Canad. Math. Bull. Vol. **38** (3), 1995 pp. 347–351

m-DIMENSIONAL SCHLÖMILCH SERIES

ALLEN R. MILLER

ABSTRACT. By using the principle of mathematical induction a simple algebraic formula is derived for an m-dimensional Schlömilch series. The result yields a countably infinite number of representations for null-functions on increasingly larger open intervals.

1. **Introduction.** In 1900 Nielsen [1] derived the following summation formula for a one-dimensional (1D) Schlömilch series:

$$\sum_{k=1}^{\infty}(-1)^k\frac{J_\nu(2xk)}{k^\nu} = -\frac{x^\nu}{2\Gamma(1+\nu)} + \frac{\sqrt{\pi}x^{-\nu}}{\Gamma(\frac{1}{2}+\nu)}\sum_{k=1}^{p}[x^2-(k-1/2)^2\pi^2]^{\nu-1/2} \tag{1.1}$$

where $\operatorname{Re}\nu > -1/2$, $x > 0$ and p is a non-negative integer such that $(p-1/2)\pi < x < (p+1/2)\pi$. Recently, motivated by a conjecture of Henkel and Weston [2], Miller [3] and Grosjean [4] using different methods derived a summation formula for the 2D Schlömilch series:

(1.2)

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{k+n}\frac{J_\nu(2x\sqrt{k^2+n^2})}{(\sqrt{k^2+n^2})^\nu}$$
$$= -\frac{x^\nu}{4\Gamma(1+\nu)} + \frac{\pi x^{-\nu}}{\Gamma(\nu)}\sum_{s=1}^{p}\sum_{t=1}^{u(s)}[x^2-(s-1/2)^2\pi^2-(t-1/2)^2\pi^2]^{\nu-1}$$

where $\operatorname{Re}\nu > 0$, $x > 0$. Here p and $u(s)$ are the largest integers such that

$$p < \frac{1}{2} + \sqrt{\frac{x^2}{\pi^2} - \frac{1}{4}}$$
$$u(s) < \frac{1}{2} + \sqrt{\frac{x^2}{\pi^2} - \left(s - \frac{1}{2}\right)^2}.$$

Note that if $0 < x < \pi/\sqrt{2}$, then $p < 1$, and the double sum over s, t in the right hand side of equation (1.2) vanishes.

When $\nu = 1/2$, equation (1.2) reduces to the trigonometric lattice sum

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{k+n}\frac{\sin(2x\sqrt{k^2+n^2})}{\sqrt{k^2+n^2}} = -\frac{x}{2}, \quad 0 < x < \pi/\sqrt{2}$$

Received by the editors February 10, 1994.
AMS subject classification: 33C10.

https://doi.org/10.4153/CMB-1995-050-x Published online by Cambridge University Press

which occurs in finite-size scaling of the three-dimensional spherical model of ferromagnetism [5].

From equations (1.1) and (1.2) respectively we easily obtain for $\operatorname{Re}\nu > -1/2, x > 0$

$$(1.3)\quad \sum_{k=-\infty}^{\infty}{}' (-1)^k \frac{J_\nu(2xk)}{(xk)^\nu} = -\frac{1}{\Gamma(1+\nu)} + \frac{4\pi^{-1/2}}{\Gamma(1/2+\nu)}\left(\frac{\pi^2}{4x^2}\right)^\nu \sum_{s \text{ odd}}^{s^2<4x^2/\pi^2} \left(\frac{4x^2}{\pi^2} - s^2\right)^{\nu-1/2}$$

and for $\operatorname{Re}\nu > 0, x > 0$

(1.4)

$$\sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' (-1)^{k+n} \frac{J_\nu(2x\sqrt{k^2+n^2})}{(x\sqrt{k^2+n^2})^\nu} = -\frac{1}{\Gamma(1+\nu)} + \frac{16\pi^{-1}}{\Gamma(\nu)}\left(\frac{\pi^2}{4x^2}\right)^\nu \sum_{s,t \text{ odd}}^{s^2+t^2<4x^2/\pi^2} \left(\frac{4x^2}{\pi^2} - s^2 - t^2\right)^{\nu-1}$$

where the summation indicies s and t are positive integers and a prime next to a summation means that the summation index is never zero.

In the present paper we shall generalize equations (1.3) and (1.4) to m-dimensional Schlömilch series. We shall then be able to obtain representations by Schlömilch series for null-functions on increasingly larger open intervals.

2. ***m*-dimensional series.** Following Allen and Pathria [6], let $\mathbf{q}(m)$ denote the vectors whose m components range over all integers (positive, negative and zero). A prime next to a summation will now mean that $\mathbf{q}(m) \neq \mathbf{0}$. Also let $\boldsymbol{\tau}(m)$ denote the constant vector whose m components have the value $1/2$. The length of the vector $\mathbf{q}(m)$ is denoted by $q \equiv |\mathbf{q}(m)|$. With this notation equations (1.3) and (1.4) may be written respectively for $m = 1, 2$ as

(2.1)

$$\sum_{\mathbf{q}(m)}{}' \cos(2\pi\mathbf{q}\cdot\boldsymbol{\tau}) \frac{J_\nu(2xq)}{(xq)^\nu} = -\frac{1}{\Gamma(1+\nu)} + \frac{4^m\pi^{-m/2}}{\Gamma(\frac{2-m}{2}+\nu)}\left(\frac{\pi^2}{4x^2}\right)^\nu \sum_{\boldsymbol{\xi}(m)}^{\xi^2<4x^2/\pi^2} \left(\frac{4x^2}{\pi^2} - \xi^2\right)^{\nu-m/2}$$

where $\operatorname{Re}\nu > m/2 - 1, x > 0$ and the m components of the vector $\boldsymbol{\xi}(m)$ range over odd positive integers subject to the condition $\xi^2 < 4x^2/\pi^2$.

Since

$$(2.2)\quad \frac{J_\nu(2z)}{z^\nu} = \frac{{}_0F_1[-;1+\nu;-z^2]}{\Gamma(1+\nu)},$$

equation (2.1) may also be written for $x > 0$

$$(2.3)\quad \sum_{\mathbf{q}(m)} \cos(2\pi\mathbf{q}\cdot\boldsymbol{\tau}) \frac{J_\nu(2xq)}{(xq)^\nu} = \frac{4^m\pi^{-m/2}}{\Gamma(\frac{2-m}{2}+\nu)}\left(\frac{\pi^2}{4x^2}\right)^\nu \sum_{\boldsymbol{\xi}(m)}^{\xi^2<4x^2/\pi^2} \left(\frac{4x^2}{\pi^2} - \xi^2\right)^{\nu-m/2}$$

where $\operatorname{Re}\nu > m/2 - 1$. We note that since both sides of equations (2.1) and (2.3) are even functions of x, these results are actually valid for $x \neq 0$.

In fact equation (2.3) is true for all positive integers m for we shall assume it is true for an arbitrary integer m and show it is also true for $m + 1$. Thus by the principle of mathematical induction (see *e.g.* [7, p. 42]), equation (2.3) and hence also equation (2.1) are valid for all positive integers m.

3. **The inductive proof.** Call the left-hand side of equation (2.3) $S(m)$. In order to compute $S(m+1)$ we shall need a special case of the addition theorem for generalized hypergeometric functions (see *i.e.* [8, p. 24]), namely:

$$\frac{J_\nu(2x\sqrt{m^2+n^2})}{(x\sqrt{m^2+n^2})^\nu} = \sum_{r=0}^{\infty} \frac{(-x^2m^2)^r}{r!} \frac{J_{\nu+r}(2xn)}{(xn)^{\nu+r}} \tag{3.1}$$

where all the parameters may be complex numbers. This result is sometimes called the addition theorem for Bessel functions of the first kind (see also [9, p. 129]).

Letting the integer ℓ denote any (fixed) component of the vector $\mathbf{q}(m+1)$, it is easy to see from equation (3.1) that

$$\frac{J_\nu\big(2xq(m+1)\big)}{\big(xq(m+1)\big)^\nu} = \sum_{r=0}^{\infty} \frac{(-x^2\ell^2)^r}{r!} \frac{J_{\nu+r}\big(2xq(m)\big)}{\big(xq(m)\big)^{\nu+r}}. \tag{3.2}$$

Thus we write

$$\begin{aligned} S(m+1) &= \sum_{\mathbf{q}(m+1)} \cos\big(2\pi\mathbf{q}(m+1)\cdot\boldsymbol{\tau}(m+1)\big)\frac{J_\nu\big(2xq(m+1)\big)}{\big(xq(m+1)\big)^\nu} \\ &= \sum_{\mathbf{q}(m+1)} \cos\big(2\pi\mathbf{q}(m+1)\cdot\boldsymbol{\tau}(m+1)\big)\sum_{r=0}^{\infty} \frac{(-x^2\ell^2)^r}{r!} \frac{J_{\nu+r}\big(2xq(m)\big)}{\big(xq(m)\big)^{\nu+r}} \\ &= \sum_{\ell=-\infty}^{\infty}(-1)^\ell \sum_{r=0}^{\infty}\frac{(-x^2\ell^2)^r}{r!}\sum_{\mathbf{q}(m)}\cos\big(2\pi\mathbf{q}(m)\cdot\boldsymbol{\tau}(m)\big)\frac{J_{\nu+r}\big(2xq(m)\big)}{\big(xq(m)\big)^{\nu+r}} \end{aligned} \tag{3.3}$$

where the later two summations have been interchanged. Now by using the induction hypothesis equation (2.3) with ν replaced by $\nu + r$ we obtain

$$\begin{aligned} S(m+1) = 4^m\pi^{-m/2}\Big(\frac{\pi^2}{4x^2}\Big)^\nu \sum_{\boldsymbol{\xi}(m)}^{\xi^2<4x^2/\pi^2}\Big(\frac{4x^2}{\pi^2}-\xi^2\Big)^{\nu-m/2} \\ \cdot \sum_{\ell=-\infty}^{\infty}(-1)^\ell\sum_{r=0}^{\infty}\frac{[-\ell^2(x^2-\pi^2\xi^2/4)]^r}{\Gamma(\frac{2-m}{2}+\nu)(\frac{2-m}{2}+\nu)_r r!}. \end{aligned}$$

Noting equation (2.2) we rewrite this as

$$\begin{aligned} S(m+1) = 4^m\pi^{-m/2}\Big(\frac{\pi^2}{4x^2}\Big)^\nu \sum_{\boldsymbol{\xi}(m)}^{\xi^2<4x^2/\pi^2}\Big(\frac{4x^2}{\pi^2}-\xi^2\Big)^{\nu-m/2} \\ \cdot \sum_{\ell=-\infty}^{\infty}(-1)^\ell\frac{J_{\nu-m/2}(2\ell\sqrt{x^2-\pi^2\xi^2/4})}{(\ell\sqrt{x^2-\pi^2\xi^2/4})^{\nu-m/2}}. \end{aligned}$$

https://doi.org/10.4153/CMB-1995-050-x Published online by Cambridge University Press

In order to evaluate the bilateral sum over the summation index ℓ, we use equation (1.3) with the prime next to the summation removed (which is just equation (2.3) for the case $m = 1$), x replaced by $\sqrt{x^2 - \pi^2\xi^2/4}$, and ν replaced by $\nu - m/2$. Thus, for $\operatorname{Re}\nu > (m-1)/2$ we have

$$S(m+1) = 4^m \pi^{-m/2} \Big(\frac{\pi^2}{4x^2}\Big)^{\nu} \sum_{\boldsymbol{\xi}(m)}^{\xi^2<4x^2/\pi^2} \Big(\frac{4x^2}{\pi^2} - \xi^2\Big)^{\nu-m/2}$$
$$\cdot \left\{ \frac{4\pi^{-1/2}}{\Gamma(\frac{1-m}{2}+\nu)} \Big[\frac{\pi^2}{4(x^2-\pi^2\xi^2/4)}\Big]^{\nu-m/2} \cdot \sum_{s \text{ odd}}^{s^2+\xi^2<4x^2/\pi^2} \Big[\frac{4}{\pi^2}(x^2-\pi^2\xi^2/4) - s^2\Big]^{\nu-m/2-1/2} \right\}$$
$$= \frac{4^{m+1}\pi^{-\frac{m+1}{2}}}{\Gamma(\frac{1-m}{2}+\nu)} \Big(\frac{\pi^2}{4x^2}\Big)^{\nu} \sum_{\boldsymbol{\xi}(m)}^{\xi^2<4x^2/\pi^2} \sum_{s \text{ odd}}^{s^2+\xi^2<4x^2/\pi^2} \Big(\frac{4x^2}{\pi^2} - \xi^2 - s^2\Big)^{\nu-\frac{m+1}{2}}$$

which simplifies to

$$S(m+1) = \frac{4^{m+1}\pi^{-\frac{m+1}{2}}}{\Gamma(\frac{1-m}{2}+\nu)} \Big(\frac{\pi^2}{4x^2}\Big)^{\nu} \sum_{\boldsymbol{\xi}(m+1)}^{\xi^2<4x^2/\pi^2} \Big(\frac{4x^2}{\pi^2} - \xi^2\Big)^{\nu-\frac{m+1}{2}}.$$

Hence comparing this with equation (3.3) we see that equation (2.3) is valid for all positive integers m by induction.

4. **Null-functions.** Recalling that the vectors $\boldsymbol{\xi}(m)$, $\boldsymbol{\tau}(m)$ are defined for $m = 1, 2, 3, \ldots$ by

$$\boldsymbol{\xi}(m) = (s_1, s_2, \ldots, s_m)$$
$$\boldsymbol{\tau}(m) = \Big(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\Big)$$

where the s_j are odd positive integers, we have

$$\xi^2 = s_1^2 + s_2^2 + \cdots + s_m^2, \quad \tau^2 = m/4.$$

Hence if $0 < x < \pi\sqrt{m}/2$, equations (2.1) and (2.3) give respectively

$$\sum_{\mathbf{q}(m)}{}' \cos(2\pi\mathbf{q}\cdot\boldsymbol{\tau}) \frac{J_\nu(2xq)}{(xq)^\nu} + \frac{1}{\Gamma(1+\nu)} = 0 \tag{4.1}$$
$$\sum_{\mathbf{q}(m)} \cos(2\pi\mathbf{q}\cdot\boldsymbol{\tau}) \frac{J_\nu(2xq)}{(xq)^\nu} = 0$$

where $\operatorname{Re}\nu > m/2 - 1$ and x is in the open interval $(0, \pi\tau)$.

Allen and Pathria, who derived equation (4.1) in [6] using their previous results in [10], have noted that the importance of equation (4.1) lies in the fact that it provides representations for null-functions over the increasingly larger intervals $(0, \pi\sqrt{m}/2)$.

In conclusion, we remark that a different approach (via L. Schwartz's distributions) to the summation of Schlömilch series may be found in [11].

ACKNOWLEDGEMENT. The author thanks the referee for several comments used to clarify the presentation.

REFERENCES

1. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1962.
2. M. Henkel and R. A. Weston, *Problem 92-11*: On alternating multiple sums*, SIAM Rev. **35**(1993), 497–500.
3. A. R. Miller, *On certain two-dimensional Schlömilch series*, J. Physics, A. to appear.
4. C. C. Grosjean, *Solving some problems posed in the SIAM Review II*, Simon Stevin, to appear.
5. M. Henkel and R. A. Weston, *Universal amplitudes in finite-size scaling: the antiperiodic* 3*D spherical model*, J. Phys. A **25**(1992), L207–L211.
6. S. Allen and R. K. Pathria, *On the conjectures of Henkel and Weston*, J. Phys. A **26**(1993), 5173–5176.
7. N. H. McCoy, *Introduction to Modern Algebra*, Allyn and Bacon, Boston, 1960.
8. H. Exton, *Multiple Hypergeometric Functions and Applications*, Ellis Horwood, Chichester, 1976.
9. W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer-Verlag, New York, 1966.
10. S. Allen and R. K. Pathria, *Analytical evaluation of a class of phase-modulated lattice sums*, J. Math. Phys. **34**(1993), 1497–1507.
11. N. Ortner and P. Wagner, *Fundamental solution of hyperbolic differential operators and the Poisson summation formula*, Integral Transforms and Special Functions **1**(1993), 183–196.

Department of Mathematics
George Washington University
Washington, DC 20052
U.S.A.