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A FURTHER PROPERTY OF SPHERICAL ISOMETRIES

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Abstract

In this paper, it is proved that every isometry between the unit spheres of two real Banach spaces preserves the frames of the unit balls. As a consequence, if X and Y are n-dimensional Banach spaces and T_0 is an isometry from the unit sphere of X onto that of Y then it maps the set of all $(n-1)$-extreme points of the unit ball of X onto that of Y.

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1. Introduction

Throughout this paper, the term 'Banach space' always means a real Banach space. Let X and Y be Banach spaces. Then the classical Mazur–Ulam theorem states that if $T : X \to Y$ is a surjective isometry then T is affine. In 1972, Mankiewicz [17] extended this result by showing that if $U \subset X$ and $V \subset Y$ are open and connected and $T_0 : U \to V$ is a surjective isometry then there exists a surjective affine isometry $T : X \to Y$ such that $T_0 = T|_U$. From this, in particular, it turns out that every isometry from the unit ball of X onto that of Y can be extended to an isometric isomorphism between X and Y. Motivated by this observation, Tingley [22] proposed in 1987 the following problem. For a normed space X, let S_X denote its unit sphere.

Tingley's problem. Let X and Y be Banach spaces. Suppose that $T_0 : S_X \to S_Y$ is a surjective isometry. Then, does T_0 have a linear isometric extension $T : X \to Y$?

As the first result on this topic, it was shown in the same paper that $T_0(-x) = -T_0 x$ for all $x \in S_X$ if both X and Y are finite dimensional. This problem is also known as the isometric extension problem. Many papers, especially in the last decade, have been devoted to the problem, and it has been solved positively for some classical Banach spaces; see, for example, [1, 6, 10, 16, 23, 24].

Recently some mathematicians began to attack the problem on more general spaces. In 2011, Cheng and Dong [3] studied somewhere-flat spaces. One year later, it was shown by Kadets and Martín [15] that the problem has an affirmative answer for finite-dimensional polyhedral Banach spaces. Ding and Li [8] studied it using the notion of a sharp corner point and Tan and Liu [19] introduced the Tingley property and obtained results on almost-CL-spaces. However, surprisingly, Tingley's problem remains open even if $X = Y$ and X is two dimensional. In [20], new methods and some results on the two-dimensional Tingley problem were given. The survey of Ding [7] is a good starting point for understanding the history of the problem.

The aim of this paper is to give a further geometric property of spherical isometries by using the frame of the unit ball of Banach spaces, which can be understood as a natural generalisation of the set of all the weakest k-extreme points. To do this, a simple characterisation of the frame of the unit ball is given. We also mention two problems which arise from our main result.

2. Preliminaries

Let X be a Banach space, and let B_X denote the unit ball of X. A subset F of B_X is said to be an exposed face if $F = f^{-1}(\{1\}) \cap B_X$ for some support functional f of B_X. Let ν be the spherical image map from S_X into S_{X^*}, that is, $\nu(x) = \{f \in S_{X^*} : f(x) = 1\}$. We remark that $f^{-1}(\{1\}) = x + \ker f$ whenever $f \in \nu(x)$. In what follows, for each $f \in \nu(x)$, the exposed face $(x + \ker f) \cap B_X$ is denoted by $F(f)$ for short. Let $E(f)$ be the relative boundary of $F(f)$ with respect to the affine hyperplane $f^{-1}(\{1\})$. Then the frame of B_X is defined by $\mathrm{frm}(B_X) = \bigcup\{E(f) : f \text{ is a support functional for } B_X\}$. This was first introduced in [18] to construct a new calculation method for the Dunkl–Williams constant (compare [14]). Its geometric and topological properties were studied in [21]. Suppose now that $\dim X \geq k + 1$. Then, an element $x \in S_X$ is said to be a k-extreme point of B_X if $\{x_i\}_{i=1}^{k+1} \subset S_X$ and $x = (k+1)^{-1} \sum_{i=1}^{k+1} x_i$ imply the linear dependence of $\{x_i\}_{i=1}^{k+1}$. The set of all k-extreme points of B_X is denoted by $\mathrm{ext}_k(B_X)$. We remark that 1-extreme points are just extreme points in the usual sense, and that $\mathrm{ext}_k(B_X) \subset \mathrm{ext}_{k+1}(B_X)$ for all $k \in \mathbb{N}$. This means that the notion of a $(k+1)$-extreme point is weaker than that of a k-extreme point for each positive integer k.

For finite-dimensional spaces, the set $\mathrm{frm}(B_X)$ has the following characterisation.

Theorem 2.1 [21]. *Let X be an n-dimensional Banach space. Then* $\mathrm{frm}(B_X) = \mathrm{ext}_{n-1}(B_X)$.

This shows that $\mathrm{frm}(B_X)$ is a natural generalisation of the set of all the weakest k-extreme points of the unit ball.

We have another characterisation which has no restrictions on the dimension of the space.

Theorem 2.2 [21]. *Let X be a Banach space. Then*

$$\mathrm{frm}(B_X) = \bigcup \{\mathrm{ext}(B_M) : M \text{ is a two-dimensional subspace of } X\}.$$

Some basic properties of $\mathrm{frm}(B_X)$ are collected in the following theorem. An element x in a Banach space X is said to be Birkhoff orthogonal to $y \in X$, denoted by $x \perp_B y$, if $\|x + ty\| \geq \|x\|$ for all $t \in \mathbb{R}$; see Birkhoff [2], Day [4, 5] and James [11–13].

THEOREM 2.3 [21]. *Let X be a Banach space.*

(i) *Suppose that $x \in S_X$. Then $x \in \mathrm{frm}(B_X)$ if and only if there exists $y \in X\backslash\{0\}$ such that $x \perp_B y$ and $\|x + ty\| > 1$ for all $t > 0$.*
(ii) *The set $\mathrm{frm}(B_X)$ is symmetric, that is, $\mathrm{frm}(B_X) = -\mathrm{frm}(B_X)$.*
(iii) *Let M be a closed subspace of X. Then $\mathrm{frm}(B_M) \subset \mathrm{frm}(B_X) \cap M$.*
(iv) *If X is infinite dimensional, then $\bigcup_{k\in\mathbb{N}} \mathrm{ext}_k(B_X) \subset \mathrm{frm}(B_X)$.*
(v) *The set $\mathrm{frm}(B_X)$ is always closed, and is connected if $\dim X \geq 3$.*

3. A further property

We start this section with the following simple characterisation of $\mathrm{frm}(B_X)$.

THEOREM 3.1. *Let X be a Banach space, and let $x \in S_X$. Then $x \notin \mathrm{frm}(B_X)$ if and only if $(x + tB_X) \cap S_X$ is convex for some $t > 0$.*

PROOF. Suppose that $x \notin \mathrm{frm}(B_X)$. Then x is a smooth point of B_X by [21, Lemma 4.1], and we have $x + rB_{\ker \nu(x)} \subset F(\nu(x))$ for some $r > 0$. Putting $t = r/(2 + r)$, it follows that $(x + tB_X) \cap S_X = x + tB_{\ker \nu(x)}$. Indeed, for each $y \in x + tB_X$, one has $y = x + tz$ for some $z \in B_X$, or

$$\begin{aligned}(2 + r)y &= (2 + r)x + rz \\ &= (2 + r + r\langle z, \nu(x)\rangle)x + r(z - \langle z, \nu(x)\rangle x) \\ &= (2 + r + r\langle z, \nu(x)\rangle)\Big(x + \frac{r}{2 + r + r\langle z, \nu(x)\rangle}(z - \langle z, \nu(x)\rangle x)\Big).\end{aligned}$$

We now remark that

$$\frac{1}{2 + r + r\langle z, \nu(x)\rangle}(z - \langle z, \nu(x)\rangle x) \in B_{\ker \nu(x)}$$

since $\|z - \langle z, \nu(x)\rangle x\| \leq 2$ and $2 + r + r\langle z, \nu(x)\rangle \geq 2$, which implies that

$$y \in \frac{2 + r + r\langle z, \nu(x)\rangle}{2 + r}(x + rB_{\ker \nu(x)}) \subset \frac{2 + r + r\langle z, \nu(x)\rangle}{2 + r} S_X.$$

This shows $y \in S_X$ if and only if $\langle z, \nu(x)\rangle = 0$, and hence we obtain $(x + tB_X) \cap S_X \subset x + tB_{\ker \nu(x)}$. The other inclusion is obvious.

Conversely, we assume that $(x + tB_X) \cap S_X$ is convex for some $t > 0$. Suppose that there exists $y \in S_X$ such that $x \perp_B y$ and $\|x + ry\| > 1$ for all $r > 0$. Let $z(r) = \|x + ry\|^{-1}(x + ry)$ for all $r \in \mathbb{R}$. Then we have $\max\{\|z(r) - x\|, \|z(-r) - x\|\} \leq t$ for some $r > 0$, that is, $z(r), z(-r) \in (x + tB_X) \cap S_X$. However, putting $k = (\|x + ry\| + \|x - ry\|)/2$ and $\lambda = \|x + ry\|/(\|x + ry\| + \|x - ry\|)$, one obtains $k > 1$ and $x = k((1 - \lambda)z(r) + \lambda z(-r))$. This is a contradiction, which, together with Theorem 2.3(i), proves the theorem. □

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As a consequence, we obtain another formulation of $\mathrm{frm}(B_X)$.

Corollary 3.2. *Let X be a Banach space. Then*

$$\mathrm{frm}(B_X) = \{x \in S_X : (x + tB_X) \cap S_X \text{ is not convex for all } t > 0\}.$$

For our purpose, we need three lemmas which can be essentially found in Cheng and Dong [3]; see also Holmes [9, Exercise 2.18] for the first one. The proofs are given only for the sake of completeness, and based on the original ones except for the former half of the third one.

For each $x \in S_X$, let $\mathrm{st}(x, S_X) = \{y \in S_X : \|x + y\| = 2\}$. Then we remark that $C \subset \mathrm{st}(x, S_X)$ whenever C is a convex subset of S_X and $x \in C$.

Lemma 3.3. *Let X be a separable Banach space. Suppose that C is a maximal convex subset of S_X. Then $C = \mathrm{st}(x, S_X)$ for some $x \in C$.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a dense subset of C, and let $x_0 = \sum_n 2^{-n} x_n$. Then, for each $f \in \nu(x_0)$, we have $f(x_n) = 1$ for all $n \in \mathbb{N}$, which implies that $C \subset F(f)$. This and the maximality of C together imply that $C = F(f)$. Now, take an arbitrary $y \in \mathrm{st}(x_0, S_X)$. For an element g of $\nu(2^{-1}(x_0 + y))$, one has $g(x_0) = g(y) = 1$, or $g \in \nu(x_0)$ and $y \in F(g) = C$. This completes the proof. □

For a subset A of a Banach space, let $[A]$ be the closed linear span of A.

Lemma 3.4. *Let X and Y be Banach spaces, and let A be a separable subset of X. Suppose that $T_0 : S_X \to S_Y$ is a surjective isometry. Then there exist separable subspaces $X_0 \subset X$ and $Y_0 \subset Y$ such that $A \subset X_0$ and $T_0(S_{X_0}) = S_{Y_0}$.*

Proof. Let $M_1 = [A]$ and $N_1 = [T_0(S_{M_1})]$, respectively. Define the closed subspaces $M_k \subset X$ and $N_k \subset Y$ inductively by $M_k = [T_0^{-1}(S_{N_{k-1}})]$ and $N_k = [T_0(S_{M_k})]$ for all $k \geq 2$. Then one can show that $M_0 = \bigcup_{k \in \mathbb{N}} M_k$ and $N_0 = \bigcup_{k \in \mathbb{N}} N_k$ have the desired properties. □

Lemma 3.5. *Let X and Y be Banach spaces. Suppose that $T_0 : S_X \to S_Y$ is a surjective isometry. If C is a maximal convex subset of S_X, then $T_0(C)$ is a maximal convex subset of S_Y.*

Proof. Once it has been proved that the lemma is true for separable Banach spaces, then one can prove the general case from it. Indeed, applying the preceding lemma to each finite subset $A \subset C$, we have separable subspaces $M_A \subset X$ and $N_A \subset Y$ such that $A \subset M_A$ and $T_0(S_{M_A}) = S_{N_A}$. Let $C_A = C \cap M_A$, and let K_A be a maximal convex subset of S_{M_A} such that $C_A \subset K_A$. Then the separability of M_A ensures that $T_0(K_A)$ is a maximal convex subset of S_{N_A}, or $\mathrm{conv}\, T_0(C_A) \subset S_{N_A}$, which in turn implies that $\mathrm{conv}\, T_0(C) \subset S_Y$. Let K be a maximal convex subset of S_Y such that $T_0(C) \subset K$. Using the above argument for T_0^{-1} and K, we also have $C \subset T_0^{-1}(K) \subset \mathrm{conv}\, T_0^{-1}(K) \subset S_X$. This shows $T_0(C) = K$, as desired.

Now, suppose that both X and Y are separable. Then Lemma 3.3 assures that $C = \mathrm{st}(x, S_X)$ for some $x \in C$. Let K be a maximal convex subset of S_Y such

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that $T_0x \in K$. Then, similarly by Lemma 3.3, there exists $y \in K$ such that $K = \mathrm{st}(y, S_Y)$. We observe that $x \in T_0^{-1}(K) = \mathrm{st}(-T_0(-y_0), S_X)$ means $-T_0(-y_0) \in C$. This and the convexity of C guarantee that $C \subset T_0^{-1}(K)$, or $\mathrm{conv}\, T_0(C) \subset S_Y$. The proof is completed by an argument similar to that at the end of the preceding paragraph. □

Remark 3.6. The finite-dimensional case of the preceding lemma is due to Tingley [22, Lemma 13].

We now present a further geometric property of spherical isometries.

Theorem 3.7. *Let X and Y be Banach spaces. Suppose that $T_0 : S_X \to S_Y$ is a surjective isometry. Then $T_0(\mathrm{frm}(B_X)) = \mathrm{frm}(B_Y)$.*

Proof. Let $x \notin \mathrm{frm}(B_X)$. Then Theorem 3.1 guarantees that $(x + tB_X) \cap S_X$ is convex for some $t > 0$. Let C be a maximal convex subset of S_X such that $(x + tB_X) \cap S_X \subset C$. From the identity $(x + tB_X) \cap S_X = ((x + tB_X) \cap S_X) \cap C$, we obtain

$$\begin{aligned}(T_0x + tB_Y) \cap S_Y &= T_0((x + tB_X) \cap S_X)\\ &= T_0(((x + tB_X) \cap S_X) \cap C)\\ &= ((T_0x + tB_Y) \cap S_Y) \cap T_0(C)\\ &= (T_0x + tB_Y) \cap T_0(C).\end{aligned}$$

Applying Lemma 3.5, one has that $T_0(C)$ is also a maximal convex subset of S_Y. Thus the set $(T_0x + tB_Y) \cap S_Y$ is convex, which, together with Theorem 3.1, implies that $T_0x \notin \mathrm{frm}(B_Y)$.

Finally, since T_0^{-1} is also a surjective isometry, we have $T_0(S_X\backslash\mathrm{frm}(B_X)) = S_Y\backslash\mathrm{frm}(B_Y)$, and the theorem follows from the bijectivity of T_0. □

By Theorems 2.1 and 3.7, we immediately have the following corollary.

Corollary 3.8. *Let X and Y be n-dimensional Banach spaces. Suppose that $T_0 : S_X \to S_Y$ is a surjective isometry. Then $T_0(\mathrm{ext}_{n-1}(B_X)) = \mathrm{ext}_{n-1}(B_Y)$.*

We wonder whether Theorem 3.7 and Corollary 3.8 remain true for the sets of all stronger k-extreme points. Namely, does every spherical isometry T_0 satisfy $T_0(\mathrm{ext}_k(B_X)) = \mathrm{ext}_k(B_Y)$ for all $k \in \mathbb{N}$? As a remark, it is known that $\mathrm{ext}(B_X) \neq \emptyset$ if and only if $\mathrm{ext}_k(B_X) \neq \emptyset$ for some $k \in \mathbb{N}$. A proof of this fact is given here only for the sake of completeness.

Let X be a Banach space, and let $k \in \mathbb{N}$. We first show that $x \notin \mathrm{ext}_k(B_X)$ if and only if there exists a subspace M such that $\dim M \geq k$ and $x + tB_M \subset S_X$ for some $t > 0$. Suppose that $x \notin \mathrm{ext}_k(B_X)$. Then $x = (k+1)^{-1} \sum_{i=1}^{k+1} x_i$ for some linearly independent subset $\{x_i\}_{i=1}^{k+1}$ of S_X. We remark that $\mathrm{conv}(\{x_i\}_{i=1}^{k+1}) \subset F(f)$ whenever $f \in \nu(x)$. Let $M = [\{-x + x_i\}_{i=1}^{k+1}]$. It is easy to see that $\{-x + x_i\}_{i=1}^{k}$ is linearly independent, and so $\dim M = k$. Moreover, it follows from the identity $x = (k+1)^{-1} \sum_{i=1}^{k+1} x_i$ that $k^{-1}(x - x_j) = k^{-1} \sum_{i \neq j}(-x + x_i) \in -x + F(f)$ for all $1 \leq j \leq k$, which, together with $0 \in -x + F(f)$, implies that $k^{-1}\,\mathrm{absconv}(\{-x + x_i\}_{i=1}^{k}) \subset -x + F(f)$. This shows $tB_M \subset -x + F(f)$ for some $t > 0$, or $x + tB_M \subset S_X$. Conversely, assume that there exists a subspace M of X such that $\dim M \geq k$ and $x + tB_M \subset S_X$ for some $t > 0$. We remark that $x \notin M$ from the assumption. Let $\{e_i\}_{i=1}^k$ be a linearly independent subset of M, and let $e_{k+1} = -\sum_{i=1}^k e_i$. Putting $L = t^{-1} \max_{1\leq i\leq k+1} \|e_i\|$, one can show that $\{x + L^{-1}e_i\}_{i=1}^{k+1} \subset S_X$ is also linearly independent and that $x = (k+1)^{-1} \sum_{i=1}^{k+1}(x + L^{-1}e_i)$. Hence it follows that $x \notin \mathrm{ext}_k(B_X)$. We note that this equivalence easily shows $\mathrm{ext}_k(B_X) \subset \mathrm{ext}_{k+1}(B_X)$ for all $k \in \mathbb{N}$.

Now, suppose that $\mathrm{ext}_k(B_X) \neq \emptyset$ for some $k \in \mathbb{N}$. Take an arbitrary $x \in \mathrm{ext}_k(B_X)$. We assume that $x \notin \mathrm{ext}(B_X)$. Then there exist two distinct elements $y, z \in S_X$ and $s \in (0,1)$ such that $x = (1-s)y + sz$. Let $z(t) = (1-t)y + tz$ for all $t \in \mathbb{R}$. Since the function $t \to \|z(t)\|$ is convex, it follows that $\{t \in \mathbb{R} : z(t) \in S_X\} = [t_1, t_2]$ for some $t_1 \leq 0$ and $1 \leq t_2$. Without loss of generality, we may assume that $t_1 = 0$ and $t_2 = 1$. It is enough to prove that $y \in \mathrm{ext}_{k-1}(B_X)$. To this end, suppose on the contrary that $y \notin \mathrm{ext}_{k-1}(B_X)$. As was shown above, there exists a subspace M of X such that $\dim M \geq k-1$ and $y + tB_M \subset S_X$ for some $t > 0$. Let $\{e_i\}_{i=1}^{k-1}$ be a linearly independent subset of S_M. Then one has $x \pm (1-s)te_i \in F(f)$ for all $1 \leq i \leq k-1$ whenever $f \in \nu(x)$. Putting $r = \min\{1-s, s, (1-s)t\}$ and $e_k = z - y$, it follows that $r > 0$ and $\mathrm{absconv}(\{re_i\}_{i=1}^k) \subset -x + F(f)$. This guarantees that there exists $r_0 > 0$ such that $x + r_0 B_N \subset S_X$, where $N = [\{e_i\}_{i=1}^k]$. Since it can be shown that $\dim N = k$, we have $x \notin \mathrm{ext}_k(B_X)$ by the argument in the preceding paragraph, which is a contradiction that proves $y \in \mathrm{ext}_{k-1}(B_X)$. Thus $\mathrm{ext}(B_X) \neq \emptyset$ follows by an induction, and so we should assume that $\mathrm{ext}(B_X) \neq \emptyset$ when considering the above problem.

We finally mention two problems which naturally arise from Theorem 3.7. The first one is a Mazur–Ulam type problem.

Problem 3.9. Let X and Y be Banach spaces. Suppose that $T_0 : \mathrm{frm}(B_X) \to \mathrm{frm}(B_Y)$ is a surjective isometry. Then, does T_0 have a linear isometric extension $T : X \to Y$?

Of course this is more difficult than Tingley's problem unless the following one is solved positively.

Problem 3.10. Let X and Y be Banach spaces. Suppose that $T_0 : \mathrm{frm}(B_X) \to \mathrm{frm}(B_Y)$ is a surjective isometry. Then, does T_0 have an isometric extension $\widetilde{T_0} : S_X \to S_Y$?

Remark 3.11. If no assumptions are added, both Problems 3.9 and 3.10 have negative answers in the case $\dim X = \dim Y = 2$. Indeed, let $X = Y = \ell_\infty^2$. Then $\mathrm{frm}(B_X) = \mathrm{ext}(B_X) = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$. Define an operator T_0 on $\mathrm{frm}(B_X)$ by $T_0(1,1) = (1,1)$, $T_0(1,-1) = (1,-1)$, $T_0(-1,1) = (-1,-1)$ and $T_0(-1,-1) = (-1,1)$. This is a counterexample of the problems since T_0 does not map antipodal pairs of points to such pairs. Hence, in the case $\dim X = \dim Y = 2$, we at least need an assumption which implies $T_0(-x) = -T_0 x$ for all $x \in \mathrm{frm}(B_X)$.

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