AMENABLE TRANSFORMATION SEMIGROUPS

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1. Introduction

For any set X denote by $m(X)$ the Banach space of all bounded real-valued functions on X, equipped with the supremum norm, and denote by $\mathfrak{S}(X)$ the semigroup (under functional composition) of all transformations of X, i.e. mappings with domain X and range contained in X. A pair (X, S), where S is a subsemigroup of $\mathfrak{S}(X)$, will be called a transformation semigroup. Important examples are obtained by letting X be the underlying set in an abstract semigroup and considering the pairs (X, S_1) and (X, S_2), where S_1 [S_2] denotes the set of left [right] multiplication mappings of X. We shall call transformation semigroups in these classes of examples l-[r-] semigroups.

A mean on $m(X)$ is a positive normalized continuous linear functional on $m(X)$, i.e. an element μ in $m(X)^*$ such that $\mu(f) \geqq 0$ if $f(x) \geqq 0$ for all x in X, and such that $\mu(1) = 1$, where 1 is the function $1(x) = 1$ for all x in X. If (X, S) is a transformation semigroup (briefly a τ-semigroup), each s in S induces a continuous linear transformation T_s in $m(X)$ defined by: $(T_s f)(x) = f(sx)$. A mean μ on $m(X)$ will be called S-invariant if $\mu(T_s f) = \mu(f)$ for all s in S and all f in $m(X)$. A τ-semigroup (X, S) will be called S-amenable (or we say that $m(X)$ has an S-invariant mean) in case there exists an S-invariant mean on $m(X)$.

The l- and r-semigroups have been studied for amenability extensively in recent years; for example see [1] or [7] for an introduction to the subject, and [3] for a survey with a rather complete bibliography. In these cases means are called left or right invariant, and semigroups having a left [right] invariant mean are called left [right] amenable. The l- and r-semigroups are very special, and certain results in the theory of amenable semigroups hold because of their special properties. For example, some results are derived from the fact that if S is an abstract semigroup, then there is associated with $X = S$ as the underlying set two τ-semigroups (left and right multiplication), and some of the theory of amenable semigroups is based on the interplay between these τ-semigroups. In this paper we study general τ-semigroups for amenability. We offer a survey for

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convenient reference of analogues of some of the important results known for the l- and r-semigroups, and we point out certain contrasts in the theories. Most of our results are not conceptually new; our main goals are to gain generality and to obtain greater insight in the special l- and r-semigroup cases.

2. Amenability of τ-semigroups

We begin by studying the connection of amenability of general τ-semigroups with that of the l- and r-semigroups. The following lemma is used in the proofs of several of the theorems.

LEMMA 1. *If X and Y are sets and A is a (continuous) linear, monotonic and normalized mapping of $m(Y)$ into $m(X)$, then the adjoint A^* preserves means.*

PROOF. If $f \in m(Y)$ and $f(y) \geqq 0$ for all y in Y, then $Af(x) \geqq 0$ for all x in X; hence if μ is a mean on $m(X)$, then $A^*\mu(f) = \mu(Af) \geqq 0$. Since $A1_Y = 1_X$, $A^*\mu(1_Y) = 1$.

THEOREM 1. *Let (X, S) be a τ-semigroup. If S, considered as an abstract semigroup, has a left invariant mean, then $m(X)$ has an S-invariant mean.*

PROOF. Denote the left translation operator in $m(S)$ corresponding to s in S by l_s. Fix x in X and define a mapping A $(= A_x)$ on $m(X)$ into $m(S)$ by:

$$Af(s) = f(sx) \quad \text{for } s \text{ in } S.$$

Then A is continuous, linear, monotonic and preserves the constant functions; by lemma 1 the adjoint A^* preserves means. Let μ be a left invariant mean on $m(S)$ and put $\nu = A^*\mu$; it remains to show that ν is S-invariant. Let $s \in S$ and $f \in m(X)$; then $A(T_s f) = l_s(Af)$, for if $t \in S$, then

$$[l_s(Af)](t) = Af(st) = f((st)x) = f(s(tx)) = T_s f(tx) = [A(T_s f)](t).$$

Hence

$$A^*\mu(T_s f) = \mu(A(T_s f)) = \mu(l_s(Af)) = \mu(Af) = A^*\mu(f).$$

The converse of theorem 1 fails, and the conclusion of theorem 1 fails if 'left' is replaced by 'right' in the hypothesis, as the following simple examples show. Take $X = \{a, b, c\}$, and define five transformations, e, s, t, u, v, according to the following table:

	e	s	t	u	v
a	a	a	a	a	b
b	b	b	c	a	b
c	c	b	c	a	b

the notation meaning, for example, that $t(b) = c$. Let $S = \{e, s, t\}$, $S' = \{e, u, v\}$. Then (X, S) and (X, S') are τ-semigroups, $m(X)$ has an S-invariant mean, $m(X)$ has no S'-invariant mean, but as abstract semigroups S and S' are isomorphic and have right invariant means but no left invariant means (these assertions will be established in section 3).

The examples given above also show that amenability of τ-semigroups is not a semigroup property in the sense of being invariant under isomorphisms of the abstract semigroups involved. Thus a stronger notation of isomorphism is needed.

DEFINITION 1. Let (X, S) and (Y, T) be τ-semigroups. A *homomorphism* of (X, S) into (Y, T) is a pair of functions (ϕ, η), $\phi : X \to Y$, $\eta : S \to T$, such that $\phi(sx) = \eta(s)\phi(x)$ for all s in S, x in X.

Call (Y, T) a *homomorphic image* of (X, S) if there exists a homomorphism (ϕ, η) such that ϕ and η are both onto.

REMARK 1. If (Y, T) is a homomorphic image of (X, S), then η is a semigroup homomorphism. For let $s_1 \in S$, $s_2 \in S$, $y \in Y$ and choose x in X such that $\phi(x) = y$; then

$$(\eta(s_1)\eta(s_2))(y) = \eta(s_1)(\eta(s_2)(y)) = \eta(s_1)(\phi(s_2 x)) = \phi(s_1 s_2 x) = \\ = \eta(s_1 s_2)\phi(x) = \eta(s_1 s_2)(y).$$

REMARK 2. The notion of homomorphism defined here has all the desirable features of homomorphisms in general. Namely, if (Y, T) is a homomorphic image of (X, S) under (ϕ, η), then X and S are both partitioned into mutually disjoint equivalence classes under the relations $x_1 \sim x_2$ iff $\phi(x_1) = \phi(x_2)$ and $s_1 \approx s_2$ iff $\eta(s_1) = \eta(s_2)$. The equivalence classes in S can be regarded as acting on those in X; i.e., if A is in a class in S and E a class in X, choose s in A, x in E and let $A(E) = F$, where F is the class of sx. The action of A at E is well defined since $x_1 \sim x_2$ and $s_1 \approx s_2$ imply $s_1 x_1 \sim s_2 x_2$:

$$\phi(s_1 x_1) = \eta(s_1)\phi(x_1) = \eta(s_2)\phi(x_2) = \phi(s_2 x_2).$$

Denote the quotients $X/\sim$ by X' and $S/\approx$ by S'. Then S' is a set of transformations on X', and it is easy to see that (X', S') is in fact a τ-semigroup. Further, the fundamental theorem on homomorphisms remains valid in this context. That is (X', S') is isomorphic to (Y, T) under the natural mappings $\psi(E) = \phi(x)$, where $x \in E$, and $\xi(A) = \eta(s)$, where $s \in A$. We indicate only one part of the proof: given A and E, choose s in A, x in E; then AE is the class containing sx, so that $\psi(AE) = \phi(sx) = \eta(s)\phi(x) = \xi(A)\psi(E)$, and this is the basic relationship in definition 1.

THEOREM 2. *Let (X, S) and (X', S') be τ-semigroups and suppose that (X', S') is a homomorphic image of (X, S). Then $m(X')$ has an S'-invariant mean if $m(X)$ has an S-invariant mean.*

PROOF. Define a mapping $A : m(X') \to m(X)$ by: $Af = f \circ \phi$. Then A is continuous, linear and monotonic, and $A(1_{X'}) = 1_X$; hence A^* preserves means. Let μ be an S-invariant mean, and denote the translation operator in $m(X')$ also by T. Then for s' in S', f in $m(X')$ and x in X we have

$$\begin{aligned}(A(T_{s'}f))(x) = (T_{s'}f)\phi(x) &= f(s'(\phi(x))) = f(\eta(s)\phi(x)) \\ &= f(\phi(sx)) = Af(sx) = (T_s(Af))(x),\end{aligned}$$

where $s \in S$ and $\eta(s) = s'$. Thus $A(T_{s'}f) = T_s(Af)$, so that

$$A^*\mu(T_{s'}f) = \mu(A(T_{s'}f)) = \mu(T_s(Af)) = \mu(Af) = A^*\mu(f).$$

The next theorem contains another sufficient condition for the existence of S-invariant means. It was proved in [1] for the l- and r-semigroups, but the proof given there cannot be carried over to the general case without significant modification.

THEOREM 3. *Let (X, S) be a τ-semigroup. If $Y \subseteq X$ such that $s[Y] \subseteq Y$ for all s in S, let R be the set of transformations of Y obtained by restricting the mappings s in S to Y, with two mappings identified if they agree on Y. If $m(X)$ has an S-invariant mean μ such that $\mu(\chi_Y) > 0$, then Y has an R-invariant mean (χ denotes characteristic function).*

PROOF. Define $A : m(Y) \to m(X)$ by:

$$Af(x) = \begin{cases} f(x) & \text{if } x \in Y \\ 0 & \text{if } x \notin Y. \end{cases}$$

Then A is continuous, linear and monotonic, and $A(1_Y) = \chi_Y$. Hence if we put $\nu = (1/\mu(\chi_Y))A^*\mu$, then ν is a mean on $m(Y)$, and it remains to show that ν is R-invariant. From this point on our proof must be different from that given in [1]. If $t \in R$, choose s in S such that $s|_Y = t$. Then for f in $m(Y)$ and x in X we have

$$(A(T_t f))(x) = \begin{cases} f(tx) = f(sx) & \text{if } x \in Y \\ 0 & \text{if } x \notin Y, \end{cases}$$

and

$$(T_s(Af))(x) = \begin{cases} f(sx) & \text{if } sx \in Y \\ 0 & \text{if } sx \notin Y. \end{cases}$$

Hence $A(T_t f)$ and $T_s(Af)$ agree except possibly on the set $E_1 = \tilde{Y} \cap s^{-1}[Y]$. Now for n an integer, $n \geqq 2$, put $E_n = s^{-1}[E_{n-1}]$. By induction, if $n \geqq 2$, then $x \in E_n$ iff $s^{n-1}x \in \tilde{Y}$ and $s^n x \in Y$. Hence the sets E_n are pairwise disjoint, and

$$\mu(\chi_{E_n}) = \mu(\chi_{s^{-1}[E_{n-1}]}) = \mu(T_s \chi_{E_{n-1}}) = \mu(\chi_{E_{n-1}}).$$

It follows that $\mu(\chi_{E_1}) = 0$. By the Riesz representation theorem, there exists a unique regular Borel measure $\bar{\mu}$ of total mass 1 on the Stone-Čech compactifica-

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tion of the discrete space X such that for each g in $m(X)$ we have $\mu(g) = \int_{\beta X} \hat{g}\, d\bar{\mu}$, where $\hat{g}$ is the unique continuous extension of g to βX. Then

$$\begin{aligned}\mu(A(T_t f)) - \mu(T_s(Af)) &= \mu(A(T_t f) - T_s(Af)) \\ &= \int_{\beta X} (A(T_t f) - T_s(Af))^{\wedge}\, d\bar{\mu} \\ &= \int_{\bar{E}_1} (A(T_t f) - T_s(Af))^{\wedge}\, d\bar{\mu},\end{aligned}$$

where $\bar{E}_1$ denotes the (w^*-) closure of E_1 in βX, since the integrand vanishes on $\beta X \sim \bar{E}_1$. Since $\bar{\mu}(\bar{E}_1) = \mu(\chi_{E_1}) = 0$, it follows that $\mu(A(T_t f)) = \mu(T_s(Af)) = \mu(Af)$, and hence ν is R-invariant.

Before turning to characterizations of amenable τ-semigroups we make two further observations. First, given τ-semigroups (X, S) and (Y, T), the pair $(X \times Y, S \times T)$ becomes a τ-semigroup with the action of (s, t) at (x, y) defined to be (sx, ty). If $m(X)$ has an S-invariant mean and $m(Y)$ has a T-invariant mean, then $m(X \times Y)$ has an $S \times T$-invariant mean, defined just as the product of two measures.

The second observation arises from an attempt to extend the notion of ideals to τ-semigroups. The natural generalization of the concept of a left ideal to (X, S) is an invariant set, i.e. a set $Y \subseteq X$ for which $s[Y] \subseteq Y$ for all s in S. Here is another point of contrast between the general and the l- and r-semigroups. For it is true that if Y is an invariant set, if T consists of the restrictions of the mappings in S to Y and if $m(X)$ has a T-invariant mean, then $m(X)$ has an S-invariant mean; a proof can be constructed along the lines of the proofs given for theorems 1–3. It was proved in [11] that the converse is valid for l-semigroups. However, the converse fails in general, as the example (X, S) given after theorem 1 shows. In that example $m(X)$ has an S-invariant mean, the set $Y = \{b, c\}$ is invariant, but $m(Y)$ does not have a T-invariant mean for T, the set of restrictions of elements of S to Y.

3. Characterizations of amenable τ-semigroups

For x in X denote by qx the evaluation functional defined on $m(X)$ by: $qx(f) = f(x)$. Then qx is a mean on $m(X)$ for all x in X. In fact, the set of all means on $m(X)$ is w^*-compact and convex, and each qx is an extreme point of this set; it is a consequence of the Kreĭn-Mil'man theorem that the set of all means coincides with $\overline{co}\, q[s]$. Further the (w^*-) closure of $q[s]$ coincides with βX, the Stone-Čech compactification of the discrete space X.

We can now establish the assertions concerning the examples given after theorem 1. A finite abstract semigroup has a left invariant mean if and only if each pair of right ideals has a nonempty intersection (Rosen [10]); in S,

$sS \cap tS = \emptyset$. The points s and t are left zeros of S and hence qs and qt are right invariant means. For (X, S) qa is an S-invariant mean since a is fixed under each element of S. Suppose $m(X)$ has an S'-invariant mean μ. Then, μ must be of the form

$$\mu = \alpha_1 qa + \alpha_2 qb + \alpha_3 qc, \qquad \alpha_1 + \alpha_2 + \alpha_3 = 1.$$

Choose $f \in m(X)$ such that $f(a) = 0$ and $f(b) = 1$. Then, $\mu(T_u f) \neq \mu(T_v f)$ since $T_u f = 0$ and $T_v f = 1$; this is a contradiction.

In the case of the l- and r-semigroups, the set of all means becomes a semigroup under the Arens multiplication, and this semigroup has a number of interesting properties (see [11] for details). For example, if X $(= S)$ has a left invariant mean, then the smallest closed two-sided ideal in the semigroup of means consists of all the left invariant means. In the general case the Arens multiplication is not available, but it is possible to define a mapping, which we denote by juxtaposition, of $m(S)^* \times m(X)^*$ into $m(X)^*$ by: $\mu\nu(f) = \xi(\phi_\nu f)$, where $\phi_\nu f$ is the element in $m(S)$ whose action at s in S is: $\phi_\nu f(s) = \nu(T_s f)$. The basic properties of the mapping defined here are given in the following lemma which we state without proof; they are easy to check.

LEMMA 2. *The operation defined above has the properties*:

(i) $||\mu\nu|| \leqq ||\mu|| \cdot ||\nu||$;

(ii) *for a fixed* μ *in* $m(S)^*$ [ν *in* $m(X)^*$] *the mapping* $\nu \to \mu\nu$ [$\mu \to \mu\nu$] *is a continuous linear mapping of* $m(X)^* \to m(X)^*$ [$m(S)^* \to m(X)^*$];

(iii) *if* $w^*\text{-lim}_n \mu_n = \mu$ *in* $m(X)^*$, *then* $w^*\text{-lim}_n \mu_n \nu = \mu\nu$ *in* $m(X)^*$ *for each* ν *in* $m(X)^*$;

(iv) *if* $w^*\text{-lim}_n \nu_n = \nu$ *in* $m(X)^*$ *and* θ *is a finite mean* (*i.e. the carrier of* θ *is finite*), *then* $w^*\text{-lim}_n \theta\nu_n = \theta\nu$ *in* $m(X)^*$;

(v) $qsqx = qsx$ *for each* s *in* S, x *in* X.

Call a net $\{\theta_n\}$ of finite means (w^*-) *convergent* [*strongly convergent*] *to S-invariance* if $w^*\text{-lim}_n(qs\theta_n - \theta_n) = 0$ [$\text{lim}_n||qs\theta_n - \theta_n|| = 0$] for each s in S. If μ is an S-invariant mean and $\{\theta_n\}$ is a net of finite means such that $w^*\text{-lim}\, \theta_n = \mu$, then $\{\theta_n\}$ converges to S-invariance. Conversely, if $\{\theta_n\}$ converges to S-invariance, then any w^*-cluster point is an S-invariant mean. Hence a τ-semigroup has an S-invariant mean if and only if there is a net of finite means converging to S-invariance.

The following theorem was first proved by Day in [1]. Namioka [9] gave an elegant proof of Day's theorem, and Namioka's proof can be carried over to τ-semigroups with only the slight modification of replacing $(l_1(S))^S$ by $(l_1(X))^S$ (see [9]).

THEOREM 4 (Day). *Let* (X, S) *be a* τ-*semigroup. Then* X *has an* S-*invariant mean if and only if there exists a net* $\{\theta_n\}$ *of finite means on* X *such that*

$$\lim ||qs\theta_n - \theta_n|| = 0.$$

In [9], Namioka also gave an elegant proof of the Følner-Frey theorem on amenable semigroups. Only a slight change is required to adapt the computations to τ-semigroups. Specifically, in place of the convolution in $l_1(S)$ define a mapping in $l_1(X)$ as follows: for each s in S and each $\theta = \sum_{i=1}^n \lambda_i \delta x_i$ in $l_1(X)$ ($\lambda_i \geqq 0$, $\sum_{i=1}^n \lambda_i = 1$ and δ is the Kronecker embedding of X into $l_1(X)$) put $s \cdot \theta = \sum_{i=1}^n \lambda_i \delta s x_i$. Under this analogue of convolution in $l_1(X)$ all of Namioka's computations are valid, and the following characterization of amenable τ-semigroups is obtained.

THEOREM 6 (Følner-Frey). *Let (X, S) be a τ-semigroup. Then $m(X)$ has an S-invariant mean if and only if given any finite set F in S and $\varepsilon > 0$, there exists a finite set A in X such that $|s[A] \sim A| < \varepsilon |A|$ for each s in F.*

In [2], Day established a characterization of abstract semigroups with left (or right) invariant means in terms of the Markov-Kakutani fixed point property. This theorem also has an analogue in τ-semigroups under the stronger notion of homomorphism.

THEOREM 7 (Markov-Kakutani-Day). *Let (X, S) be a τ-semigroup. A necessary and sufficient condition that $m(X)$ have an S-invariant mean is that whenever K is a compact convex set in a locally convex linear topological space E and S' is a semigroup (under composition) of continuous affine mappings of K such that (K, S') is a homomorphic image of (X, S), there is a point y in K such that $ty = y$ for all t in S'.*

PROOF. *Sufficiency.* The pair (ϕ, η), where $\phi = q$, the evaluation mapping, and $\eta(s) = T_s^*$, is a homomorphism (each T_s^* is restricted to the set of means on $m(X)$, which is equipped with the w^*-topology); the common fixed point of the mappings T_s^* is an S-invariant mean on $m(X)$.

Necessity. Denote the canonical embedding of E into E^{**} by Q; then Q is an affine homeomorphism of K into $Q[K]$. Let (ϕ, η) be the homeomorphism of (X, S) onto (K, S'), and define $A : E^* \to m(X)$ by: $Af = f \circ \phi$. Then $A^*qx = Q\phi(x)$, so that $Q^{-1}A^*q = \phi$. Moreover $Q^{-1}A^*$ is a continuous affine mapping of the set of means on $m(X)$, and if μ is an S-invariant mean on $m(X)$, then $Q^{-1}A^*\mu$ is a common fixed point for S'. For let $\{\theta_n\}$ be a net of finite means converging (w^*) to μ. Each θ_n is of the form

$$\theta_n = \sum_{i=1}^{N(n)} \lambda_i^n q x_i^n,$$

with each $\lambda_i^n \geqq 0$ and $\sum_{i=1}^{N(n)} \lambda_i^n = 1$ for each n. Then

$$Q^{-1}A^*\theta_n = \sum_{i=1}^{N(n)} \lambda_i^n \phi(x_i^n),$$

and $\theta^{-1}A^*\theta_n \to Q^{-1}A^*\mu$. Moreover, if $t \in S'$, then $t(Q^{-1}A^*\theta_n) \to t(Q^{-1}A^*\mu)$, and

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$$t(Q^{-1}A^*\theta_n) = \sum_{i=1}^{N(n)} \lambda_i^n t\phi(x_i^n) = \sum_{i=1}^{N(n)} \lambda_i^n \phi(sx_i^n),$$

where $s \in S$ and $\eta(s) = t$. Put

$$\psi_n = \sum_{i=1}^{N(n)} \lambda_i^n qsx_i^n;$$

then $t(Q^{-1}A^*\theta_n) = Q^{-1}A^*\psi_n$, and $\psi_n \to \mu$ since μ is S-invariant. Hence $t(Q^{-1}A^*\theta_n) \to Q^{-1}A^*\mu$, and therefore $t(Q^{-1}A^*\mu) = Q^{-1}A^*\mu$.

REMARK 3. Theorem 6 includes the form of the Markov-Kakutani theorem given by Day in [2] as a special case. In this case S is an abstract semigroup, and we take $X = S$ as the underlying set and define the action of s in S at x in X by $s(x) = sx$. Let h be a homomorphism of S onto S'. Choose y from K and define a homomorphism (ϕ, η) of (X, S) into (K, S') by: $\phi(x) = (h(x))(y)$ and $\eta(s) = h(s)$. The pair (ϕ, η) is in fact a homomorphism since

$$\begin{aligned}\phi(sx) = (h(sx))(y) &= (h(s)h(x))(y) = h(s)((h(x))(y)) \\ &= h(s)\phi(x) = \eta(x)\phi(s).\end{aligned}$$

The concept of extremely amenable semigroups was introduced by Mitchell in [8] and studied extensively by Granirer in [4], [5], [6]. Using an argument analogous to that given in theorem 6, we obtain the corresponding result for extremely amenable τ-semigroups, i.e. τ-semigroups (X, S) where $m(X)$ has an S-invariant mean which lies in βX.

THEOREM 8. *Let (X, S) be a τ-semigroup. A necessary and sufficient condition that $m(X)$ have an S-invariant mean μ in βX is that whenever Y is a compact Hausdorff space and T is a semigroup of continuous mappings on Y such that (Y, T) is a homomorphic image of (X, S), there is a point y in Y such that $ty = y$ for all t in T.*

REMARK 4. Another striking difference between the theories of amenability of general τ-semigroups and of the l- and r-semigroups appears in connection with multiplicative invariant means. Granirer [4] characterized extremely left amenable semigroups by the property: given s, t in S, there exists u in S such that $su = tu = u$. Thus a nontrivial semigroup with right cancellation cannot be extremely left amenable. The situation is different for general τ-semigroups. For take $X = \{a, b, c\}$, s the identity on X and t defined by $t(a) = a$, $t(b) = c$, $t(c) = b$. Then $(X, \{s, t\})$ is a τ-semigroup, $S = \{s, t\}$ forms a group, and the integral with respect to unit mass at a is S-invariant.

REMARK 5. Granirer also showed in [4] that the existence of a multiplicative left invariant mean is equivalent to the existence of a net of point measures converging strongly to left invariance (in the sense of theorem 4 above). In simple cases such as finite τ-semigroups an S-invariant mean in βX must be the integral with respect to unit mass concentrated at a common fixed point of all s in S. An interesting problem is to determine whether either of Granirer's characterizations hold for τ-semigroups. In this connection we note that for a τ-semigroup (X, S) if S, when considered as an abstract semigroup, has a multiplicative left invariant mean, then $m(X)$ has an S-invariant mean in βX. This follows from the proof of theorem 1 together with the observation that for each s in S $A^*qs = q(sx)$ (see lemma 2 and theorem 1 for notation).

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