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Asymptotic profiles for positive solutions of diffusive logistic equations

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In this paper, we study the asymptotic profiles of positive solutions for diffusive logistic equations. The aim is to study the sharp effect of linear growth and nonlinear function. Both the classical reaction-diffusion equation and nonlocal dispersal equation are investigated. Our main results reveal that the linear and nonlinear parts of reaction term play quite different roles in the study of positive solutions.

Keywords: Reaction-diffusion equation; positive solution; logistic

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1. Introduction and main results

In this paper, we consider the diffusive logistic equation

$$\begin{cases} \Delta u + \lambda u - a(x)u^p = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a $C^{2+\mu}$ bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$, $\lambda > 0$ is a real parameter, $p > 1$ is constant, the boundary operator B is given by

$$Bu = \alpha u_\nu + \beta u,$$

here ν is the unit outward normal to $\partial\Omega$ and either $\alpha = 0$, $\beta = 1$ (the Dirichlet boundary condition) or $\alpha = 1$, $\beta \geqslant 0$ (the Neumann or Robin boundary conditions). The function $a \in C^\mu(\bar{\Omega})$ and $a(x) > 0$ for $x \in \bar{\Omega}$. Problem (1.1) is a basic reaction-diffusion model used in the study of diversity phenomena in the applied sciences (see, e.g. [**1**, **3**, **4**, **15**]). It is also the paradigmatic model in population dynamics, the diffusive logistic model [**7**, **8**, **13**, **16**, **17**]. The function $a(x)$ measures the capacity of Ω to support the species $u(x)$. Under the above assumptions, the semilinear problem (1.1) was well studied, see [**15**, **18**] and references therein.

In the case of $a(x) \equiv 0$, then (1.1) reduces to the following linear eigenvalue equation

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

We know that (1.2) admits a unique positive principal eigenvalue $\lambda_1^B(\Omega)$ associated with a positive solution $\phi(x)$. Further, (1.1) admits a unique positive solution $u(x)$ if and only if $\lambda > \lambda_1^B(\Omega)$. However, we can see that (1.2) admits positive solutions if and only if $\lambda = \lambda_1^B(\Omega)$.

In the previous work [**20**], the sharp profiles of positive solutions to (1.1) for $\lambda > \lambda_1^B(\Omega)$ have been well investigated. In this paper, we shall consider the sharp changes of positive solutions between (1.1) and (1.2). To do this, we consider the following diffusive logistic problem

$$\begin{cases} \Delta u + (\lambda_1^B(\Omega) + \varepsilon^\alpha)u - a_\varepsilon(x)u^p = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\varepsilon > 0$ is a parameter, $\alpha > 0$ is a given constant, $a_\varepsilon \in C^\mu(\Omega)$ is positive in $\bar{\Omega}$ and there exist $\beta > 0$ and $a \in C^\mu(\Omega)$ such that $a(x) > 0$ for $x \in \bar{\Omega}$ and

$$\lim_{\varepsilon \to 0+} \frac{a_\varepsilon(x)}{\varepsilon^\beta} = a(x) \text{ uniformly in } \bar{\Omega}. \tag{1.4}$$

In (1.4), the constant β is the quenching rate of nonlinear function. It follows from the classical results of reaction-diffusion equation that (1.3) admits a unique positive solution $\theta_\varepsilon \in C^{2+\mu}(\Omega)$ for every $\varepsilon > 0$, see e.g. [**9**, **15**, **16**]. According to (1.3), one may think that $\theta_\varepsilon(x)$ tends to the trivial solution or the positive eigenfunction of (1.2). However, our investigations reveal that $\theta_\varepsilon(x)$ admits quite different profiles, determined by various choices of α and β. In the present paper, we shall investigate the sharp profiles by the classical regularity estimates and uniform estimates of solutions [**13**, **16**]. More precisely, we prove the following result.

THEOREM 1.1. *Let $\theta_\varepsilon \in C^{2+\mu}(\Omega)$ be the unique positive solution of (1.4) for $\varepsilon > 0$ and Ω_* be a compact subset of Ω.*

(i) *If $\alpha < \beta$, then*

$$\lim_{\varepsilon \to 0+} \theta_\varepsilon(x) = \infty \text{ uniformly in } \Omega_*. \tag{1.5}$$

Further, for any $x \in \Omega_*$, there exist positive constants c, C such that

$$c \leqslant \liminf_{\varepsilon \to 0+} \varepsilon^{\frac{\beta-\alpha}{p-1}} \theta_\varepsilon(x) \leqslant \limsup_{\varepsilon \to 0+} \varepsilon^{\frac{\beta-\alpha}{p-1}} \theta_\varepsilon(x) \leqslant C. \tag{1.6}$$

(ii) *If $\alpha > \beta$, then*

$$\lim_{\varepsilon \to 0+} \theta_\varepsilon(x) = 0 \text{ uniformly in } \Omega_*. \tag{1.7}$$

Further, for any $x \in \Omega_*$, there exist positive constants c_1, C_1 such that

$$c_1 \leqslant \liminf_{\varepsilon \to 0+} \varepsilon^{\frac{\beta-\alpha}{p-1}} \theta_\varepsilon(x) \leqslant \limsup_{\varepsilon \to 0+} \varepsilon^{\frac{\beta-\alpha}{p-1}} \theta_\varepsilon(x) \leqslant C_1. \tag{1.8}$$

(iii) *If $\alpha = \beta$, subject to a subsequence, we have*

$$\lim_{\varepsilon \to 0+} \theta_\varepsilon(x) = c_0 \phi(x) \text{ uniformly in } \bar{\Omega}$$

for some positive constant c_0.

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Remark 1.2. It follows from theorem 1.1 that the linear term and nonlinear reaction function play quite different roles in the limiting behaviour of positive solutions of (1.1). We know from (1.5) and (1.6) that the blow-up phenomenon only occurs if the nonlinear function admits a quicker quenching speed, i.e. $\alpha < \beta$. It is interesting to point out that the blow-up phenomenon appears in the diffusive logistic equation with spatial degeneracy, see [**8**, **15**]. However, if the linear term has a quicker quenching speed to the critical value $\lambda_1^B(\Omega)$, we get from (1.7) and (1.8) that the solution will tend to the trivial solution.

Since the diffusion may take place between nonadjoint places, the research in nonlocal dispersal equation has attracted much attention in recent years. Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative and symmetric function. It is known that the nonlocal dispersal equation

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)[u(y,t) - u(x,t)]\,\mathrm{d}y \text{ in } \mathbb{R}^N \times (0,\infty), \tag{1.9}$$

and variations of it, arise in the study of different dispersal process in material science, ecology, neurology and genetics (see, for instance, [**2**, **5**, **12**]). As stated in [**10**], if $u(y,\, t)$ is thought of as the density at location y at time t, and $J(x-y)$ is thought of as the probability distribution of jumping from y to x, then $\int_{\mathbb{R}^N} J(x-y)u(y,\, t)\,\mathrm{d}y$ denotes the rate at which individuals are arriving to location x from all other places and $\int_{\mathbb{R}^N} J(y-x)u(x,\, t)\,\mathrm{d}y$ is the rate at which they are leaving location x to all other places. Thus the right-hand side of (1.9) is the change of density $u(x,\, t)$. There has been attracted considerable interest in the study of nonlocal dispersal equations recently, for example, the papers [**6**, **11**, **14**, **19**, **21**–**23**] and references therein.

Let us consider the nonlocal dispersal logistic equation

$$\int_\Omega J(x-y)u(y)\,\mathrm{d}y - u(x) + (\lambda_p(\Omega) + \varepsilon^\alpha)u - a_\varepsilon(x)u^p(x) = 0 \text{ in } \bar{\Omega}, \tag{1.10}$$

where $\varepsilon > 0$ is a parameter, $\alpha > 0$ and $a_\varepsilon \in C(\bar{\Omega})$ satisfies (1.4). In (1.10), the dispersal kernel function $J \in C(\mathbb{R}^N)$ is nonnegative, symmetric such that

$$\int_{\mathbb{R}^N} J(y)\,\mathrm{d}y = 1 \text{ and } J(0) > 0,$$

and $\lambda_p(\Omega)$ stands for the unique principal eigenvalue of

$$\int_\Omega J(x-y)u(y)\,\mathrm{d}y - u(x) = -\lambda u(x) \text{ in } \bar{\Omega}.$$

In the rest of paper, we denoted by $\psi(x)$ the positive eigenfunction of $\lambda_p(\Omega)$. Then for any $\varepsilon > 0$, we know that (1.10) admits a unique positive solution $\omega_\varepsilon(x)$, see [**11**, **22**].

Since the nonlocal dispersal equation shares many properties with the reaction-diffusion equation, it is interesting to investigate the sharp behaviour of positive solutions of (1.10) as $\varepsilon \to 0$. However, there is a deficiency of regularity theory and compact property for nonlocal dispersal operators, the study of sharp behaviour

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of (1.10) is quite different to (1.3), [**1**, **6**, **13**]. We shall obtain the asymptotic behaviour for nonlocal dispersal problem (1.10) by the means of nonlocal estimates and comparison arguments.

In the case of nonlocal dispersal logistic equation, we have the following result.

THEOREM 1.3. *Let $\omega_\varepsilon \in C(\bar{\Omega})$ be the unique positive solution of (1.10) for $\varepsilon > 0$.*

(i) *If $\alpha < \beta$, then*

$$\lim_{\varepsilon \to 0+} \omega_\varepsilon(x) = \infty \text{ uniformly in } \bar{\Omega}. \tag{1.11}$$

Further, there exist positive constants c, C such that

$$c \leqslant \liminf_{\varepsilon \to 0+} \varepsilon^{\frac{\beta-\alpha}{p-1}} \omega_\varepsilon(x) \leqslant \limsup_{\varepsilon \to 0+} \varepsilon^{\frac{\beta-\alpha}{p-1}} \omega_\varepsilon(x) \leqslant C$$

for any $x \in \bar{\Omega}$.

(ii) *If $\alpha > \beta$, then*

$$\lim_{\varepsilon \to 0+} \omega_\varepsilon(x) = 0 \text{ uniformly in } \bar{\Omega}. \tag{1.12}$$

Further, there exist positive constants c_1, C_1 such that

$$c_1 \leqslant \liminf_{\varepsilon \to 0+} \varepsilon^{\frac{\beta-\alpha}{p-1}} \omega_\varepsilon(x) \leqslant \limsup_{\varepsilon \to 0+} \varepsilon^{\frac{\beta-\alpha}{p-1}} \omega_\varepsilon(x) \leqslant C_1$$

for any $x \in \bar{\Omega}$.

(iii) *If $\alpha = \beta$, then, subject to a subsequence, we have*

$$\lim_{\varepsilon \to 0+} \omega_\varepsilon(x) = c_0 \psi(x) \text{ uniformly in } \bar{\Omega}$$

for some positive constant c_0.

The conclusions in theorem 1.3 provide us how the sharp profiles of positive solutions to (1.10) is determined by α and β. We also know that the profile for nonlocal problem is different to the classical reaction-diffusion equation. By (1.11), we obtain that the positive solution for nonlocal problem (1.10) will blow-up in the whole domain Ω when $\alpha < \beta$. Similarly, by (1.12), we know that quenching occurs for all $x \in \bar{\Omega}$.

The rest of this paper is organized as follows. In § 2, we investigate the profiles of reaction-diffusion equation (1.3). Section 3 is devoted to the sharp profiles of nonlocal dispersal logistic equations.

2. Profiles for reaction-diffusion equations

In this section, we investigate the limiting behaviour of positive solutions for the diffusive logistic equation (1.3). It follows from the classical results [**4**, **13**] that there exists a unique positive solution $\theta_\varepsilon \in C^{2+\mu}(\Omega)$ to (1.3) for every $\varepsilon > 0$. Moreover,

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the positive solution θ_ε is continuous with respect to ε. In what follows, we always assume that a_ε, $a \in C^\mu(\Omega)$ are positive in $\bar{\Omega}$ and

$$\lim_{\varepsilon \to 0+} \frac{a_\varepsilon(x)}{\varepsilon^\alpha} = a(x) \text{ uniformly in } \bar{\Omega}.$$

We first study the following diffusive logistic equation

$$\begin{cases} \Delta u + \lambda u - a(x)u^p = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We can see that (2.1) admits a unique positive solution $\theta_\lambda(x)$ if and only if $\lambda > \lambda_1^B(\Omega)$. Moreover, $\theta_\lambda(x)$ is continuous with respect to λ and

$$\lim_{\lambda \to \lambda_1^B(\Omega)+} \theta_\lambda(x) = 0 \text{ locally uniformly in } \Omega.$$

We shall give the decay estimates of $\theta_\lambda(x)$ near $\lambda_1^B(\Omega)$ as follows.

Lemma 2.1. *Suppose that Ω_* is a subdomain of Ω such that $\bar{\Omega}_* \subset \Omega$. Let $\theta_\lambda(x)$ be the unique positive solution of (2.1) for $\lambda \in (\lambda_1^B(\Omega), \lambda_1^B(\Omega) + 1]$, then there exist positive constants c and C, independent of λ such that*

$$c\left[\frac{\lambda - \lambda_1^B(\Omega)}{\max_{\bar{\Omega}} a(x)}\right]^{\frac{1}{p-1}} \leqslant \theta_\lambda(x) \leqslant C\left[\frac{\lambda - \lambda_1^B(\Omega)}{\min_{\bar{\Omega}} a(x)}\right]^{\frac{1}{p-1}} \tag{2.2}$$

for $x \in \bar{\Omega}_*$.

Proof. By the uniqueness of positive solution to (2.1), we can find positive constant M, independent of λ such that

$$0 < \max_{\bar{\Omega}} \theta_\lambda(x) \leqslant M - 1. \tag{2.3}$$

Let $\phi(x)$ be a positive eigenfunction of $\lambda_1^B(\Omega)$ such that $\|\phi\|_{L^\infty(\Omega)} = 1$. Denote

$$\Omega^* = \left\{x \in \bar{\Omega} : dist(x, \Omega_*) > \inf_{x \in \partial\Omega, y \in \partial\Omega_*} \frac{|x-y|}{2}\right\},$$

and take $C_1 > 0$ such that

$$C_1\phi(x) > \left[\frac{\lambda - \lambda_1^B(\Omega)}{\min_{\bar{\Omega}} a(x)}\right]^{\frac{1}{p-1}} \tag{2.4}$$

for $x \in \Omega_*$. Using (2.3) and (2.4), we know that there exists smooth function $u(x)$ such that

$$u(x) = \begin{cases} C_1\phi(x) & \text{if } x \in \bar{\Omega}_*, \\ M & \text{if } x \in \bar{\Omega}^*, \end{cases}$$

and $u(x)$ is an upper-solution to (2.1). Since $\phi(x)$ is independent to λ, we know from the comparison principle that the right-hand side of (2.2) holds.

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On the other hand, we define $v(x) = c_1\phi(x)$, where

$$c_1 = \left[\frac{\lambda - \lambda_1^B(\Omega)}{\max_{\bar{\Omega}} a(x)}\right]^{\frac{1}{p-1}}.$$

It is easy to see that $v(x)$ is a lower-solution to (2.1) and we obtain the left-hand side of (2.2) by the uniqueness of positive solutions to (2.1). The proof is completed. □

LEMMA 2.2. *Let $u_\varepsilon \in C^{2+\mu}(\Omega)$ be the unique positive solution of*

$$\begin{cases} \Delta u + (\lambda_1^B(\Omega) + \varepsilon^\alpha)u - \varepsilon^\beta a(x)u^p = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

for $\varepsilon > 0$.

(i) *If $\alpha < \beta$, then*

$$\lim_{\varepsilon\to 0+} u_\varepsilon(x) = \infty \text{ locally uniformly in } \Omega.$$

Further, there exist positive constants c, C such that

$$c \leqslant \liminf_{\varepsilon\to 0+} \varepsilon^{\frac{\beta-\alpha}{p-1}} u_\varepsilon(x) \leqslant \limsup_{\varepsilon\to 0+} \varepsilon^{\frac{\beta-\alpha}{p-1}} u_\varepsilon(x) \leqslant C.$$

(ii) *If $\alpha > \beta$, then*

$$\lim_{\varepsilon\to 0+} u_\varepsilon(x) = 0 \text{ locally uniformly in } \Omega.$$

Further, there exist positive constants c_1, C_1 such that

$$c_1 \leqslant \liminf_{\varepsilon\to 0+} \varepsilon^{\frac{\beta-\alpha}{p-1}} u_\varepsilon(x) \leqslant \limsup_{\varepsilon\to 0+} \varepsilon^{\frac{\beta-\alpha}{p-1}} u_\varepsilon(x) \leqslant C_1.$$

(iii) *If $\alpha = \beta$, then, subject to a subsequence, we have*

$$\lim_{\varepsilon\to 0+} u_\varepsilon(x) = c_0\phi(x) \text{ uniformly in } \bar{\Omega}$$

for some positive constant c_0.

Proof. Set $v_\varepsilon(x) = \varepsilon^{\frac{\beta}{p-1}} u_\varepsilon(x)$, it becomes apparent that $v_\varepsilon(x)$ is the unique positive solution of

$$\begin{cases} \Delta u + (\lambda_1^B(\Omega) + \varepsilon^\alpha)u - a(x)u^p = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega. \end{cases}$$

Let Ω_* be a compact subset of Ω, thanks to lemma 2.1, we know that there exist c_0, C_0 such that

$$c_0\varepsilon^{\frac{\alpha}{p-1}} \leqslant v_\varepsilon(x) \leqslant C_0\varepsilon^{\frac{\alpha}{p-1}}$$

for $x \in \bar{\Omega}_*$. Hence we obtain

$$c_0\varepsilon^{\frac{\alpha-\beta}{p-1}} \leqslant u_\varepsilon(x) \leqslant C_0\varepsilon^{\frac{\alpha-\beta}{p-1}} \tag{2.6}$$

for $x \in \bar{\Omega}_*$. According to (2.6), we obtain the conclusions (i) and (ii).

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By standard interior estimates and (2.6), there exists a positive constant $\tilde{C} = \tilde{C}(\Omega_*)$ such that

$$\|u_\varepsilon\|_{C^{2+\mu}(\bar{\Omega}_*)} \leqslant \tilde{C}.$$

Therefore, by passing to a subsequence and the diagonal argument, there exists $u \in L^2(\Omega)$ such that

$$\lim_{\varepsilon\to 0+} u_\varepsilon(x) = u(x) \text{ weakly in } W^{1,2}(\Omega) \quad \text{and strongly in } L^2(\Omega).$$

Thanks to (2.5), we know that $u(x)$ is a positive weak solution of

$$\begin{cases} \Delta u + \lambda_1^B(\Omega)u = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

By elliptic regularity, it must be a strong solution. By the uniqueness of the positive solution of (2.7), $u(x) = c_0\phi(x)$ for some positive constant c_0. As this argument is independent of the sequence ε, it is apparent from Sobolev imbedding theorem that

$$\lim_{\varepsilon\to 0+} u_\varepsilon(x) = c_0\phi(x) \quad \text{uniformly in } \bar{\Omega}.$$

Thus the proof is completed. □

At the end of this section, we prove the main result theorem 1.1.

Proof of theorem 1.1. We first take $\delta > 0$ such that

$$a(x) > \delta > 0$$

for $x \in \bar{\Omega}$. Then we choose $\varepsilon > 0$ small, denoted by $\varepsilon < \varepsilon_0$ such that

$$a(x) + 1 \geqslant \frac{a_\varepsilon(x)}{\varepsilon^\alpha} \geqslant a(x) - \delta > 0$$

for $x \in \bar{\Omega}$.

Now let $\hat{u}(x)$ be the unique positive solution of

$$\begin{cases} \Delta u + (\lambda_1^B(\Omega) + \varepsilon^\alpha)u - \varepsilon^\beta[a(x) - \delta]u^p = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\bar{u}(x)$ be the unique positive solution of

$$\begin{cases} \Delta u + (\lambda_1^B(\Omega) + \varepsilon^\alpha)u - \varepsilon^\beta[a(x) + 1]u^p = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega \end{cases}$$

for $\varepsilon > 0$, respectively. A simple argument from upper–lower solutions gives

$$0 < \bar{u}(x) \leqslant \theta_\varepsilon(x) \leqslant \hat{u}(x) \tag{2.8}$$

for $x \in \Omega$.

Thus we know from (2.8) and lemma 2.2 that the conclusions (i)–(iii) of theorem 1.1 are true. □

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3. Profiles for nonlocal dispersal logistic equation

In this section, we investigate the limiting behaviour of positive solutions of (1.10) as $\varepsilon \to 0+$. It follows from [**11**, **21**] that there exists a unique positive solution $\omega_\varepsilon \in C(\bar{\Omega})$ to (1.10) for every $\varepsilon > 0$ and θ_ε is continuous with respect to ε. In the rest of this section, for simplicity, we always assume that a_ε, $a \in C(\bar{\Omega})$ are positive in $\bar{\Omega}$ and

$$\lim_{\varepsilon \to 0+} \frac{a_\varepsilon(x)}{\varepsilon^\alpha} = a(x) \text{ uniformly in } \bar{\Omega}.$$

We first give some estimates for the positive solution of

$$\int_\Omega J(x-y)u(y)\,\mathrm{d}y - u(x) + \lambda u - a(x)u^p(x) = 0 \text{ in } \bar{\Omega}. \tag{3.1}$$

The positive solution problem (3.1) has been well investigated, see e.g. [**11**, **20**–**22**].

Lemma 3.1. *Let $\omega_\lambda(x)$ be the unique positive solution of (1.10) for $\lambda \in (\lambda_p(\Omega), \lambda_p(\Omega)+1]$, then there exist positive constants c and C, independent of λ such that*

$$c\left[\frac{\lambda - \lambda_p(\Omega)}{\max_{\bar{\Omega}} a(x)}\right]^{\frac{1}{p-1}} \leqslant \omega_\lambda(x) \leqslant C\left[\frac{\lambda - \lambda_p(\Omega)}{\min_{\bar{\Omega}} a(x)}\right]^{\frac{1}{p-1}}$$

for $x \in \bar{\Omega}$.

Proof. By the uniqueness of positive solution to (3.1), we can find positive constant M, independent of λ such that

$$0 < \max_{\bar{\Omega}} \theta_\lambda(x) \leqslant M.$$

Let $\psi(x)$ be a positive eigenfunction of $\lambda_p(\Omega)$ such that $\|\psi\|_{L^\infty(\Omega)} = 1$. Since $\psi(x) > 0$ for $x \in \bar{\Omega}$, we can take $C_1 > 0$ such that

$$C_1\phi(x) \geqslant \left[\frac{\lambda - \lambda_p(\Omega)}{\min_{\bar{\Omega}} a(x)}\right]^{\frac{1}{p-1}}$$

for $x \in \bar{\Omega}$. Then a direct computation gives that $C_1\phi(x)$ is an upper-solution to (3.1) and we know from the uniqueness of positive solution that

$$\omega_\lambda(x) \leqslant C_1\phi(x)$$

for $x \in \bar{\Omega}$. Hence we obtain

$$\omega_\lambda(x) \leqslant C_1\phi(x) \leqslant C\left[\frac{\lambda - \lambda_p(\Omega)}{\min_{\bar{\Omega}} a(x)}\right]^{\frac{1}{p-1}},$$

by taking $C = [\min_{\bar{\Omega}} \phi(x)]^{-1}$ and

$$C_1 = [\min_{\bar{\Omega}} \phi(x)]\left[\frac{\lambda - \lambda_p(\Omega)}{\min_{\bar{\Omega}} a(x)}\right]^{\frac{1}{p-1}}.$$

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On the other hand, we define

$$v(x) = \left[\frac{\lambda - \lambda_p(\Omega)}{\max_{\bar{\Omega}} a(x)}\right]^{\frac{1}{p-1}} \psi(x).$$

It is easy to see that $v(x)$ is a lower-solution to (3.1). But $\psi(x)$ is independent to λ, it follows from the comparison principle that there exists $c > 0$ such that

$$\omega_\lambda(x) \geqslant c\left[\frac{\lambda - \lambda_p(\Omega)}{\max_{\bar{\Omega}} a(x)}\right]^{\frac{1}{p-1}}$$

for $x \in \bar{\Omega}$. □

Lemma 3.2. *Let $u_\varepsilon \in C(\bar{\Omega})$ be the unique positive solution of*

$$\int_\Omega J(x-y)u(y)\,\mathrm{d}y - u(x) + (\lambda_p(\Omega) + \varepsilon^\alpha)u - \varepsilon^\beta a(x)u^p(x) = 0 \text{ in } \bar{\Omega} \tag{3.2}$$

for $\varepsilon > 0$.

(i) *If $\alpha < \beta$, then*

$$\lim_{\varepsilon \to 0+} u_\varepsilon(x) = \infty \text{ uniformly in } \bar{\Omega}.$$

Further, there exist positive constants c, C such that

$$c \leqslant \liminf_{\varepsilon \to 0+} \varepsilon^{\frac{\beta-\alpha}{p-1}} u_\varepsilon(x) \leqslant \limsup_{\varepsilon \to 0+} \varepsilon^{\frac{\beta-\alpha}{p-1}} u_\varepsilon(x) \leqslant C.$$

(ii) *If $\alpha > \beta$, then*

$$\lim_{\varepsilon \to 0+} u_\varepsilon(x) = 0 \text{ uniformly in } \bar{\Omega}.$$

Further, there exist positive constants c_1, C_1 such that

$$c_1 \leqslant \liminf_{\varepsilon \to 0+} \varepsilon^{\frac{\beta-\alpha}{p-1}} u_\varepsilon(x) \leqslant \limsup_{\varepsilon \to 0+} \varepsilon^{\frac{\beta-\alpha}{p-1}} u_\varepsilon(x) \leqslant C_1.$$

(iii) *If $\alpha = \beta$, then, subject to a subsequence, we have*

$$\lim_{\varepsilon \to 0+} u_\varepsilon(x) = c_0\psi(x) \text{ uniformly in } \bar{\Omega} \tag{3.3}$$

for some positive constant c_0.

Proof. Set $v_\varepsilon(x) = \varepsilon^{\frac{\beta}{p-1}} u_\varepsilon(x)$, then we can see that $v_\varepsilon(x)$ is the unique positive solution of

$$\int_\Omega J(x-y)u(y)\,\mathrm{d}y - u(x) + (\lambda_p(\Omega) + \varepsilon^\alpha)u - a(x)u^p(x) = 0 \text{ in } \bar{\Omega}.$$

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Then we know from lemma 3.1 that there exist c_0, C_0 such that

$$c_0\varepsilon^{\frac{\alpha}{p-1}} \leqslant v_\varepsilon(x) \leqslant C_0\varepsilon^{\frac{\alpha}{p-1}}$$

for $x \in \bar{\Omega}$. Hence we obtain

$$c_0\varepsilon^{\frac{\alpha-\beta}{p-1}} \leqslant u_\varepsilon(x) \leqslant C_0\varepsilon^{\frac{\alpha-\beta}{p-1}}$$

for $x \in \bar{\Omega}$ and the conclusions (i) and (ii) are followed.

At last, we prove (3.3). In this case, we still have

$$c_0 \leqslant u_\varepsilon(x) \leqslant C_0$$

for $x \in \bar{\Omega}$. Since $\lambda_p(\Omega) \in (0, 1)$ and

$$\left[1 - \lambda_p(\Omega) - \varepsilon^\alpha + \varepsilon^\beta a(x)(u_\varepsilon(x))^{p-1}\right] u_\varepsilon(x) = \int_\Omega J(x-y)u_\varepsilon(y)\,\mathrm{d}y \text{ in } \bar{\Omega},$$

we know that there exists $\rho > 0$ which is independent to ε such that

$$1 - \lambda_p(\Omega) - \varepsilon^\alpha + \varepsilon^\beta a(x)(u_\varepsilon(x))^{p-1} \geqslant \rho \tag{3.4}$$

for $x \in \bar{\Omega}$, provided $\varepsilon \in (0, 1)$ is small. Then for any x_1, $x_2 \in \bar{\Omega}$, without loss of generality, we may assume that $u_\varepsilon(x_1) > u_\varepsilon(x_2)$. A direct computation from (3.2)–(3.4) shows that

$$\begin{aligned}
&(1 - \lambda_p(\Omega) - \varepsilon^\alpha + p\varepsilon^\beta a(x_2)\theta_\varepsilon^{p-1})[u_\varepsilon(x_1) - u_\varepsilon(x_2)] \\
&\quad = \int_\Omega (J(x_1, y) - J(x_2, y))u_\varepsilon(y)\,\mathrm{d}y + \varepsilon^\beta(a(x_2) - a(x_1))u_\varepsilon^p(x_1) \\
&\quad \leqslant C_0 \int_\Omega |J(x_1, y) - J(x_2, y)|\,\mathrm{d}y + C_0^p|(a(x_2) - a(x_1))|,
\end{aligned}$$

here θ_ε is between $u_\varepsilon(x_2)$ and $u_\varepsilon(x_1)$. Thus we obtain that

$$|u_\varepsilon(x_1) - u_\varepsilon(x_2)| \leqslant \frac{C_0 \int_\Omega |J(x_1, y) - J(x_2, y)|\,\mathrm{d}y + C_0^p|(a(x_2) - a(x_1))|}{\rho} \tag{3.5}$$

for x_1, $x_2 \in \bar{\Omega}$. It follows from (3.5) and a compact argument that we can extract a subsequence still denoted by ε and there exists positive function $V \in C(\bar{\Omega})$ such that

$$\lim_{\varepsilon \to 0+} u_\varepsilon(x) = V(x) \quad \text{uniformly in } \bar{\Omega},$$

and

$$\int_\Omega J(x-y)V(y)\,\mathrm{d}y - V(x) + \lambda_p(\Omega)V(x) = 0 \text{ in } \bar{\Omega}. \tag{3.6}$$

Note that $\lambda_p(\Omega)$ is the unique principal eigenvalue of (3.6), we know that (3.3) holds. □

We are ready to prove the main result theorem 1.3.

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Proof of theorem 1.3. We first take $\delta > 0$ such that

$$a(x) > \delta > 0$$

for $x \in \bar{\Omega}$. Then we can choose $\varepsilon > 0$ small, denoted by $\varepsilon < \varepsilon_0$ such that

$$a(x) + 1 \geqslant \frac{a_\varepsilon(x)}{\varepsilon^\alpha} \geqslant a(x) - \delta > 0$$

for $x \in \bar{\Omega}$. Let $\hat{u}(x)$ be the unique positive solution of

$$\int_\Omega J(x-y)u(y)\,\mathrm{d}y - u(x) + (\lambda_p(\Omega) + \varepsilon^\alpha)u - \varepsilon^\beta[a(x) - \delta]u^p = 0 \text{ in } \bar{\Omega},$$

and $\bar{u}(x)$ be the unique positive solution of

$$\int_\Omega J(x-y)u(y)\,\mathrm{d}y - u(x) + (\lambda_p(\Omega) + \varepsilon^\alpha)u - \varepsilon^\beta[a(x) + 1]u^p = 0 \text{ in } \bar{\Omega}$$

for $\varepsilon > 0$, respectively. Thus we get from the comparison principle that

$$0 < \bar{u}(x) \leqslant \omega_\varepsilon(x) \leqslant \hat{u}(x)$$

for $x \in \bar{\Omega}$.

The conclusions (i)–(iii) of theorem 1.3 are followed by lemma 3.2. □

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