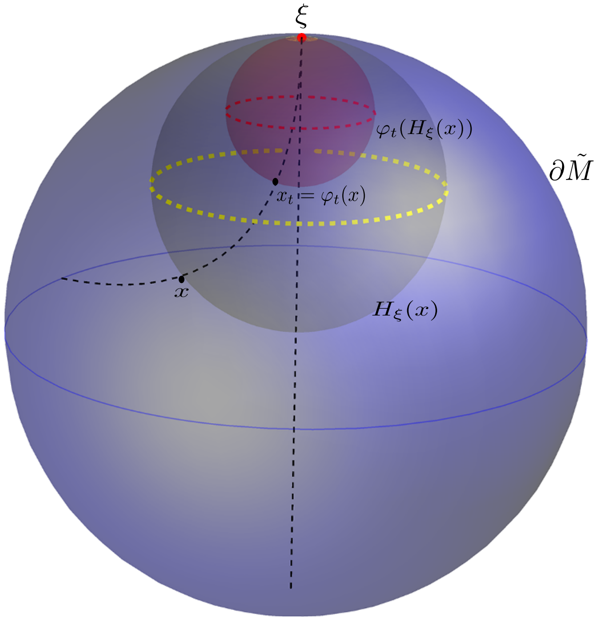

FIGURE 1. Horospheres and action of $\varphi_t = \varphi_{t,\xi}$.

Notice that condition (2) tells that this stable holonomy, if it exists, is 'flat' and condition (4) that the stable holonomy is equivariant under the action of the fundamental group of M on the set of horospheres.

Let us choose a point $\xi \in \partial\tilde{M}$. After this section, we will set $\varphi_t := \varphi_{t,\xi}$, $t \in \mathbb{R}$ and $\tilde{p}_\xi = \tilde{p}$. Recall that the induced Riemannian metric on $H_\xi(t)$ is denoted by h_t, and let ∇^t denote the Levi-Civita connection associated to h_t. The parallel transport with respect to ∇^t, along any path joining any two points x and y in $H_\xi(t)$, is an isometry between $T_x H_\xi(t)$ and $T_y H_\xi(t)$. The isometry *a priori* depends on the path. However, if x, y in $H_\xi(t)$ are at a distance less than the injectivity radius of $H_\xi(t)$, there exists a unique geodesic segment joining x and y and we will therefore denote by

$$P_t^\xi(x, y) \tag{3.2}$$

the parallel transport along this segment.

We now turn to the main proposition of this section that will grant us the existence of the stable holonomy along horospheres. It is a reformulation of [17, Proposition 4.2] or of [8, Proposition 2.2]. Since we will use the construction later on, we will briefly describe it. We first need two lemmas.

The first lemma gives uniform contraction properties of the maps φ_t under the strong $1/4$-pinching condition on the curvature of M. Let us normalize the sectional curvature K of M, so that the following inequalities are satisfied for some constant $1 > \tau > 0$:

$$-4(1-\tau) \le K \le -1. \tag{3.3}$$

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LEMMA 3.2. *Let x, y be two points on $H_\xi(s)$ and let X be a tangent vector in $T_x H_\xi(s)$. Then, for any $t \geq 0$, the following estimates hold:*

(1) $\|D\varphi_t(x)(X)\|_{h_{s+t}} \leq e^{-t}\|X\|_{h_s}$;

(2) $\|D\varphi_t^{-1}(x)(X)\|_{h_{s-t}} \leq e^{(2\sqrt{1-\tau})t}\|X\|_{h_s} \leq e^{(2-\tau)t}\|X\|_{h_s}$; *and*

(3) $d_{h_{s+t}}(\varphi_t(x), \varphi_t(y)) \leq e^{-t} d_{h_s}(x, y)$.

Proof. The norm and the distance we use above are computed with respect to the induced Riemannian metric on the corresponding horosphere. Recall that a *stable* Jacobi field $Y(t)$ along a geodesic ray $c_{\tilde{v}}(t)$, $t > 0$, is a bounded Jacobi field, see [**15**, Definition 2.1]. The proof of these inequalities is a direct consequence of the estimate of the growth of the stable Jacobi fields, as done in [**15**, Theorem 2.4].

In fact, we only need to show that $D\varphi_t(X)$ is a stable Jacobi field. This follows from the Anosov property of the geodesic flow of M, see [**2**, Appendix 21]. Indeed, if X is a tangent vector in $T_x H_\xi(s)$ at the point x, then $X = D\tilde{p}(\tilde{v})(V)$, where $V \in E^{ss}(\tilde{v}) \subset T_{\tilde{v}} T^1\tilde{M}$, and $\tilde{v}$ is the unit vector in $T_x\tilde{M}$ perpendicular to $H_\xi(s)$ and pointing toward ξ. Therefore, by applying the chain rule to equation (3.1) and recalling that $x = \tilde{p}(\tilde{v})$, we obtain that

$$D\varphi_t(x)(X) = D\tilde{p}(\tilde{g}_t(\tilde{v})) \circ D\tilde{g}_t(\tilde{v})(V). \tag{3.4}$$

Since the geodesic flow of M is Anosov and $V \in E^{ss}(\tilde{v})$, it follows that

$$\lim_{t\to\infty} \|D\tilde{g}_t(\tilde{v})(V)\| = 0, \tag{3.5}$$

which implies that $\lim_{t\to\infty} \|D\varphi_t(x)(X)\| = 0$. Indeed, the map $\tilde{p} : T^1\tilde{M} \to \tilde{M}$ is defined on the quotient (by $\pi_1(M)$) by $p : T^1M \to M$, and the compactness of M grants us that $\tilde{p}$ as well as $D\tilde{p}$ are bounded. Hence, it follows that $D\varphi_t(X)$ is a stable Jacobi field and this concludes the proof of the first assertion of Lemma 3.2. The other assertions follow easily. □

Since $(\tilde{M}, \tilde{g})$ covers the closed manifold (M, g), for each $\sigma \in [0, 1]$, we are able to obtain a uniform control on the action of φ_σ as follows. We first study the behavior of the family of horospheres $H_\xi(s)$, $s \in \mathbb{R}$, orthogonal to the geodesic $c_{\tilde{v}}(s)$ such that $c_{\tilde{v}}(+\infty) = \xi$. By assertion (3) of Proposition 2.1, we will assume from now on that the injectivity radius of every horosphere is bounded below by $\rho > 0$. For each $x \in H_\xi(s)$, we denote c_x as the geodesic passing through x asymptotic to ξ, i.e., $c_x(+\infty) = c_{\tilde{v}}(+\infty) = \xi$ parameterized in such a way that $c_x(s) = x$.

LEMMA 3.3. *For every $R > 0$, there exists a constant $C_R > 0$ such that for any $s \in \mathbb{R}$, any $\sigma \in [0, R]$, any two points $x, y \in H_\xi(s)$ such that $d_{H_\xi(s)}(x, y) < \rho$, and any $X \in T_x H_\xi(s)$, the following holds:*

$$\begin{aligned}\|(D\varphi_\sigma^{-1}(\varphi_\sigma(y)) \circ P^\xi_{s+\sigma}(\varphi_\sigma(x), \varphi_\sigma(y)) \circ D\varphi_\sigma(x) - P^\xi_s(x, y))(X)\|_{h_s}\\ \leq C_R d_{h_s}(x, y)\, \|X\|_{h_s}.\end{aligned} \tag{3.6}$$

Proof. Let us first assume that $X \in T_x H_\xi(s)$ has a unit norm. Define $X_\sigma := D\varphi_\sigma(x)X$ and let $c : [0, d] \to H_\xi(s)$ be the geodesic segment of $H_\xi(s)$ between x and y, where $d = d_{h_s}(x, y)$. Let $c_\sigma(u) : [0, d] \longrightarrow H_\xi(s + \sigma)$ be the geodesic segment, parameterized with constant speed, joining $\varphi_\sigma(x)$ and $\varphi_\sigma(y)$ which exists by Lemma 3.2(3). Notice that also by Lemma 3.2, we have

$$e^{-(2-\tau)} \leq |\dot{c}_\sigma| \leq 1. \tag{3.7}$$

We have

$$\begin{aligned} &D\varphi_\sigma^{-1}(\varphi_\sigma(y)) \circ P^\xi_{s+\sigma}(\varphi_\sigma(x), \varphi_\sigma(y)) \circ D\varphi_\sigma(x) - P^\xi_s(x, y) \\ &= \int_0^d \frac{d}{du}(D\varphi_\sigma^{-1}(c_\sigma(u)) \circ (P^\xi_{s+\sigma}(\varphi_\sigma(x), c_\sigma(u)) \circ D\varphi_\sigma(x) \\ &\quad - D\varphi_\sigma(c(u)) \circ P^\xi_s(x, c(u)))) \, du. \end{aligned}$$

By compactness of M and by equation (3.1), the norm of every covariant derivative of φ^ξ_σ and $(\varphi^\xi_\sigma)^{-1}$, $\xi \in \partial\tilde{M}$ and $\sigma \in [0, R]$ is bounded above by a constant depending on the degree of derivation. In particular, there exists a constant $C_R > 0$ such that the integrand in the right-hand side term is bounded above by C_R.

We deduce that

$$\|P^\xi_s(x, y)(X) - D\varphi_\sigma^{-1}(\varphi_\sigma(y)) \circ P^\xi_{s+\sigma}(\varphi_\sigma(x), \varphi_\sigma(y)) \circ D\varphi_\sigma(x)(X)\|_{h_s} \leq C d_{h_s}(x, y). \tag{3.8}$$

If the norm of X is not equal to 1, the desired inequality follows by simple modifications of the proof above. □

Remark 3.4. Notice that the constant C in the above proposition does not depend on the horosphere $H_\xi(s)$ nor even on ξ. More precisely, in equation 3.8, the parallel transport operators are isometries, and hence their norms are bounded by one. Only the differential of ϕ_σ matters. These maps, for $\sigma \in [0, 1]$, are projections, by $\tilde{p}$ to $\tilde{M}$, of the geodesic flow on $T^1\tilde{M}$ restricted to the submanifolds $\tilde{W}^s(\xi)$. Now by compactness of M, $T^1(M)$, and $[0, 1]$, $\tilde{p}$ and the geodesic flow on $T^1\tilde{M}$ have bounded derivatives at any order. Finally, the arguments in §2.1 show that the manifolds $\tilde{W}^s(\xi)$ have uniformly bounded geometry at any order with constants independent of ξ. Notice, however, that independence on ξ is not really needed in our argument.

We now turn to prove the existence of a stable holonomy. In the following proposition, we assume that the sectional curvature satisfies either the strong 1/4-pinching or relative 1/2-pinching assumption. We will then describe possible generalizations based on the results in [**11**]. However, stable holonomy may exist without any pinching assumption but just under the negativity of the sectional curvature. We do not know any counterexample to this. For every $\tilde{v} \in T^1\tilde{M}$, we consider the family of horospheres centered at $\xi := c_{\tilde{v}}(+\infty)$, which we parameterize as $H_\xi(t)$, $t \in \mathbb{R}$, where the parameter $t = 0$ corresponds to the horosphere containing the base point of $\tilde{v}$.

PROPOSITION 3.5. *Let M be a closed Riemannian manifold with pinched negative curvature satisfying either the strong 1/4-pinching condition $-4(1-\tau) \leq \kappa \leq -1$ or the relative 1/2-pinching condition. Let $\tilde{v}$ be a unit vector tangent to $\tilde{M}$. Let $\xi = \lim_{t\to+\infty} c_{\tilde{v}}(t) \in \partial\tilde{M}$. Then:*

(i) *for every* $s \in \mathbb{R}$, $x, y \in H_\xi(s)$, *there exists a linear map*

$$\Pi_s^\xi(x, y) : T_x H_\xi(s) \to T_y H_\xi(s)$$

satisfying conditions (1), (2), and (3) in Definition 3.1;

(ii) $\|\Pi_s^\xi(x, y) - P_s^\xi(x, y)\| \le C d_{h_s}(x, y)$ *for all* x, y *such that* $d_{h_s}(x, y) < \rho$*;*

(iii) *properties (i) and (ii) uniquely determine the stable holonomy;*

(iv) *the stable holonomy is* $\pi_1(M)$*-equivariant, i.e., for every* $\gamma \in \pi_1(M)$*, we have*

$$\Pi_{s+B_{\gamma\xi}(\gamma x_0)}^{\gamma\xi}(\gamma x, \gamma y) = D\gamma(y) \circ \Pi_s^\xi(x, t) \circ (D\gamma(x))^{-1}.$$

Proof. The proof follows closely the methods given in [**17**, Proposition 4.2]. We reproduce here only the part of the construction, modified to our setting, which we will need in what follows. The proof is organized into the two cases corresponding to the different curvature assumptions.

The case of strong $1/4$*-pinching of the curvature.* Recall that we have denoted $\varphi_t := \varphi_{\xi,t}$. Let us consider $x, y \in H_\xi(s)$ such that $d_{H_\xi(s)}(x, y) \le R$ for some fixed R. For every $x \in H_\xi(s)$, denote $x_t := \varphi_t(x) \in H_\xi(s+t)$. By Lemma 3.2(3), there exists $t_0 := t_0(R) \ge 0$ such that $d_{H_\xi(s+t_0)}(x_{t_0}, y_{t_0}) < \rho$.

Let us turn to proving assertion (i). For every $t \in \mathbb{R}$, define

$$c_t : [0, 1] \to H_\xi(s+t)$$

as the geodesic segment, parameterized with constant speed, between x_t and y_t which is well defined when their distance is less than ρ.

For $x, y \in H_\xi(s)$, we define

$$\Pi_s^\xi(x, y) = \lim_{t\to\infty} d\varphi_t^{-1}(y_t) \circ P_t^\xi(x_t, y_t) \circ d\varphi_t(x). \tag{3.9}$$

Note that the term $P_t^\xi(x_t, y_t)$ in the limit is well defined for all $t \ge t_0$, since the distance between x_t and y_t is decreasing. Let us show that the above limit exists. Define for $j \ge 0$, $x, y \in H_\xi(s)$,

$$\Pi_{s,j}^\xi(x, y) := d\varphi_{t_0+j}^{-1}(y_{t_0+j}) \circ P_{s+t_0+j}^\xi(x_{t_0+j}, y_{t_0+j}) \circ d\varphi_{t_0+j}(x).$$

We have for every $N \ge 0$,

$$\Pi_{s,N}^\xi(x, y) = \Pi_{s,0}^\xi(x, y) + \sum_{j=0}^{N-1} (\Pi_{s,j+1}^\xi(x, y) - \Pi_{s,j}^\xi(x, y)). \tag{3.10}$$

Each term in the above sum is expanded as

$$\begin{aligned}(\Pi_{s,j+1}^\xi - \Pi_{s,j}^\xi)(x, y) &= D\varphi_{t_0+j}^{-1}(y_{t_0+j}) \circ \big[D\varphi_1^{-1}(y_{t_0+j+1}) \circ P_{s+t_0+j+1}^\xi(x_{t_0+j+1}, y_{t_0+j+1}) \circ D\varphi_1(x_{t_0+j}) \\ &\quad - P_{c_j}(x_{t_0+j}, y_{t_0+j})\big] \circ D\varphi_{t_0+j}(x),\end{aligned}$$

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and hence, by Lemma 3.3, we get

$$\|(\Pi^{\xi}_{s,j+1} - \Pi^{\xi}_{s,j})(x, y)\| \le C\|D\varphi^{-1}_{t_0+j}(y_{t_0+j})\|\|D\varphi_{t_0+j}(x)\|d_{h_{s+t_0+j}}(x_{t_0+j}, y_{t_0+j}). \tag{3.11}$$

Assertion (3) of Lemma 3.2 implies that

$$d_{h_{t_0+s+j}}(x_{t_0+j}, y_{t_0+j}) \le e^{-(t_0+j)}d_{h_s}(x, y). \tag{3.12}$$

Substituting this inequality back in equation (3.11) and using the estimates (1) and (2) of Lemma 3.2 yield that

$$\|(\Pi^{\xi}_{s,j+1} - \Pi^{\xi}_{s,j})(x, y)\| \le Ce^{-\tau(t_0+j)}d_{h_s}(x, y). \tag{3.13}$$

Therefore, the limit in equation (3.9) exists and is well defined. The $\pi_1(M)$-invariance is obvious and proofs of the others parts of this proposition are the same as in those of [17, Theorem 4.2].

The case of relative $1/2$*-pinching of the curvature.* In this case, we use a result of B. Hasselblatt stating that the geodesic flow g_t of a closed manifold with a relatively $1/2$-pinched negative curvature satisfies the following 'bunching' condition [12, Theorem 6]. The geodesic flow g_t on T^1M is α-bunched, $\alpha > 0$, if there exists functions $\mu_\pm : T^1M \times \mathbb{R}_+ \to (0, 1)$ such that for every $v \in T^1M$, $X \in E^{ss}(v)$, and $t > 0$,

$$\mu_-(v, t)\|X\| \le \|Dg_t(X)\| \le \mu_+(v, t)\|X\| \tag{3.14}$$

and

$$\lim_{t\to\infty} \sup_{v\in T^1M} \mu_+(v, t)^{2/\alpha}\mu_-(v, t)^{-1} = 0. \tag{3.15}$$

THEOREM 3.6. [12] *Let M be a closed Riemannian manifold with relative* $1/2$*-pinched negative curvature. Then the geodesic flow of M is* $(1 + \epsilon)$*-bunched for some* $\epsilon > 0$.

It turns out that in the proof of this theorem it is shown that the convergence in equation (3.15) is exponential, i.e., there exists $\tau > 0$ and $A > 0$ such that

$$\sup_{v\in T^1M} \mu_+(v, t)^{2/\alpha}\mu_-(v, t)^{-1} \le A\, e^{-\tau t}. \tag{3.16}$$

The proof of the proposition in the relative pinching case in Proposition 3.5 is then similar to that under the strong pinching assumption. We first notice that equations (3.14) and (3.15) lift the universal cover into

$$\tilde\mu_-(\tilde v, t)\|X\| \le \|D\tilde g_t(X)\| \le \tilde\mu_+(\tilde v, t)\|X\| \tag{3.17}$$

and

$$\sup_{\tilde v\in T^1\tilde M} \tilde\mu_+(\tilde v, t)^{2/\alpha}\tilde\mu_-(\tilde v, t)^{-1} \le A\, e^{-\tau t}, \tag{3.18}$$

where $t > 0$, $X \in E^{ss}(\tilde v)$, and the functions $\tilde\mu_\pm : T^1\tilde M \times \mathbb{R}_+ \to (0, 1)$ are invariant under the action of the fundamental group of M on the first variable. By equation (3.1), recall that

$$\varphi_{t,\xi} = \tilde p_\xi \circ \tilde g_t \circ \tilde p_\xi^{-1}.$$

For $\tilde{v} \in T^1\tilde{M}$, $x = \pi(\tilde{v})$, and $\xi = c_{\tilde{v}}(+\infty)$, denote

$$\mu_{\pm}^{\xi}(x,t) := \tilde{\mu}_{\pm}(\tilde{v},t).$$

Since there exists $C > 0$ such that $C^{-1} \leq \|D\tilde{p}_{\xi}^{\pm 1}\| \leq C$ by compactness of M, the above equations (3.17) and (3.18) translate into

$$C^{-2}\mu_{-}^{\xi}(x,t)\|X\| \leq \|D\varphi_{t,\xi}(X)\| \leq C^{2}\mu_{+}^{\xi}(x,t)\|X\| \tag{3.19}$$

and

$$\sup_{x\in\tilde{M},\xi\in\partial\tilde{M}} \mu_{+}^{\xi}(x,t)^{2/\alpha}\mu_{-}^{\xi}(x,t)^{-1} \leq A\, e^{-\tau t}, \tag{3.20}$$

where X is a vector tangent at x to the horosphere centered at ξ and passing through x. Since $\alpha = 1+\epsilon$ and $\mu_{+}^{\xi}(x,t) = \tilde{\mu}_{+}(\tilde{v},t) < 1$, we can choose by equation (3.20) $t_0 > 0$ and $0 < \theta < 1$ such that for every $x \in \tilde{M}$,

$$\mu_{+}^{\xi}(x,t_0)^{2}\mu_{-}^{\xi}(x,t_0)^{-1} \leq \theta. \tag{3.21}$$

We now argue as in the previous case. We define

$$\Pi_{s,j}^{\xi}(x,y) := d\varphi_{t_0(1+j)}^{-1}(y_{t_0(1+j)}) \circ P_{s+t_0(1+j)}^{\xi}(x_{t_0(1+j)}, y_{t_0(1+j)}) \circ d\varphi_{t_0(1+j)}(x)$$

and get similarly as in equation (3.11)

$$\begin{aligned}\|(\Pi_{s,j+1}^{\xi} - \Pi_{s,j}^{\xi})(x,y)\| \\ \leq C_1\|(D\varphi_{t_0(1+j)}(y))^{-1}\|\|D\varphi_{t_0(1+j)}(x)\| d_{h_{s+t_0(1+j)}}(x_{t_0(1+j)}, y_{t_0(1+j)}).\end{aligned} \tag{3.22}$$

There is here a slight difference with the previous case coming from the fact that the term $\|(D\varphi_{t_0(1+j)}(y))^{-1}\|\|D\varphi_{t_0(1+j)}(x)\|$ in equation (3.22) cannot be estimated through the estimates of Lemma 3.2, which are uniform in x and y. Instead, we argue like in [**17**, Lemma 4.3]: denoting $x_k := \varphi_{t_0(1+k)}(x)$ and similarly replacing x by y, we have

$$D\varphi_{t_0(1+j)}(x) = D\varphi_{t_0}(x_{t_0(1+j-1)}) \circ \cdots \circ D\varphi_{t_0}(x)$$

and

$$D\varphi_{t_0(1+j)}(y) = D\varphi_{t_0}(y_{t_0(1+j-1)}) \circ \cdots \circ D\varphi_{t_0}(y),$$

and hence

$$\|(D\varphi_{t_0(1+j)}(y))^{-1}\|\|D\varphi_{t_0(1+j)}(x)\| \leq \Pi_{k=0}^{k=j-1}(\|D\varphi_{t_0}(y_k)^{-1}\|\|D\varphi_{t_0}(x_k)\|); \tag{3.23}$$

therefore,

$$\begin{aligned}\|(D\varphi_{t_0(1+j)}(y))^{-1}\|\|D\varphi_{t_0(1+j)}(x)\| \\ \leq \Pi_{k=0}^{k=j-1}(\|D\varphi_{t_0}(y_k)^{-1}\|\|D\varphi_{t_0}(y_k)\|)\Pi_{k=0}^{k=j-1}\frac{\|D\varphi_{t_0}(x_k)\|}{\|D\varphi_{t_0}(y_k)\|}.\end{aligned} \tag{3.24}$$

Let us estimate the last product in equation (3.24). Since for every $x \in \tilde{M}$, we have $C_2^{-1} \leq \|D\varphi_{t_0}(x)\| \leq C_2$ for $C_2 = C^2 \sup_x(\mu_{+}^{\xi}(x,t_0))$ by equation (3.19), we deduce from Lemma 3.3 that

$$\begin{aligned}\|(P^{\xi}_{s+t_0}(\varphi_{t_0}(x), \varphi_{t_0}(y)) \circ D\varphi_{t_0}(x) - D\varphi_{t_0}(y) \circ P^{\xi}_{s}(x, y))(X)\|_{h_s} \\ \leq C_3\, C_R d_{h_s}(x, y)\, \|X\|_{h_s},\end{aligned} \tag{3.25}$$

and since the parallel transport is an isometry, we have

$$|\|D\varphi_{t_0}(x)\| - \|D\varphi_{t_0}(y)\|| \leq C_3\, C_R d_{h_s}(x, y). \tag{3.26}$$

We therefore get

$$\left|\left(1 - \frac{\|D\varphi_{t_0}(x)\|}{\|D\varphi_{t_0}(y)\|}\right)\right| \leq C_4\, C_R d_{h_s}(x, y), \tag{3.27}$$

where $C_4 = C_2\, C_3$. Now, recalling that the sectional curvature of M satisfies $K \leq -a^2 < 0$ for some a, we deduce that $d(x_k, y_k) \leq e^{-at_0(1+k)}$ and thus that there exists $C_5 > 0$ such that for every k,

$$\Pi_{k=0}^{k=j-1} \frac{\|D\varphi_{t_0}(x_k)\|}{\|D\varphi_{t_0}(y_k)\|} \leq C_5. \tag{3.28}$$

From equations (3.22), (3.24), and (3.28), we obtain

$$\begin{aligned}&\|(\Pi^{\xi}_{s,j+1} - \Pi^{\xi}_{s,j})(x, y)\| \\ &\leq C_6 \Pi_{k=0}^{k=j-1} (\|D\varphi_{t_0}(y_k)^{-1}\| \|D\varphi_{t_0}(y_k)\|) d_{h_{s+t_0(1+j)}}(x_{t_0(1+j)}, y_{t_0(1+j)}).\end{aligned} \tag{3.29}$$

From equations (3.19) and (3.29), we get

$$\begin{aligned}&\|(\Pi^{\xi}_{s,j+1} - \Pi^{\xi}_{s,j})(x, y)\| \\ &\leq C_6 \Pi_{k=0}^{k=j-1} ((\mu^{\xi}_{-}(y_k, t_0))^{-1} \mu^{\xi}_{+}(y_k, t_0)) d_{h_{s+t_0(1+j)}}(x_{t_0(1+j)}, y_{t_0(1+j)}).\end{aligned} \tag{3.30}$$

Now, similarly as in [7, Lemma 1.1], we see that

$$d_{h_{s+t_0(1+j)}}(x_{t_0(1+j)}, y_{t_0(1+j)}) \leq C_7\, \Pi_{k=0}^{k=j-1} \mu^{\xi}_{+}(y_k, t_0), \tag{3.31}$$

and hence from equation (3.30), we deduce

$$\|(\Pi^{\xi}_{s,j+1} - \Pi^{\xi}_{s,j})(x, y)\| \leq C_8 \Pi_{k=0}^{k=j-1} ((\mu^{\xi}_{-}(y_k, t_0))^{-1} (\mu^{\xi}_{+}(y_k, t_0))^2). \tag{3.32}$$

By equation (3.21), we therefore have

$$\|(\Pi^{\xi}_{s,j+1} - \Pi^{\xi}_{s,j})(x, y)\| \leq C_9\, \theta^{j+1} \tag{3.33}$$

and we conclude as in the previous case. □

3.1. *General relative pinching.* Theorem 3.6 can be extended; indeed, following [11], one could consider the situation of a more general relative pinching. To get a result with the same technique than in [12], that is, comparison theorems for the Riccati equations, we need to combined strong pinching and relative pinching. More precisely, let us recall the statement of [11, Theorem 4.3].

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THEOREM 3.7. *Let a and b satisfy* $0 \leq b \leq a \leq 1$. *The geodesic flow of a closed negatively curved Riemannian manifold which is b-pinched and relatively a-pinched is*

$$C(a, b) + \epsilon\text{-bunched}$$

for some positive ϵ, *where* $C(a, b) = a - b + \sqrt{(a+b)^2 + 4(1-a)b}$.

Like in [12], the $\epsilon > 0$ is small and it appears because of how the pinching assumptions are strict. Notice that when the upper sectional curvature approaches zero, even though a relative pinching is given, the comparison may not give the bunching, so the role played by the strong pinching is then to circumvent this difficulty. In [11], some interesting explicit solutions and evidence of the optimality of this result are given.

Now, if a and b are chosen so that $C(a, b) = 1$, we get $(1 + \epsilon)$-bunching and are able to apply a proof similar to the one given for Theorem 3.6. A direct computation shows that $C(a, b) = 1$ is equivalent to $a + b = 1/2$. The following remark then gives some useful information.

Remark 3.8. Let us first remark (see [11, Remark 1.5]) that, under the hypotheses given in Theorem 3.7: $C(a, b) \geq 2a$ and $C(a, a) = 2\sqrt{a}$. It is furthermore obvious to check that if $a = 1/2$, then $b = 0$ yields $(1 + \epsilon)$-bunching which means nothing more than the sectional curvature is non-positive, this is our second case. Also, if $a = 1/4$, then $b = 1/4$, which means that we have a strong quarter-pinching and not only a relative one; this is our first case. These are the only cases for which only one pinching condition is necessary. To go further, note that the bunching condition satisfies a monotonicity property; indeed, if $\alpha \geq \beta$, then α-bunched implies β-bunched. A straightforward consequence is that the condition we really need to make the construction of the stable holonomy is that $C(a, b) \geq 1 + \epsilon$, for some small ϵ, which is equivalent, by the same direct computation, to $a + b > 1/2$. Consequently, $a = 1/4 + \eta'$ and $b = 1/4 - \eta$ with $\eta' > \eta$ makes possible the construction. Here, η' can takes all values in the interval $]0, 1/4[$. The two extreme cases are given by our case 1 and case 2, and this remark yields situations that interpolate between them. The price to pay for these new cases is to combine the two pinching conditions, strong and relative.

Remark 3.9. In the proof of the above proposition, the following fact, which will be useful later, was applied several times.

CLAIM 3.10. *For every* $\epsilon > 0$ *and* $d > 0$, *there exists N such that for every* $\xi \in \partial\tilde{M}$ *and every pair of points in a horosphere* H_ξ *such that* $d_{H_\xi}(x, y) \leq d$, *then*

$$\|\Pi^\xi(x, y) - \Pi^\xi_N(x, y)\| \leq \epsilon,$$

where $\Pi^\xi_N(x, y)$ *is defined in equation (3.10) with* $s = 0$.

Proof. Let us prove the claim. In the case of strong 1/4-pinching, it follows by equation (3.10) that

$$\Pi^\xi(x, y) - \Pi^\xi_N(x, y) = \sum_{j=N}^{\infty} (\Pi^\xi_{j+1}(x, y) - \Pi^\xi_j(x, y)),$$

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and by equation (3.13), we obtain

$$\|\Pi^{\xi}(x, y) - \Pi^{\xi}_{N}(x, y)\| \leq C \sum_{j=N}^{\infty} e^{-\tau(t_0+j)} d_{h_s}(x, y).$$

This concludes the proof of the claim since the rest of the series satisfies

$$\sum_{j=N}^{\infty} e^{-\tau(t_0+j)} d_{h_s}(x, y) \leq d \sum_{j=N}^{\infty} e^{-j\tau} \leq \epsilon$$

for N large enough. □

In the case of relative 1/2-pinching, the proof is similar, replacing equation (3.13) by equation (3.33) in the last step.

We now wish to compare the stable holonomy with the parallel transport of the Levi-Civita connection on horospheres. Consider two points x, y on a horosphere H_ξ in $\tilde{M}$ centered at $\xi \in \partial\tilde{M}$. Assume that $d_{H_\xi}(x, y) < \rho$ is smaller than the injectivity radius of H_ξ. We recall that, by Proposition 2.1(3), the injectivity radius of every horosphere is bounded below by a constant $\rho > 0$. The stable holonomy $\Pi^{\xi}(x, y)$ and the parallel transport $P^{\xi}(x, y)$ along the unique geodesic segment joining x and y *a priori* do not coincide. We insist on the fact that the stable holonomy is a dynamical object, whereas the Levi-Civita connection is geometric. Assuming that they coincide locally on a horosphere has the following strong implication.

PROPOSITION 3.11. *Let M be a closed Riemannian manifold with sectional curvature satisfying either the strong 1/4-pinching or relative 1/2-pinching assumption. Let ξ be a point in $\partial\tilde{M}$ and $x_0 \in H_\xi$ be a point in a horosphere centered at ξ. Assume that for every $x, y \in B_{H_\xi}(x_0, \rho/2)$, the stable holonomy $\Pi^{\xi}(x, y)$ coincides with the parallel transport $P^{\xi}(x, y)$ of the Levi-Civita connection of H_ξ. Then, the induced metric on H_ξ restricted to $B_{H_\xi}(x_0, \rho/2)$ is flat.*

Proof. Since any pair of points in $B_{H_\xi}(x_0, \rho/2)$ are at distance less than ρ, there is a unique geodesic segment joining them, and by our coincidence assumption and assertion (2) of Definition 3.1, it follows that

$$P^{\xi}(x, y) = P^{\xi}(z, y) \circ P^{\xi}(x, z).$$

From the classical formula of the curvature in terms of the parallel transport, see for instance [**18**, Theorem 7.1], we deduce that the curvature of the induced metric of H_ξ restricted to $B_{H_\xi}(x_0, \rho/2)$ is identically zero. □

The goal of what follows is to show that if the stable holonomy and the parallel transport of the Levi-Civita connection locally coincide on a given horosphere H_ξ, then the same property holds on all horospheres. To accomplish this, we need to establish the continuity of the stable holonomy. Let $\tilde{v}$ be a unit vector tangent to $\tilde{M}$ and $\tilde{v}_k \in T^1\tilde{M}$ a sequence of unit tangent vectors such that $\lim_k \tilde{v}_k = \tilde{v}$. Let $\xi_{\tilde{v}} = c_{\tilde{v}}(+\infty)$ be the associated point on $\partial\tilde{M}$. Denote by $H_{\tilde{v}}$ the horosphere centered at $\xi_{\tilde{v}}$ passing through the base point of

$\tilde{v}$. Let $\tilde{Q}_k$ and $\tilde{Q}$ be the lifts of the plaques Q_k and Q of the strong stable foliation W^{ss} embedded in a chart $U \subset T^1M$ and containing $\tilde{v}_k$ and $\tilde{v}$, respectively. Recall that, from Proposition 2.1, the sequence of diffeomorphisms

$$\pi^{-1} \circ p \circ \Theta(v_k) : D^n \to \tilde{p}(\tilde{Q}_k) \tag{3.34}$$

converges in the C^r-topology to

$$\pi^{-1} \circ p \circ \Theta(v) : D^n \to \tilde{p}(\tilde{Q}). \tag{3.35}$$

PROPOSITION 3.12. *Let $\tilde{v}_k \in T^1\tilde{M}$ be a sequence of unit tangent vectors such that $\lim_k \tilde{v}_k = \tilde{v}$. Let $x = \pi^{-1} \circ p \circ \Theta(v)(q_x)$, $y = \pi^{-1} \circ p \circ \Theta(v)(q_y)$ be a pair of points in $\tilde{p}(\tilde{Q})$ and $x_k = \pi^{-1} \circ p \circ \Theta(v_k)(q_{x_k})$, $y_k = \pi^{-1} \circ p \circ \Theta(v_k)(q_{y_k})$ in $\tilde{p}(\tilde{Q}_k)$. Then,*

$$\lim_k \Pi^{\xi_{\tilde{v}_k}}(x_k, y_k) = \Pi^{\xi_v}(x, y).$$

Proof. Let us fix $\epsilon > 0$. By Claim 3.10, we can choose N such that for every $x, y \in \tilde{p}(\tilde{Q})$ and every $x_k, y_k \in \tilde{p}(\tilde{Q}_k)$, we have

$$||\Pi^{\xi_{\tilde{v}}}(x, y) - \Pi_N^{\xi_{\tilde{v}}}(x, y)|| \le \epsilon \tag{3.36}$$

and similarly,

$$||\Pi^{\xi_{\tilde{v}_k}}(x_k, y_k) - \Pi_N^{\xi_{\tilde{v}_k}}(x_k, y_k)|| \le \epsilon. \tag{3.37}$$

By the above convergence of equation (3.34) to equation (3.35), the points x_k and y_k converge to x and y, and the unit normals to $\tilde{p}(\tilde{Q}_k)$ at x_k and y_k converge to the unit normals to $\tilde{p}(\tilde{Q})$ at x and y, respectively. Therefore, the flows $(\varphi_t^{\xi_{\tilde{v}_k}})_{|\tilde{p}(\tilde{Q}_k)}$ converge to $(\varphi_t^{\xi_{\tilde{v}}})_{|\tilde{p}(\tilde{Q})}$ uniformly for $t \in [0, T]$ for every T. Now, the way $\Pi_N^{\xi_{\tilde{v}}}(x, y)$ depends on $\varphi_t^{\xi_{\tilde{v}}}$, $t \le N$, and the fact that $t_0 \le \log \rho$ implies that $\Pi_N^{\xi_{\tilde{v}_k}}(x_k, y_k)$ converges to $\Pi_N^{\xi_{\tilde{v}}}(x, y)$. Therefore, there exists $K > 0$ such that for all $k \ge K$,

$$\|\Pi_N^{\xi_{\tilde{v}}}(x, y) - \Pi_N^{\xi_{\tilde{v}_k}}(x_k, y_k)\| \le \epsilon.$$

We then deduce that for N and $k \ge K$,

$$\begin{aligned} &\|\Pi^{\xi_{\tilde{v}}}(x, y) - \Pi^{\xi_{\tilde{v}_k}}(x_k, y_k)\| \\ &\quad \le \|\Pi^{\xi_{\tilde{v}}}(x, y) - \Pi_N^{\xi_{\tilde{v}}}(x, y)\| + \|\Pi_N^{\xi_{\tilde{v}}}(x, y) - \Pi_N^{\xi_{\tilde{v}_k}}(x_k, y_k)\| \\ &\quad\quad + \|\Pi_N^{\xi_{\tilde{v}_k}}(x_k, y_k) - \Pi^{\xi_{\tilde{v}_k}}(x_k, y_k)\| \end{aligned}$$

and thus, $\|\Pi^{\xi_{\tilde{v}}}(x, y) - \Pi^{\xi_{\tilde{v}_k}}(x_k, y_k)\| \le 3\epsilon$, which concludes the proof. □

We can now state the main result of this section.

PROPOSITION 3.13. *Let M be a closed Riemannian manifold with sectional curvature satisfying either the strong $1/4$-pinching or the $1/2$-relative pinching assumption. Let $\tilde{v}$ be a unit tangent vector in $T^1\tilde{M}$ and $\xi_{\tilde{v}} = c_{\tilde{v}}(+\infty)$ the corresponding point in $\partial\tilde{M}$. Assume that the stable holonomy $\Pi^{\xi_{\tilde{v}}}(x, y)$ and the parallel transport for the Levi-Civita connection $P^{\xi_{\tilde{v}}}(x, y)$ coincide on every ball of radius $\rho/2$ of the horosphere $H_{\xi_{\tilde{v}}}$. Then,*

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for every horosphere $H_{\xi_{\tilde{w}}}$, $\tilde{w} \in T^1\tilde{M}$, *and every* $z \in H_{\xi_{\tilde{w}}}$, *there exists a neighborhood* $\mathcal{V}(z) \subset H_{\xi_{\tilde{w}}}$ *of* z *such that the stable holonomy* $\Pi^{\xi_{\tilde{v}}}(x, y)$ *and the parallel transport for the Levi-Civita connection* $P^{\xi_{\tilde{v}}}(x, y)$ *coincide for all points* $x, y \in \mathcal{V}(z)$.

Proof. Suppose that $H_{\tilde{v}}$ satisfies the assumption of the proposition and let us consider a different horosphere $H_{\tilde{w}}$. We will prove that locally around $\tilde{p}(\tilde{w})$ on $H_{\tilde{w}}$, the stable holonomy and the Levi-Civita parallel transport coincide. As mentioned in the proof of assertion (2) in Proposition 2.1, each leaf of the strong stable foliation $W^{ss} \subset T^1M$, in particular, $W^{ss}(v)$, is dense in T^1M, where $v = d\tilde{\pi}(\tilde{v})$. Moreover, thanks to equations (2.5) and (2.6) in Proposition 2.1, the lift $\tilde{p}(\tilde{Q}) \subset H_{\tilde{w}}$ is the C^r limit of the sequence of sets $\tilde{p}(\tilde{Q}_l)$, where $\tilde{Q}_l$ are lifts of Q_l. These lifts $\tilde{Q}_l$ are subsets of translates, by elements of the fundamental group of M, of $H_{\tilde{v}}$. By the $\pi_1(M)$-equivariance of the stable holonomy (coming from Proposition 3.5) and of the Levi-Civita connection, we get from our assumption that the stable holonomy and the parallel transport of the Levi-Civita connection coincide on $\tilde{p}(\tilde{Q}_l)$. The proof then follows from the continuity properties of Propositions 3.12 and 2.1(4). □

COROLLARY 3.14. *Let* M *be a closed Riemannian manifold with sectional curvature satisfying either the strong* $1/4$*-pinching or relative* $1/2$*-pinching assumption. If the stable holonomy and the parallel transport of the induced Levi-Civita connection coincide on every ball of radius* $\rho/2$ *of one horosphere* $H_{\xi_{\tilde{v}}}$, *then the induced metric on each horosphere of* $\tilde{M}$ *is isometric to a Euclidean metric. Moreover, for every* $\tilde{w} \in T^1\tilde{M}$, $x, y \in H_{\xi_{\tilde{w}}}$, *we have* $\Pi_s^{\xi_{\tilde{w}}}(x, y) = P_s^{\xi_{\tilde{w}}}(x, y)$, *in other words, the stable holonomy and the parallel transport associated to the Euclidean metric coincide on every horosphere. In particular, the parallel transport associated to the Euclidean metric is invariant by the geodesic flow.*

Proof. By Proposition 3.13, for every horosphere $H_{\xi_{\tilde{w}}}$ and $x \in H_{\xi_{\tilde{w}}}$, the stable holonomy and the parallel transport associated to the Levi-Civita connection coincide on a neighbohood $\mathcal{V}(x)$ of x. Thanks to Proposition 3.11 applied to $\mathcal{V}(x)$, we deduce that the induced metric on every horosphere has a flat Levi-Civita connection, and hence is an Euclidean metric. This proves the first part. Let us prove the second part of the corollary. Let us consider $x, y \in H_{\xi_{\tilde{w}}}$. Choose a continuous path $c : [0, 1] \to H_{\xi_{\tilde{w}}}$ such that $c(0) = x$ and $c(1) = y$. There exists $t_0 = 0 < t_1 < \cdots < t_{2k} = 1$ such that $\{\mathcal{V}(c(t_{2i}))\}_{i=0}^{k}$ is a finite covering of $c([0, 1])$ and $c(t_{2i+1}) \in \mathcal{V}(c(t_{2i})) \cap \mathcal{V}(c(t_{2(i+1)}))$. Since the Levi-Civita connection on the metric of $H_{\xi_{\tilde{w}}}$ is flat, we have

$$P_s^{\xi_{\tilde{w}}}(x, y) = P_s^{\xi_{\tilde{w}}}(c(t_0), c(t_1)) \circ P_s^{\xi_{\tilde{w}}}(c(t_1, c(t_2)) \circ \cdots \circ P_s^{\xi_{\tilde{w}}}(c(t_{2k-1}, c(t_{2k}))$$

and similarly, thanks to property (2) of Definition 3.1,

$$\Pi_s^{\xi_{\tilde{w}}}(x, y) = \Pi_s^{\xi_{\tilde{w}}}(c(t_0), c(t_1)) \circ \Pi_s^{\xi_{\tilde{w}}}(c(t_1, c(t_2)) \circ \cdots \circ \Pi_s^{\xi_{\tilde{w}}}(c(t_{2k-1}, c(t_{2k})).$$

We then conclude that $P_s^{\xi_{\tilde{w}}}(x, y) = \Pi_s^{\xi_{\tilde{w}}}(x, y)$ since $P_s^{\xi_{\tilde{w}}}(c(t_j), c(t_{j+1})) = \Pi_s^{\xi_{\tilde{w}}}(c(t_j), c(t_{j+1}))$. □

4. *A quasi-isometry between $\tilde{M}$ and a Heintze group*

In this section, the main theorem of this article, Theorem 1.2, will be proved. As we explained in §1, the proof amounts to proving Theorem 1.4. Henceforth, M is assumed to satisfy either the strong 1/4-pinching or relative 1/2-pinching assumption and to be of dimension greater than or equal to 3. Furthermore, by Corollary 3.14, we may assume that all the horospheres in $\tilde{M}$ are isometric to the Euclidean space and that the associated parallel transport is invariant by the geodesic flow. We will therefore be able to prove below the following. Given a geodesic $c_{\tilde{v}}(t)$ in $\tilde{M}$ which projects to a closed geodesic in M, there exists a quasi-isometry between the universal cover $\tilde{M}$ of M and the Heintze group G_A, where A is the derivative of the first return Poincaré map along the closed geodesic. Theorem 1.5, will then imply that the eigenvalues of A all have the same modulus, and hence conclude the proof of Theorem 1.4.

Let us choose a geodesic $c_{\tilde{v}}(t)$ in $\tilde{M}$ with end point $\xi = c_{\tilde{v}}(\infty) \in \partial\tilde{M}$, which projects to a closed geodesic in M. We consider the horosphere $H_\xi(0)$ centered at ξ and passing through the base point $x_0 = c_{\tilde{v}}(0)$. For each $p \in \tilde{M}$, the geodesic c joining p and ξ intersects $H_\xi(0)$ at a point $x = c(0)$. The pair, $(t, x) \in \mathbb{R} \times H_\xi(0)$, is the horospherical coordinates of p.

Keeping the same notation as in §3, we recall that $\{\varphi_t\}_{t\in\mathbb{R}}$ is a one-parameter group of diffeomorphisms of $\tilde{M}$ which sends $H_\xi(0)$ diffeomorphically onto $H_\xi(t)$ (see Definition 3.1) and the above horospherical coordinates realize the following diffeomorphism Φ : $\mathbb{R} \times H_\xi(0) \to \tilde{M}$ defined by

$$(t, x) \to \varphi_t(x) \quad \text{for } t \in \mathbb{R} \text{ and } x \in H_\xi(0). \tag{4.1}$$

Therefore, in horospherical coordinates, the pull back by Φ of the metric $\tilde{g}$ on $\tilde{M}$ at (t, x) writes as the orthogonal sum:

$$\Phi^*(\tilde{g}) = dt^2 + \varphi_t^* h_t(x), \tag{4.2}$$

where $\varphi_t^* h_t$ is a flat metric on $H_\xi(0)$. Note that φ_t acts by translation on geodesics, and hence, there is no effect on the dt^2 factor.

As before, since the horosphere $(H_\xi(0), h_0)$ is flat, we will identify it with the Euclidean space $(\mathbb{R}^n, h_{\mathrm{eucl}})$. The geodesic $c_{\tilde{v}}$ projects to a closed geodesic on M of period l. Let γ be the element of the fundamental group of M with axis $c_{\tilde{v}}$ such that $D\gamma(\tilde{g}_l(\tilde{v})) = \tilde{v}$. The map $\psi = \gamma \circ \varphi_l$ is a diffeomorphism of $\tilde{M}$ (see Definition 4.5). When restricted to $H_\xi(0)$, ψ can be considered as a diffeomorphism of $\mathbb{R}^n$ fixing x_0, and $d\psi(x_0)$ as a linear operator of $\mathbb{R}^n$ which we will denote by T, see Definitions 4.5 and 4.7, where $T = T^1$. Up to replacing T by T^2, we can assume that T is contained in a one-parameter group in $GL(n, \mathbb{R})$, that is, $T = e^{lA}$ for some matrix A (see [9]). Indeed, replacing T with $T^2 = D\psi(x_0)^2$, we simply work with twice the periodic orbit of period $2l$ and the argument is rigorously the same. We thus can assume from now on that $T = e^{lA}$. Let us consider the Heintze group G_A associated to the matrix A and recall from §2 that $G_A = \mathbb{R} \ltimes_A \mathbb{R}^n$ is the solvable group endowed with the multiplication law

$$(s, x) \cdot (t, y) = (s + t, x + e^{-sA} y) \quad \text{for all } s, t \in \mathbb{R}, x, y \in \mathbb{R}^n. \tag{4.3}$$

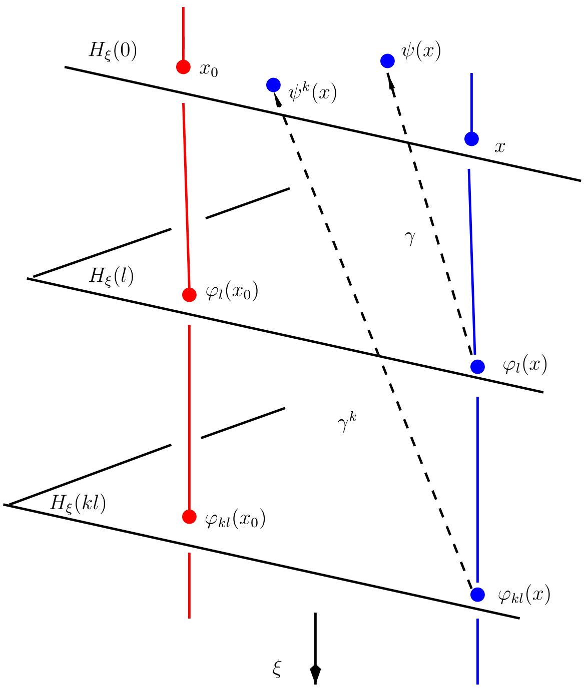

FIGURE 2. The action of ψ on horospheres.

The group G_A is diffeomorphic to $\mathbb{R} \times \mathbb{R}^n$, and the tangent space at each point (s, x) of G_A splits as $\mathbb{R} \times \mathbb{R}^n$. Let us consider the left invariant metric g_A on G_A which is defined to be the standard Euclidean metric at $(0, 0) \in G_A$, where $\mathbb{R} \times \{0\}$ is orthogonal to $\{0\} \times \mathbb{R}^n$. Since the inverse of the left multiplication is given by $L_{(s,x)^{-1}}(t, y) = (t - s, -e^{sA}x + e^{sA}y)$, an easy computation shows that the metric g_A is then defined, for a vector $Z = (a, X)$ which is tangent to G_A at an arbitrary point $(s, x) \in G_A$, by

$$g_A(s, x)(Z, Z) = a^2 + h_{\mathrm{eucl}}(e^{sA}X, e^{sA}X). \tag{4.4}$$

We start by identifying the flat horosphere $(H_\xi(0), h_0)$ with the Euclidean space $(\mathbb{R}^n, d_{\mathrm{eucl}})$. Let us recall that $c_{\tilde{v}}$ is a geodesic in $\tilde{M}$ with $\xi = c_{\tilde{v}}(\infty) \in \partial\tilde{M}$, which projects to a closed geodesic in M of period l. We do not require that this geodesic is primitive; in fact, we will later replace the corresponding element γ of the fundamental group by a large enough power of it.

We now consider the diffeomorphism of $H_\xi(0)$ defined by

$$\psi(x) = \gamma \circ \varphi_l(x) \quad \text{for } x \in H_\xi(0). \tag{4.5}$$

For all $k \geq 1$, let $\psi^k = \psi \circ \psi \cdots \circ \psi$ denote the kth power of ψ (see Figure 2). For $x \in H_\xi(0)$, we define

$$T_k(x) = d\psi(\psi^{k-1}(x)) \tag{4.6}$$

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and

$$T^k(x) = T_k(x) \cdot T_{k-1}(x) \cdots T_1(x). \tag{4.7}$$

Since γ and φ_t commute for all $t \in \mathbb{R}$, it follows that

$$\psi^k(x) = \gamma^k \circ \varphi_{kl}(x) \quad \text{and} \quad T^k(x) = D\psi^k(x) = D\gamma^k \circ D\varphi_{kl}(x). \tag{4.8}$$

As explained at the beginning of the section, we recall that $T^1(x_0) = e^A$ for A being a $(n \times n)$-matrix. In particular,

$$T^k(x_0) = D\psi^k(x_0) = D\gamma^k \circ D\varphi_{kl}(x_0) = e^{lkA}. \tag{4.9}$$

The main result of this section is the following.

THEOREM 4.1. *With the notation above, $(\tilde{M}, \tilde{g})$ is bi-Lipschitz diffeomorphic, and hence quasi-isometric, to (G_A, g_A).*

Proof. In fact, we will show that there is a bi-Lipschitz diffeomorphism between G_A and $\tilde{M}$. Recall that the map $\Phi : \mathbb{R} \times H_\xi(0) \to \tilde{M}$ defined by $\Phi(s, x) = \varphi_s(x)$ is a diffeomorphism.

By Corollary 3.14, the horosphere $H_\xi(0)$ endowed with the induced metric from $\tilde{M}$ is flat, and hence, $\mathbb{R} \times H_\xi(0) = \mathbb{R} \times \mathbb{R}^n$ and, therefore, we can see Φ as a diffeomorphism between G_A and $\tilde{M}$.

We first show that the two metrics $\Phi^*\tilde{g}$ and g_A coincide at points with coordinates (lk, y) where k is an integer.

LEMMA 4.2. *For every $k \in \mathbb{Z}$ and $y \in \mathbb{R}^n$, we have $\Phi^*\tilde{g}(lk, y) = g_A(lk, y)$.*

Proof. It is clear that for tangent vectors of the form $Z = (a, 0)$, we have $\tilde{g}(Z, Z) = g_A(Z, Z) = a^2$ at *any* point of coordinate (t, x). Therefore, we now focus on tangent vectors of the type $Z = (0, X)$, where $X \in \mathbb{R}^n$ is a vector tangent to $H_\xi(0) = \mathbb{R}^n$ at x. By equation (4.4), it suffices to show that

$$\Phi^*\tilde{g}(lk, x)(Z, Z) = h_{\text{eucl}}(e^{lkA}X, e^{lkA}X). \tag{4.10}$$

In fact, it follows from equation (4.2) that

$$\Phi^*\tilde{g}(lk, x)(Z, Z) = h_{lk}(d\varphi_{lk}(X), d\varphi_{lk}(X)), \tag{4.11}$$

where $d\varphi_{lk}(X)$ is a vector tangent to $H_\xi(lk)$ at $x_{lk} = \varphi_{lk}(x)$, and h_{lk} is the flat metric of $H_\xi(lk)$. Note that the tangent vector X can be extended to a constant vector field on $\mathbb{R}^n$, which we will still denote by X.

Recall (see §3) that for each integer k, P^ξ_{lk} is the parallel transport associated to the flat metric h_{lk} on $H_\xi(lk)$, and that $x_0 = c_v(0)$ is the unique point on $H_\xi(0)$ which lies on the axis of γ. Let us denote by q_{lk} the point $\varphi_{lk}(x_0)$. We thus have

$$h_{lk}(d\varphi_{lk}(X), d\varphi_{lk}(X)) = h_{lk}(P^\xi_{lk}(x_{lk}, q_{lk})(d\varphi_{lk}(X)), P^\xi_{lk}(x_{lk}, q_{lk})(d\varphi_{lk}(X))). \tag{4.12}$$

By assumption (ii) of Theorem 1.2, the parallel transport of the flat metric $h_0 = h_{eucl}$ (h_{lk}) coincides with the stable holonomy Π^ξ_0 (Π^ξ_{lk}). In particular, the commutation property (3)

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of Definition 3.1 holds:

$$d\varphi_{lk}(x_0) \circ P_0^{\xi}(x, x_0)(X) = P_{lk}^{\xi}(x_{lk}, q_{lk})(d\varphi_{lk}(X)). \tag{4.13}$$

Note that equation (4.13) relies on the fact that the family of parallel transports of the Levi-Civita connections coincide with the stable holonomies, and hence is invariant by the geodesic flow and that it is the only place in the proof where we use it. We now deduce from equation (4.13) that

$$h_{lk}(d\varphi_{lk}(X), d\varphi_{lk}(X)) = h_{lk}(d\varphi_{lk}(x_0) \circ P_0^{\xi}(x, x_0)(X), d\varphi_{lk}(x_0) \circ P_0^{\xi}(x, x_0)(X)). \tag{4.14}$$

Since for every k, γ^k is an isometry, we obtain

$$\begin{aligned} & h_{lk}(d\varphi_{lk}(X), d\varphi_{lk}(X)) \\ & \quad = h_0(d\gamma^k \circ d\varphi_{lk}(x_0)(P_0^{\xi}(x, x_0)(X)), d\gamma^k \circ d\varphi_{lk}(x_0)(P_0^{\xi}(x, x_0)(X))), \end{aligned} \tag{4.15}$$

and thus, by equation (4.9),

$$h_{lk}(d\varphi_{lk}(X), d\varphi_{lk}(X)) = h_0(e^{lkA}(P_0^{\xi}(x, x_0)(X)), e^{lkA}(P_0^{\xi}(x, x_0)(X))). \tag{4.16}$$

Since $H_{\xi}(0)$ with the induced metric from $\tilde{M}$ is identified with $\mathbb{R}^n$, h_0 with the standard Euclidean metric h_{eucl}, and X is a constant vector field, we have $P_0^{\xi}(x, x_0)(X) = X$ and

$$h_0(e^{lkA}(P_0^{\xi}(x, x_0)(X)), e^{lkA}(P_0^{\xi}(x, x_0)(X))) = h_{\mathrm{eucl}}(e^{lkA}X, e^{lkA}X), \tag{4.17}$$

which implies by equations (4.11) and (4.16) that

$$\Phi^*\tilde{g}(lk, x)(Z, Z) = h_{\mathrm{eucl}}(e^{lkA}X, e^{lkA}X) = g_A(lk, x)(Z, Z), \tag{4.18}$$

which completes the proof of Lemma 4.2. □

For $t \in \mathbb{R}$, let k be the integer part of t/l. We now compare $g_A(t, x)$ and $g_A(lk, x)$ at any $x \in \mathbb{R}^n$. Let us set $\sigma = t/l - k$ with $\sigma \in [0, 1[$. For $Z = (0, X)$, we have

$$g_A(t, x)(Z, Z) = h_{\mathrm{eucl}}(e^{tA}X, e^{tA}X) = h_{\mathrm{eucl}}(e^{l\sigma A}e^{lkA}X, e^{l\sigma A}e^{lkA}X). \tag{4.19}$$

Recall that $e^{lA} = D(\gamma \circ \varphi_l)(x_0) = D\psi(x_0)$ is a fixed $n \times n$ matrix, so that there exists a constant C such that $\|e^{\pm l\sigma A}\|^2 \leq C$ for every $\sigma \in [0, 1[$. Therefore, we deduce from equation (4.19) that

$$C^{-1}g_A(lk, x) \leq g_A(t, x) \leq C g_A(lk, x) \tag{4.20}$$

for every $lk \leq t < (k+1)l$. However, we have

$$h_t(D\varphi_t X, D\varphi_t X) = h_t(D\varphi_{l\sigma} \circ D\varphi_{lk}X, d\varphi_{l\sigma} \circ D\varphi_{lk}X)$$

and the same argument as before yields

$$C^{-1}\Phi^*\tilde{g}(lk, x) \leq \Phi^*\tilde{g}(t, x) \leq C\Phi^*\tilde{g}(lk, x). \tag{4.21}$$

Then equations (4.20) and (4.21) and Lemma 4.2 conclude the proof of Theorem 4.1. □

COROLLARY 4.3. *All the eigenvalues of $T = D\psi(x_0)$ have the same modulus.*

Proof. By Theorem 4.1, (G_A, g_A) is quasi-isometric to $(\tilde{M}, \tilde{g})$. Since M is closed, $(\tilde{M}, \tilde{g})$ is quasi-isometric to the finitely generated group $\pi_1(M)$ endowed with the word metric, which is therefore a hyperbolic group. We thus deduce that G_A is quasi-isometric to a hyperbolic group and by Theorem 1.5, this can occur only if the real part of the complex eigenvalues of A are equal. Recall that A has been chosen so that either $T = e^{lA}$ or $T^2 = e^{lA}$, where $T = D\psi(x_0)$. We deduce that the eigenvalues of T have the same modulus. □

We are now in position to prove Theorem 1.4, and thus, complete the proof of Theorem 1.2.

Proof of Theorem 1.4. Theorem 4.1 holds for any choice of a closed geodesic, or equivalently of an element γ of the fundamental group of M, and so does Corollary 4.3. This implies that for any such choice, the moduli of the complex eigenvalues of $T = D\psi(x_0)$ coincide.

Recall that

$$D\psi(x_0) = e^{l(v)A} = D\tilde{p} \circ (D(\gamma \circ \tilde{g}_{l(v)})(\tilde{v})|E^{ss}(\tilde{v})) \circ D\tilde{p}^{-1}, \tag{4.22}$$

so that $Dg_{l(v)}|E^{ss}$ and $D\psi(x_0)$ are conjugate matrices; therefore, we conclude that the eigenvalues of $Dg_{l(v)}|E^{ss}$ have the same modulus. □

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A. *Appendix. Pinching, bunching and stable holonomies*

The goal of this appendix is twofold. We first will show that the strong 1/4-pinching assumption implies the bunching of the stable cocycle of the geodesic flow defined in [17]. Then we will show that the stable holonomy on the horospheres is conjugate to the stable holonomy defined on the strong stable leaves of the geodesic flow.

A.1. *Strong* 1/4*-pinching and bunching.*

Under the assumption $-4(1 - \tau) \leq K \leq -1$, the strong stable bundle $P : E^{ss} \to T^1M$ is C^1 (see [16, p. 226]). We choose a C^1-metric on T^1M such that the splitting $TT^1M = E^{ss} \oplus \mathbb{R}\, Z \oplus E^{su}$ is orthogonal, the generator Z of the geodesic flow satisfies $|Z| = 1$, and the metric on E^{ss} and E^{su} are obtained by pulling back the metric of M by the projection $p : T^1M \to M$. We consider now the diffeomorphism $f := g_1 : T^1M \to T^1M$. The linear stable cocycle over f defined as

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$F := (Dg_1)_{|E^{ss}}$ is also C^1 and satisfies

$$\|F(v)\| \leq e^{-1} \quad \text{and} \quad \|(F(v))^{-1}\| \leq e^{2(1-\tau)^{1/2}}. \tag{A.1}$$

With the notation of [17, §2], denoting $\nu(v) := e^{-1}$, $\hat{\nu}(v) := e^{-1}$, we then have $\|Df(v)\| \leq \nu(v)$, $\|(Df(v))^{-1}\| \geq (\hat{\nu}(v))^{-1}$, and $|Df(Z)| = 1$, and hence

$$\|F(v)\|\|(F(v))^{-1}\|\nu(v) \leq e^{-1}\, e^{2(1-\tau)^{1/2}} e^{-1} < 1. \tag{A.2}$$

This inequality $\|F(v)\|\|(F(v))^{-1}\|\nu(v) < 1$ coincides with the bunching condition of [17, equation (3.1)] since we can take the Hölder coefficient $\beta = 1$ since the strong stable bundle is C^1.

A.2. *Conjugation of stable holonomies.* Recall that the map $\Phi : T^1\tilde{M} \to \tilde{M} \times \partial\tilde{M}$ is defined by $\Phi(\tilde{v}) = (x, \xi)$, where $x = \pi(\tilde{v})$ and $\xi = c_{\tilde{v}}(+\infty)$ is a homeomorphism. By abuse of notation, we will write $\tilde{v} = (x, \xi)$. Given $\tilde{v} = (x, \xi)$, the projection $\tilde{p} : T^1\tilde{M} \to \tilde{M}$ induces a diffeomorphism between the strong stable leaf $W^{ss}(\tilde{v})$ of $\tilde{v}$ and the horosphere $H_\xi(x)$ centered at ξ and passing through x. In particular, $D\tilde{p}(\tilde{v})$ induces an isomorphism between $E^{ss}(\tilde{v}) = T_{\tilde{v}}W^{ss}(\tilde{v})$ and $T_x H_\xi(x)$.

LEMMA A.1. *Let $\tilde{v} = (x, \xi)$ and $\tilde{w} = (y, \xi)$ be on a same strong stable leaf $W^{ss}(\tilde{v}) \subset T^1\tilde{M}$, and $H_\xi(x)$ the horosphere centered at ξ and passing through x and y. Then the stable holonomy $\mathcal{H}(\tilde{v}, \tilde{w})$ on $W^{ss}(\tilde{v})$ (respectively $\Pi^\xi(x, y)$ on $H_\xi(x)$) is conjugate,*

$$\mathcal{H}(\tilde{v}, \tilde{w}) = (D\tilde{p}(\tilde{w}))^{-1} \circ \Pi^\xi(x, y) \circ D\tilde{p}(\tilde{v}).$$

Proof. Define $\mathcal{H}(\tilde{v}, \tilde{w}) = (D\tilde{p}(\tilde{w}))^{-1} \circ \Pi^\xi(x, y) \circ D\tilde{p}(\tilde{v})$. The properties (1), (2), and (3) of Definition 3.1 for $\mathcal{H}(\tilde{v}, \tilde{w})$ are consequences of the corresponding properties of $\Pi^\xi(x, y)$ stated in Proposition 3.5. As stated in [17, Proposition 4.2(c)], the stable holonomy $\mathcal{H}$ is uniquely determined by the property that

$$\|\mathcal{H}(\tilde{v}, \tilde{w}) - I(\tilde{v}, \tilde{w})\| \leq C d(\tilde{v}, \tilde{w}), \tag{A.3}$$

where $I(\tilde{v}, \tilde{w})$ is a local identification between $E^{ss}(\tilde{v})$ and $E^{ss}(\tilde{w})$. However, as noticed at the bottom of [17, p. 173], holonomies do not depend on the choice of the local identifications. By defining

$$I(\tilde{v}, \tilde{w}) := (D\tilde{p}(\tilde{w}))^{-1} \circ P(x, y) \circ D\tilde{p}(\tilde{v}),$$

where $P(x, y)$ is the parallel transport along $H_\xi(x, y)$, we see that property (ii) of Proposition 3.5 implies equation (A.3). Therefore, properties (a), (b), and (c) of [17, Proposition 4.2] are satisfied, which concludes the proof of this lemma. □

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