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ORLICZ–POINCARÉ INEQUALITIES

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Abstract Corresponding to known results on Orlicz–Sobolev inequalities which are stronger than the Poincaré inequality, this paper studies the weaker Orlicz–Poincaré inequality. More precisely, for any Young function Φ whose growth is slower than quadric, the Orlicz–Poincaré inequality

$$\|f\|_\Phi^2 \leqslant C\mathcal{E}(f,f), \qquad \mu(f) := \int f \,\mathrm{d}\mu = 0$$

is studied by using the well-developed weak Poincaré inequalities, where $\mathcal{E}$ is a conservative Dirichlet form on $L^2(\mu)$ for some probability measure μ. In particular, criteria and concrete sharp examples of this inequality are presented for $\Phi(r) = r^p$ ($p \in [1,2)$) and $\Phi(r) = r^2 \log^{-\delta}(\mathrm{e} + r^2)$ ($\delta > 0$). Concentration of measures and analogous results for non-conservative Dirichlet forms are also obtained. As an application, the convergence rate of porous media equations is described.

Keywords: Orlicz–Poincaré inequality; weak Poincaré inequality; Poincaré inequality

2000 *Mathematics subject classification:* Primary 60H10; 47D07

1. Introduction

Let $(E,\mathcal{F},\mu)$ be a probability space and let $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ be a conservative Dirichlet form on $L^2(\mu)$. The well-known Poincaré inequality is

$$\mu(f^2) \leqslant C\mathcal{E}(f,f), \quad \mu(f) = 0,\ f \in \mathcal{D}(\mathcal{E}), \tag{1.1}$$

where $\mu(f) := \int_E f\,\mathrm{d}\mu$ and $C > 0$ is a constant that provides a lower bound of the spectral gap.

In recent years, a stronger version of (1.1), using Orlicz norms in place of the L^1-norm of f^2, was studied intensively (see [**14**] and references therein). More precisely, for a Young function Φ, i.e. $\Phi \in C(\mathbb{R})$ is convex, even and positive such that $\Phi(s) = 0$ if and only if $s = 0$, $\Phi(s)/s \to 0$ as $s \to 0$, and $\Phi(s)/|s| \to \infty$ as $|s| \to \infty$, we consider the inequality

$$\|f^2\|_\Phi \leqslant C\mathcal{E}(f,f), \quad \mu(f) = 0,\ f \in \mathcal{D}(\mathcal{E}), \tag{1.2}$$

where $\|\cdot\|_\Phi$ is the Orlicz norm induced by Φ as follows:

$$\|f\|_\Phi := \inf\{\lambda > 0 : \mu(\Phi(f/\lambda)) \leqslant 1\}, \quad \inf\emptyset := \infty.$$

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Since $\Phi(r)/r \to \infty$ as $r \to \infty$, this inequality is stronger than the Poincaré inequality. In particular, if $\Phi(x) := |x|^p$ for $p > 1$, then (1.2) is the well-known Sobolev inequality, while for $\Phi(x) := |x| \log(1 + |x|)$ it corresponds to Gross's log-Sobolev inequality introduced in [**10**] (see [**2**]). Owing to these facts, (1.2) was called the (Φ-)Orlicz–Sobolev inequality in [**14**].

In general, (1.2) can be described by the following F-Sobolev inequality (see [**16, 17**])

$$\mu(f^2 F(f^2)) \leqslant C_1 \mathcal{E}(f, f) + C_2, \quad \mu(f^2) = 1,\ f \in \mathcal{D}(\mathcal{E}), \tag{1.3}$$

for a proper function F with $F(r) \uparrow \infty$ as $r \uparrow \infty$. For instance, for any $\theta \in (0, 1]$, (1.2) with $\Phi(x) = |x| \log^\theta(1 + |x|)$ holds if and only if (1.3) with $F(r) = r \log^\theta(1 + r)$ and (1.1) hold (see [**2**] for $\theta = 1$ and [**18, 21**] for $\theta \in (0, 1)$). Moreover, in [**20**] (1.3) is related to a class of inequalities which interpolate between the Poincaré inequality and the log-Sobolev inequality (see also [**4, 11**]). Therefore, in some cases (1.2) can be described by known results on (1.3) (or equivalent versions); see [**21**] and references therein. In particular, various criteria for (1.2) to hold have been addressed in [**5, 6**] for one-dimensional situations.

Since the Orlicz–Sobolev inequality (1.2) has been comprehensively studied in [**14**], we do not investigate this further here. As a supplement to the study of (1.2), we consider here the following Orlicz–Poincaré inequality:

$$\|f\|_\Phi^2 \leqslant C\mathcal{E}(f, f), \quad \mu(f) = 0,\ f \in \mathcal{D}(\mathcal{E}), \tag{1.4}$$

which has not yet been studied for Φ growing slower than quadric. Since for Φ growing faster than quadric this inequality may be reduced to (1.2) by replacing $\Phi(s)$ by $\Phi(\sqrt{s})$ (see [**14**]), we consider only the case where $\Phi(|x|)/|x|^2 \to 0$ as $|x| \to \infty$. In particular, for $\Phi(r) = r^p$ with $p \in [1, 2)$, this becomes

$$\mu(|f|^p)^{2/p} \leqslant C\mathcal{E}(f, f), \quad \mu(f) = 0,\ f \in \mathcal{D}(\mathcal{E}). \tag{1.5}$$

To distinguish (1.4) from the stronger inequality (1.2), we call it the (Φ-)Orlicz–Poincaré inequality.

Since we are restricted to (1.4), which is weaker than the Poincaré inequality (1.1), it should be reasonable to adopt the following weak Poincaré inequality introduced in [**15**]:

$$\mu(f^2) \leqslant \alpha(r)\mathcal{E}(f, f) + r\|f\|_\infty^2, \quad r > 0, \qquad \mu(f) = 0, \tag{1.6}$$

where $\alpha : (0, \infty) \to (0, \infty)$ is a decreasing function which describes the convergence rate of the associated Markov semigroup P_t (see [**15**] for details). Since (1.6) is easy to check in applications and is powerful in the study of long-time behaviours of P_t, for the study of (1.4) it is useful to make a connection to (1.6).

In fact, to derive (1.4) from (1.6), we do not need the fact that Φ is a Young function but rather we use the following assumption:

(Φ) $\Phi(\cdot)$ is continuous, even and strictly increasing in $|\cdot|$, $\Phi(0) = 0$, $s^2/\Phi(s) \uparrow \infty$ as $|s| \uparrow \infty$, and $\liminf_{s\to\infty} \Phi(s)/s > 0$.

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Remark 1.1. The condition $s^2/\Phi(s) \uparrow \infty$ as $|s| \uparrow \infty$ is to make (1.4) weaker than (1.1) for the purpose explained above. Owing to the restriction that $\mu(f) = 0$, which requires *a priori* that $f \in L^1(\mu)$, we are only able to treat (1.4) for Φ with growth not slower than linear (see the proof of Theorem 1.3). This is ensured by the condition that $\liminf_{|x|\to\infty} \Phi(x)/|x| > 0$. We note that the other restriction, $f \in \mathcal{D}(\mathcal{E})$ in (1.4), can be replaced by $f \in \mathcal{D}(\mathcal{E}_{\mathrm{e}})$, the extended domain of the Dirichlet form (see [**8**]), which is not necessarily included by $L^1(\mu)$.

Remark 1.2. To study the Orlicz–Poincaré inequality for Φ with growth slower than linear, we must drop the restriction $\mu(f) = 0$. In this case the Dirichlet form is non-conservative, i.e. either $1 \notin \mathcal{D}(\mathcal{E})$ or $\mathcal{E}(1,1) > 0$. We return to this situation in § 4.

Theorem 1.3. *Assume that* (Φ) *holds.*

(1) *If there exists* $K > 0$ *such that*

$$\sum_{n=1}^{\infty} \alpha^{-1}(K\Phi(2^{n+2})^{-1}4^n)\Phi(2^{n+2}) < \infty, \tag{1.7}$$

then (1.6) implies (1.4) for some constant $C > 0$.

(2) *If (1.4) holds, then (1.6) holds for* $\alpha(r) = Cr\Phi^{-1}(r^{-1})^2$.

To illustrate Theorem 1.3, we consider below two specific cases, where (1.6) holds with either $\alpha(r) = cr^{-\theta}$ or $\alpha(r) = c\log^{\theta}(1 + r^{-1})$ for some $c, \theta > 0$. According to [**15**, Corollary 2.4], these two situations correspond to the algebraic and the sub-exponential convergence of P_t, respectively.

Corollary 1.4.

(1) *Let* $\delta > 0$. *Then (1.6) holds for* $\alpha(r) = c\log^{\delta}(1 + r^{-1})$ *for some* $c > 0$ *if and only if (1.4) holds for* $\Phi(r) = r^2 \log^{-\delta}(\mathrm{e} + r^2)$ *and some* $C > 0$. *Both inequalities are equivalent to the sub-exponential convergence*

$$\|P_t - \mu\|_{\infty\to 2} \leqslant c_1 \exp[-c_2 t^{1/(1+\delta)}], \quad t > 0,$$

for some $c_1, c_2 > 0$, *where* $\|\cdot\|_{\infty\to 2}$ *is the operator norm from* $L^{\infty}(\mu)$ *to* $L^2(\mu)$.

(2) *Let* $\theta \in [0, 1]$. *Then (1.6) with* $\alpha(r) = cr^{-\theta}$, *which is equivalent to* $\|P_t - \mu\|_{\infty\to 2} \leqslant c' t^{-1/\theta}$ *for some* $c' > 0$ *and all* $t > 0$, *implies (1.5) for any* $p \in [1, 2/(1+\theta))$ *with some* C *depending on* p. *On the other hand, (1.5) with* $p = 2/(1+\theta)$ *implies (1.6) with* $\alpha(r) = cr^{-\theta}$ *for some* $c > 0$.

The above results will be proved in the next section. To check their sharpness, necessary conditions of (1.4) through concentrations of μ are addressed in § 3. Section 4 contains analogous results for non-conservative Dirichlet forms. Finally, the convergence rate of porous media equations is described in § 5 by using (1.5).

To illustrate our results, we present two specific examples. In particular, the first example indicates that, in general, (1.6) with $\alpha(r) = cr^{-\theta}$ does not imply (1.4) with $\Phi(r) = r^{2/(\theta+1)}$. Thus, the statement in Theorem 1.3 (2) is somewhat optimal.

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Example 1.5. Let $E=\mathbb{R}$ and $\mu(\mathrm{d}x)=Z^{-1}(1+|x|)^{-(1+\theta)}\,\mathrm{d}x$, where $\sigma>0$ and Z is the normalization. Let $\mathcal{E}(f,g)=\mu(f'g')$ with $\mathcal{D}(\mathcal{E})=W^{2,1}(\mu)$. Then (1.6) holds for $\alpha(r)=cr^{-2/\sigma}$ for some $c>0$. Thus, by Corollary 1.4 (2), (1.5) holds for $p\in[1,2\sigma/(\sigma+2))$. However, according to Proposition 3.5 and Remark 3.2, with $\rho=|\cdot|$, for any $C>0$, (1.5) does not hold for $p=2\sigma/(\sigma+2)$.

Example 1.6. Let $E=\mathbb{R}^d$ and $\mu(\mathrm{d}x)=Z^{-1}\mathrm{e}^{-|x|^\sigma}\,\mathrm{d}x$, where $\sigma>0$ and Z is the normalization. Let $\mathcal{E}(f,g)=\mu(\langle\nabla f,\nabla g\rangle)$ with $\mathcal{D}(\mathcal{E})=W^{2,1}(\mu)$. It is well known that (1.1) holds provided that $\sigma\geqslant 1$. If $\sigma\in(0,1)$, then (1.4) holds with

$$\Phi(x)=|x|^2\log^{-4(1-\sigma)/\sigma}(\mathrm{e}+|x|^2)$$

and some $C>0$, i.e.

$$\mu(f^2\log^{-4(1-\sigma)/\sigma}(\mathrm{e}+f^2/\|f\|_\Phi^2))\leqslant C\mu(|\nabla f|^2),\quad \mu(f)=0,\ f\in\mathcal{D}(\mathcal{E}).$$

According to Proposition 3.4, this inequality is sharp in the sense that (1.4) does not hold for $\Phi(x)=|x|^2\log^{-\delta}(\mathrm{e}+|x|^2)$ if $\delta<4(1-\sigma)/\sigma$.

Next, we consider the corresponding examples in the context of birth–death processes. Let $E=\mathbb{Z}_+$ and let $\mu=(\mu_i>0)_{i\geqslant 0}$ be a probability measure on E. Let $a_i,b_i\geqslant 0$ satisfy $\mu_i b_i=\mu_{i+1}a_{i+1}$, $i\geqslant 0$. Then the Dirichlet form for the birth–death process with birth rate b_i and death rate a_i is

$$\mathcal{E}(f,g)=\sum_{i=0}^{\infty}(f_{i+1}-f_i)(g_{i+1}-g_i)b_i\mu_i$$

for $f,g\in\mathcal{D}(\mathcal{E}):=\{f\in L^2(\mu):\mathcal{E}(f,f)<\infty\}$.

Example 1.7. Let $a_i=b_i>0$ for $i\geqslant 1$ and $a_0=0$, $b_0=1$. We have $\mu_i=a_i^{-1}\mu_0$, $i\geqslant 1$.

(1) Let $a_i=i^\delta$, $i\geqslant 1$, for some $\delta\in(1,2)$. By [**15**, Example 5.5 (1)], (1.6) holds for $\alpha(r)=cr^{-(2-\delta)/(\delta-1)}$ for some $c>0$. Thus, by Corollary 1.4 (2), if $\delta\in(\frac{3}{2},2)$, then for any $p\in[1,2(\delta-1))$, (1.5) holds for some $C=C(p)>0$. But, as in Example 1.5, for any $C>0$, (1.4) does not hold for $p=2(\delta-1)$. To see this from Proposition 3.5, we define

$$\rho_0=0,\qquad \rho_1=\sqrt{2},\qquad \rho_{i+1}=\sqrt{2}+\sum_{k=1}^{i}a_{k+1}^{-1/2},\quad i\geqslant 1. \tag{1.8}$$

By Remark 3.3,

$$L_{\mathcal{E}}(\rho)^2=\tfrac{1}{2}\sup_{i\geqslant 0}\{b_i(\rho_{i+1}-\rho_i)^2+a_i(\rho_i-\rho_{i-1})^2\}=1$$

provided that a_i is increasing in i. Obviously, in the present case $\rho_i=O(i^{(2-\delta)/2})$ as $i\to\infty$. Therefore, $\mu(\rho^\varepsilon)<\infty$ for small $\varepsilon>0$. By Proposition 3.5, if (1.4) holds

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for $p = 2(\delta - 1)$, then $\mu(\rho^{2p/(2-p)}) < \infty$. But

$$\mu(\rho^{2p/(2-p)}) = \sum_{i=0}^{\infty} \mu_i \rho_i^{2(\delta-1)/(2-\delta)} \geqslant c_1 \sum_{i=1}^{\infty} i^{-1} = \infty$$

for some $c_1 > 0$, so (1.4) does not hold for $p = 2(\delta - 1)$.

(2) Let $a_i = i^2 \log^{-\delta}(1+i)$, $i \geqslant 1$, for some $\delta > 0$. By [**15**, Example 5.5 (3)], (1.6) holds for $\alpha(r) = c\log^{\delta}(1 + r^{-1})$ for some $c > 0$. Thus, according to Corollary 1.4 (1), (1.4) holds for $\Phi(r) = r^2 \log^{-\delta}(\mathrm{e} + r^2)$. This Φ is sharp since, for any $p < \delta$, (1.4) does not hold for $\Phi(r) = r^2 \log^{-p}(\mathrm{e} + r^2)$. Indeed, for ρ defined in (1.8) we have $\rho_i \geqslant c_1 \log^{(\delta+2)/2}(1 + i)$ for some $c_1 > 0$ and all $i \geqslant 1$. By Proposition 3.4, (1.4) with the above Φ for $p < \delta$ implies that, for some $\lambda > 0$,

$$\begin{aligned}\infty &> \sum_{i=1}^{\infty} \mu_i \exp[\lambda \log^{(\delta+2)/(p+2)}(1 + i)] \\ &= \sum_{i=1}^{\infty} i^{-2} \log^{\delta}(1 + i) \exp[\lambda \log^{(\delta+2)/(p+2)}(1 + i)],\end{aligned}$$

which is, however, infinite since $(\delta + 2)/(p + 2) > 1$.

2. Proofs of Theorem 1.3 and Corollary 1.4

To derive (1.4) from (1.6), we shall adopt a cut-off argument to control the term $\|f\|_\infty$. To this end, we need the following lemma (see [**9**, Lemma 2.2] or the proof of [**12**, Proposition I.4.11]).

Lemma 2.1. *Let $(E, \mathcal{F}, \mu)$ be a (not necessarily finite) measure space and let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Dirichlet form on $L^2(\mu)$. For any $f \in \mathcal{D}(\mathcal{E})$ and a sequence $\{f_n\}_{n\geqslant1} \subset L^2(\mu)$ such that*

$$\sum_{n=1}^{n} |f_n(x) - f_n(y)| \leqslant |f(x) - f(y)|, \quad \sum_{n=1}^{\infty} |f_n| \leqslant |f|, \quad x, y \in E,$$

we have $f_n \in \mathcal{D}(\mathcal{E})$ for all $n \geqslant 1$ and $\sum_{n=1}^{\infty} \mathcal{E}(f_n, f_n) \leqslant \mathcal{E}(f, f)$.

Proof of Theorem 1.3 (1). (i) For any $f \in \mathcal{D}(\mathcal{E})$, define

$$f_n = \operatorname{sgn}(f)\{(|f| - 2^n)^+ \wedge 2^n\}, \quad \bar{f}_n = (f \wedge 2^n) \vee (-2^n), \quad n \geqslant 1.$$

It is easy to see that, for every $N \geqslant 1$,

$$\bar{f}_N + \sum_{n=N}^{\infty} f_n = f. \tag{2.1}$$

We only verify this at points where $f \geqslant 0$ (otherwise, we simply use $-f$ in place of f). If $f \leqslant 2^N$, then the equation is trivial. If $f \in (2^m, 2^{m+1}]$ for some $m \geqslant N$, then the desired

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equation becomes

$$2^N + \sum_{n=N}^{m-1} 2^n + f - 2^m = f,$$

which is also trivial. Next, for any $x, y \in E$ with $f(x) \geqslant f(y)$, we have $f_n(x) \geqslant f_n(y)$ and $\bar{f}_n(x) \geqslant \bar{f}_n(y)$, $n \geqslant 1$. Then (2.1) implies that

$$\begin{aligned}|\bar{f}_N(x) - \bar{f}_N(y)| + \sum_{n=N}^{\infty} |f_n(x) - f_n(y)| &= \bar{f}_N(x) - \bar{f}_N(y) + \sum_{n=N}^{\infty} (f_n(x) - f_n(y)) \\ &= |f(x) - f(y)|.\end{aligned}$$

Therefore, by Lemma 2.1 we obtain

$$\mathcal{E}(\bar{f}_N, \bar{f}_N) + \sum_{n=N}^{\infty} \mathcal{E}(f_n, f_n) \leqslant \mathcal{E}(f, f). \tag{2.2}$$

(ii) We claim that there exists $c_1 > 0$ and a sequence $\{C_N > 0\}_{N \geqslant 1}$ such that

$$\mu(\Phi(\bar{f}_N)) \leqslant \tfrac{1}{4} + C_N \mathcal{E}(\bar{f}_N, \bar{f}_N) + c_1 \sum_{n=N-1}^{\infty} 4^{-n} \Phi(2^{n+2}) \mu(f_n^2) \tag{2.3}$$

holds for all $N \geqslant 1$ and $f \in \mathcal{D}(\mathcal{E})$ with $\mu(f) = 0$, $\|f\|_\Phi = 1$. Firstly, by (Φ) there exists $c_2 > 0$ such that

$$\Phi(f) \leqslant \tfrac{1}{4} + c_2 f^2.$$

Then (1.6) implies that

$$\mu(\Phi(\bar{f}_N)) \leqslant \tfrac{1}{4} + c_2 \alpha(r) \mathcal{E}(\bar{f}_N, \bar{f}_N) + c_2 r 4^N + c_2 \mu(\bar{f}_N)^2, \quad r > 0. \tag{2.4}$$

Next, let $c_3 > 0$ be such that $\Phi(s) \geqslant c_3 s$ for $s \geqslant 2$. Since $\mu(f) = 0$ and $\mu(\Phi(f)) = 1$, we obtain

$$\mu(\bar{f}_N)^2 = \mu(\operatorname{sgn}(f)(|f| - 2^N)^+)^2 \leqslant c_3 \mu(\Phi(f) \mathbf{1}_{\{|f| > 2^N\}}).$$

Therefore, to prove (2.3), it remains to note that

$$\Phi(f) \mathbf{1}_{\{|f| > 2^N\}} \leqslant \sum_{n=N-1}^{\infty} 4^{-n} \Phi(2^{n+2}) \mu(f_n^2).$$

This follows from the fact that

$$\Phi(f) \leqslant \Phi(2^{m+1}) \leqslant 4^{1-m} \Phi(2^{m+1}) f_{m-1}^2 \quad \text{if } |f| \in (2^m, 2^{m+1}]. \tag{2.5}$$

(iii) We now consider f_n. For any $n \geqslant 1$, (1.6) implies that

$$\begin{aligned}\mu(f_n^2) &\leqslant \alpha(r) \mathcal{E}(f_n, f_n) + r 4^n + \mu(f_n)^2 \\ &\leqslant \alpha(r) \mathcal{E}(f_n, f_n) + r 4^n + \mu(f_n^2) \mu(|f| > 2^n) \\ &\leqslant \alpha(r) \mathcal{E}(f_n, f_n) + r 4^n + \mu(f_n^2) \Phi(2^n)^{-1}, \quad r > 0.\end{aligned}$$

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Let $N_0 \geqslant 1$ be such that $\Phi(2^{N_0}) \geqslant 2$. We obtain

$$\mu(f_n^2) \leqslant 2\alpha(r)\mathcal{E}(f_n, f_n) + 2r4^n, \quad r > 0,\ n \geqslant N_0. \tag{2.6}$$

Next, by (2.5) we have

$$\Phi(f) \leqslant \Phi(\bar{f}_N) + \sum_{n=N-1}^{\infty} 4^{-n}\Phi(2^{n+2})\mu(f_n^2).$$

Combining this with (2.3) and (2.6) and noting that $\mu(\Phi(f)) = 1$, for every $N \geqslant N_0 + 1$ we obtain that

$$1 \leqslant \tfrac{1}{4} + C_N\mathcal{E}(\bar{f}_N, \bar{f}_N) + 2(1 + c_1)\sum_{n=N-1}^{\infty}\{\alpha(r_n)4^{-n}\Phi(2^{n+2})\mathcal{E}(f_n, f_n) + r_n\Phi(2^{n+2})\},$$

where $r_n > 0$ for $n \geqslant N - 1$. Taking $r_n = \alpha^{-1}(K4^n\Phi(2^{n+2})^{-1})$, we arrive at

$$\begin{aligned} 1 \leqslant \tfrac{1}{4} + C_N\mathcal{E}(\bar{f}_N, \bar{f}_N) + 2(1 + c_1)K &\sum_{n=N-1}^{\infty}\mathcal{E}(f_n, f_n) \\ &+ 2(1 + c_1)\sum_{n=N-1}^{\infty}\Phi(2^{n+2})\alpha^{-1}(K4^n\Phi(2^{n+2})^{-1}). \end{aligned}$$

Combining this with (2.2) and noting that (1.7) implies that

$$\lim_{N\to\infty}\sum_{n=N-1}^{\infty}\Phi(2^{n+2})\alpha^{-1}(K4^n\Phi(2^{n+2})^{-1}) = 0,$$

we may find a sufficiently large N such that

$$1 \leqslant \tfrac{1}{2} + (C_N + 2 + 2c_1)\mathcal{E}(f, f), \quad \mu(f) = 0,\ \|f\|_\Phi = 1,\ f \in \mathcal{D}(\mathcal{E}).$$

This implies (1.4) for $C = 2C_N + 4 + 4c_1$. □

Proof of Theorem 1.3 (2). Let $f \in \mathcal{D}(\mathcal{E})$ be such that $\mu(f) = 0$ and $\mu(\Phi(f)) = 1$. Let $t := \mu(f^2) > 0$. Since $r^2\Phi(r)^{-1}$ is increasing in $r > 0$, if $\|f\|_\infty^2 \leqslant t/r$, then

$$t = \mu(f^2) \leqslant \mu(\Phi(f))\left\|\frac{f^2}{\Phi(f)}\right\|_\infty \leqslant \frac{t}{r\Phi(\sqrt{t/r})}.$$

This implies that

$$\mu(f^2) \leqslant r\Phi^{-1}(1/r)^2\|f\|_\Phi^2 + r\|f\|_\infty^2, \quad \mu(f) = 0,\ f \in \mathcal{D}(\mathcal{E}).$$

The proof is completed by combining this with (1.4). □

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Remark 2.2. In general, for any sequence $\delta_n \uparrow \infty$ as $n \uparrow \infty$, by repeating the proof of Theorem 1.3 with

$$f_n := \operatorname{sgn}(f)\{(|f| - \delta_n)^+ \wedge (\delta_{n+1} - \delta_n)\}, \quad n \geqslant 1,$$

one may obtain a corresponding sufficient condition generalizing (1.7) for (1.4) to hold. Here we take $\delta_n := 2^n$, since it makes representations much simpler and, more importantly, the resulting estimates of Φ are sharp, as illustrated by the examples in § 1.

Proof of Corollary 1.4. We prove only (1), since the proof of (2) is similar and simpler. Moreover, since the claimed equivalence between the convergence of P_t and (1.6) is included in [**15**, Corollary 2.4], we need only to prove the equivalence of (1.4) and (1.6) for the specific $\alpha(r) := c\log^{\delta}(1 + r^{-1})$ and $\Phi(r) := r^2 \log^{-\delta}(\mathrm{e} + r^2)$. For any $K > 0$,

$$\begin{aligned}\alpha^{-1}(K\Phi(2^{n+2})^{-1}4^n)\Phi(2^{n+2}) &= \alpha^{-1}(K4^{-1}\log^{\delta}(\mathrm{e} + 4^{n+2}))4^{n+2}\log^{-\delta}(\mathrm{e} + 4^{n+2}) \\ &\leqslant \frac{4^{n+2}}{\exp[(K/4c)^{1/\delta}\log(\mathrm{e} + 4^{n+2})] - 1}.\end{aligned}$$

Therefore, for any $K > 4c$, (1.7) holds and, hence, (1.6) implies (1.4).

On the other hand, if (1.4) holds, then Theorem 1.3 (2) implies (1.6) for

$$\alpha(r) = Cr\Phi^{-1}(r^{-1})^2 \leqslant c\log^{\delta}(1 + r^{-1})$$

for some $c > 0$ and all $r \in (0, 1]$. The proof is thus complete, since (1.6) is trivial for $r \geqslant 1$ and any $\alpha(r) \geqslant 0$. □

3. Concentration of measure μ

In this section we aim to derive concentration estimates of μ from the inequality (1.4) (or (1.5)). To this end, as in [**1**], we first introduce the class of distance-like reference functions.

Definition 3.1. Let $(E, \mathcal{F}, \mu)$ be a probability space and let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a conservative symmetric Dirichlet form on $L^2(\mu)$. Let

$$\mathcal{D}(\boldsymbol{L}_{\mathcal{E}}) := \{f \in \mathcal{F} : f_n := (f \wedge n) \vee (-n) \in \mathcal{D}(\mathcal{E}),\ n \geqslant 1\}.$$

For $f \in \mathcal{D}(\boldsymbol{L}_{\mathcal{E}})$, the following quantity is called the *$\mathcal{E}$-Lipschitz constant* of f:

$$\boldsymbol{L}_{\mathcal{E}}(f) := \varliminf_{n \to \infty} \sup\{\mathcal{E}(f_n g, f_n) - \tfrac{1}{2}\mathcal{E}(f_n^2, g) : g \in \mathcal{D}(\mathcal{E}),\ \mu(|g|) \leqslant 1\}^{1/2}.$$

Remark 3.2. Let E be a complete Riemannian manifold with μ a probability measure equivalent to the volume measure $\mathrm{d}x$. Let

$$\mathcal{E}(f, g) := \mu(\langle \nabla f, \nabla g \rangle), \quad f, g \in W^{2,1}(\mu).$$

Then $\boldsymbol{L}_{\mathcal{E}}(f) = \|\nabla f\|_\infty = \operatorname{Lip}(f)$ for any Lipschitz continuous function f (see [**21**, Proposition 1.2.2]).

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Remark 3.3. Let $q \in L^1(\mu \times \mu)$ be non-negative such that $q(x,y) = q(y,x)$ for $x, y \in E$. Define

$$\mathcal{E}(f,g) := \frac{1}{2}\int_{E\times E}(f(x)-f(y))(g(x)-g(y))q(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y),$$
$$\mathcal{D}(\mathcal{E}) := \{f \in L^2(\mu) : \mathcal{E}(f,f) < \infty\}.$$

Then $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a symmetric Dirichlet form and $\mathcal{D}(\boldsymbol{L}_{\mathcal{E}})$ coincides with the class of measurable functions on E. Moreover, for $f \in \mathcal{D}(\boldsymbol{L}_{\mathcal{E}})$ one has (see [**21**, Proposition 1.2.3])

$$\boldsymbol{L}_{\mathcal{E}}(f)^2 = \tfrac{1}{2}\operatorname*{ess\,sup}_{\mu}\sup_{x}\int |f(x)-f(y)|^2 q(x,y)\mu(\mathrm{d}y).$$

To study the concentration of μ in terms of ρ with $L_{\mathcal{E}}(\rho) \leqslant 1$, we make use of the following assumption instead of (Φ):

(Φ') Φ is convex, continuous and strictly increasing, $\Phi(0) = 0$, $\lim_{s\to\infty}\Phi(s) = \infty$ and $\lim_{s\to\infty}\Phi(s)^{-1}s^2 = \infty$.

Proposition 3.4. *Assume that (Φ') holds and let*

$$\Psi(r) = \int_r^1 \Phi^{-1}(2s^{-1})^2\,\mathrm{d}s, \quad r \in (0,1).$$

Then, for any $\rho \geqslant 0$ with $\boldsymbol{L}_{\mathcal{E}}(\rho) \leqslant 1$, (1.4) implies that

$$\mu(\rho \geqslant t) \leqslant \Psi^{-1}(ct), \quad t \geqslant N,$$

for some $c, N > 0$. Consequently, if $\Phi(s) = s^2\log^{-\delta}(\mathrm{e}+s^2)$ for some $\delta > 0$, then

$$\mu(\rho \geqslant t) \leqslant \exp[-\lambda t^{2/(2+\delta)}]$$

for some $\lambda > 0$ and all sufficiently large t.

Proof. (i) For any $t, r > 0$, let $\rho_{t,r} = (\rho - t)^+ \wedge r$. We have (see [**1**] or [**21**, Lemma 1.2.4])

$$\mathcal{E}(\rho_{t,r}, \rho_{t,r}) \leqslant 2\mu(t \leqslant \rho \leqslant t+r), \quad t, r > 0.$$

Thus, it follows from (1.4) that

$$\|\rho_{t,r} - \mu(\rho_{t,r})\|_{\Phi}^2 \leqslant 2C\mu(t \leqslant \rho \leqslant t+r).$$

On the other hand, if $fg = 0$, then

$$\mu\biggl(\Phi\biggl(\frac{f+g}{\lambda}\biggr)\biggr) = \mu\biggl(\Phi\biggl(\frac{f}{\lambda}\biggr)\biggr) + \mu\biggl(\Phi\biggl(\frac{g}{\lambda}\biggr)\biggr), \quad \lambda > 0.$$

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We obtain $\|f+g\|_\Phi \geqslant \|f\|_\Phi \vee \|g\|_\Phi$. Moreover, the convexity of Φ implies the triangle inequality for $\|\cdot\|_\Phi$. Hence,

$$\begin{aligned}\|\rho_{t,r}-\mu(\rho_{t,r})\|_\Phi &= \|\mu(\rho_{t,r})\mathbf{1}_{\{\rho\leqslant t\}} + (\rho_{t,r}-\mu(\rho_{t,r}))\mathbf{1}_{\{\rho>t\}}\|_\Phi \\ &\geqslant \tfrac{1}{2}(\|\mu(\rho_{t,r})\mathbf{1}_{\{\rho\leqslant t\}}\|_\Phi + \|(\rho_{t,r}-\mu(\rho_{t,r}))\mathbf{1}_{\{\rho>t\}}\|_\Phi) \\ &\geqslant \tfrac{1}{2}\{\mu(\rho_{t,r})\mu(\rho\leqslant t) + \|\rho_{t,r}\|_\Phi - \mu(\rho_{t,r})\Phi(1)\mu(\rho>t)\} \geqslant \tfrac{1}{2}\|\rho_{t,r}\|_\Phi\end{aligned}$$

provided that $\mu(\rho>t)\leqslant 1/(1+\Phi(1))$. We thus obtain

$$\|\rho_{t,r}\|_\Phi^2 \leqslant C_1\mu(t\leqslant\rho\leqslant t+r), \quad \text{if } r>0,\ \mu(\rho\geqslant t)\leqslant \frac{1}{1+\Phi(1)}. \tag{3.1}$$

Next, since $\Phi(t)^{-1}t^{-2}\to 0$ as $t\to\infty$ according to (Φ'), we have $\lim_{t\to\infty}\Phi^{-1}(2t)^2t^{-1}=\infty$. Therefore, there exists $\varepsilon_0\in(0,1/(1+\Phi(1))]$ such that

$$\Phi^{-1}(2s^{-1})^2 \geqslant 2s^{-1}, \quad s\in(0,\varepsilon_0]. \tag{3.2}$$

(ii) Now let $t_0>0$ such that $\mu(\rho\geqslant t_0)\leqslant\varepsilon_0$, and $t_i := t_0+i$, $i\geqslant 0$. Without loss of generality, we assume that $\mu(\rho=t_i)=0$ for all $i\geqslant 0$, since the set $\{t>0:\mu(\rho=t)>0\}$ is either finite or countable, and $\{t_i\}$ can be uniformly approximated by sequences in $\{t>0:\mu(\rho=t)=0\}$.

Let $a_i := \mu(\rho\geqslant t_i)$, $i\geqslant 0$. By (3.1) and setting $\mu(\rho=t_i)=0$, we obtain

$$\Phi^{-1}(a_{i+1}^{-1})^{-2} = \|\mathbf{1}_{[t_{i+1},\infty)}(\rho)\|_\Phi^2 \leqslant \|\rho_{t_i,1}\|_\Phi^2 \leqslant C_1(a_i-a_{i+1}), \quad i\geqslant 0.$$

This implies that

$$C_1^{-1} \leqslant \Phi^{-1}(a_{i+1}^{-1})^2(a_i-a_{i+1}), \quad i\geqslant 0.$$

If $a_{i+1}\geqslant\frac{1}{2}a_i$, then

$$\frac{1}{C_1} \leqslant \Phi^{-1}(2a_i^{-1})^2(a_i-a_{i+1}) \leqslant \int_{a_{i+1}}^{a_i}\Phi^{-1}(2s^{-1})^2\,\mathrm{d}s. \tag{3.3}$$

If $a_{i+1}<\frac{1}{2}a_i$, then

$$\int_{a_{i+1}}^{a_i}\frac{2}{s}\,\mathrm{d}s \geqslant \frac{2(a_i-a_{i+1})}{a_i} \geqslant 1. \tag{3.4}$$

Combining (3.3) and (3.4) with (3.2), we obtain

$$c := 1\wedge\frac{1}{C_1} \leqslant \int_{a_{i+1}}^{a_i}\left\{\Phi^{-1}(2s^{-1})^2\vee\frac{2}{s}\right\}\mathrm{d}s = \int_{a_{i+1}}^{a_i}\Phi^{-1}(2s^{-1})^2\,\mathrm{d}s, \quad i\geqslant 0. \tag{3.5}$$

Now, for any $t\geqslant t_0+1$, let $k:=\min\{n\in\mathbb{N}: n\leqslant t-t_0\}$. By (3.5) we have

$$c(t-t_0-1) \leqslant \sum_{i=0}^{k-1}\int_{a_{i+1}}^{a_i}\Phi^{-1}(2s^{-1})^2\,\mathrm{d}s \leqslant \int_{\mu(\rho\geqslant t)}^{a_0}\Phi^{-1}(2s^{-1})^2\,\mathrm{d}s.$$

This implies that

$$\Psi(\mu(\rho\geqslant t)) \geqslant c_1 t$$

for some constant $c_1>0$ and sufficiently large t. Hence, the first assertion holds.

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(iii) If $\Phi(s) = s^2 \log^{-\delta}(\mathrm{e} + s^2)$, then there exist $c_2, c_3 > 0$ such that

$$\begin{aligned}\Psi(r) &= \int_r^1 \Phi^{-1}(2s^{-1})^2 \,\mathrm{d}s \\ &\geqslant c_2 \int_r^1 \frac{1}{s} \log^{\delta/2}(\mathrm{e} + s^{-2}) \,\mathrm{d}s \\ &\geqslant c_2 \int_r^1 \frac{\mathrm{d}}{\mathrm{d}s}\biggl\{ -\frac{1}{(\delta+2)(\mathrm{e}+1)} \log^{(\delta+2)/2}(\mathrm{e} + s^{-2}) \biggr\} \,\mathrm{d}s \\ &\geqslant c_3 \log^{(\delta+2)/2}(\mathrm{e} + r^{-2})\end{aligned}$$

for sufficiently small $r > 0$. Therefore, there exists $c_4 > 0$ such that, for sufficiently large $t > 0$,

$$\Psi^{-1}(t) \leqslant \{\exp[(t/c_3)^{2/(2+\delta)}] - \mathrm{e}\}^{-1/2} \leqslant \exp[-c_4 t^{2/(\delta+2)}].$$

Then the second assertion follows from the first. □

It is easy to see from Proposition 3.4 that if (1.5) holds for some $p \in [1, 2)$, then $\mu(\rho \geqslant t) \leqslant ct^{-p/(p-2)}$ for some $c > 0$ and all $t > 0$. The next result is considerably better than this, and is sharp, as illustrated by the examples in § 1.

Proposition 3.5. *Let $\rho \geqslant 0$ such that $\boldsymbol{L}_{\mathcal{E}}(\rho) \leqslant 1$. If $\mu(\rho^{\varepsilon}) < \infty$ for some $\varepsilon > 0$ and (1.5) holds for some $p \in [1, 2)$, then $\mu(\rho^{2p/(2-p)}) < \infty$.*

Proof. Without loss of generality, we assume that $\rho \geqslant 1$ (otherwise, we use $1 + \rho$ in place of ρ). Let

$$h(t) := \mu(\rho^t), \quad h_n(t) := \mu((\rho \wedge n)^t), \quad n \geqslant 1,\ t > 0.$$

By (1.5) and [**21**, Lemma 1.2.4],

$$\mu(|(\rho \wedge n)^t - h_n(t)|^p) \leqslant [2C(t-1)^2 h_n(2(t-1))]^{p/2}. \tag{3.6}$$

Since $p < 2$, there exists $c(p) > 0$ such that

$$[2C(t-1)^2 h_n(2(t-1))]^{p/2} \leqslant \frac{1}{2p} h_n(tp) + c(p), \quad t \leqslant \frac{2}{2-p}. \tag{3.7}$$

Here, we have used the fact that $\rho^{2(t-1)} \leqslant \rho^{tp}$ for $t \leqslant 2/(2-p)$ since $\rho \geqslant 1$. Moreover,

$$\mu(|(\rho \wedge n)^t - h_n(t)|^p) + h_n(t)^p \geqslant \frac{1}{p} \mu((\rho \wedge n)^{tp}).$$

Combining this with (3.6) and (3.7), we obtain

$$h_n(tp) \leqslant C(p)(1 + h_n(t))^p, \quad t \in \biggl[\varepsilon, \frac{2}{2-p}\biggr],$$

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for some constant $C(p) > 0$. Letting $n \to \infty$, we have

$$h(tp) \leqslant C(p)(1 + h(t))^p, \quad t \in \left[\varepsilon, \frac{2}{2-p}\right]. \tag{3.8}$$

Since $h(\varepsilon) < \infty$, this implies that $h(\varepsilon p^{k+1}) < \infty$ for all $k \geqslant 0$ such that $\varepsilon p^k \leqslant 2/(2-p)$. Let k_0 be the largest integer such that $\varepsilon p^{k_0} \leqslant 2/(2-p)$. We have $h(\varepsilon p^{k_0+1}) < \infty$ and $\varepsilon p^{k_0+1} > 2/(2-p)$. Therefore, $h(2/(2-p)) < \infty$. The proof is thus completed by taking $t = 2/(2-p)$ in (3.8). □

4. Non-conservative Dirichlet forms

For non-conservative Dirichlet forms we have either $1 \notin \mathcal{D}(\mathcal{E})$ (e.g. the Dirichlet form of the Dirichlet Laplacian on a manifold with boundary) or $1 \in \mathcal{D}(\mathcal{E})$ but $\mathcal{E}(1,1) > 0$ (e.g. the Dirichlet form of a Schrödinger operator with negative potential). In these cases we consider the following inequality instead of (1.4):

$$\|f\|_{\Phi}^2 \leqslant C\mathcal{E}(f,f), \quad f \in \mathcal{D}(\mathcal{E}). \tag{4.1}$$

To study this inequality, we use the following type of weak Poincaré inequality introduced in [**19**]:

$$\mu(f^2) \leqslant \alpha(r)\mathcal{E}(f,f) + r\|f\|_\infty^2, \quad r > 0,\ f \in \mathcal{D}(\mathcal{E}). \tag{4.2}$$

Theorem 4.1. *Assume (Φ) but without the condition $\liminf_{s\to\infty} \Phi(s)/s > 0$. Then all results in Theorem 1.3 hold for (4.1) and (4.2) in place of (1.4) and (1.6), respectively. Consequently, we have the following.*

(1) *Let $\delta > 0$. Then (4.2) holds for $\alpha(r) = c\log^\delta(1 + r^{-1})$ for some $c > 0$ if and only if (4.1) holds for $\Phi(r) = r^2\log^{-\delta}(\mathrm{e} + r^2)$ and some $C > 0$. Both inequalities are equivalent to the sub-exponential convergence*

$$\|P_t\|_{\infty\to 2} \leqslant c_1 \exp[-c_2 t^{1/(1+\delta)}], \quad t > 0,$$

for some $c_1, c_2 > 0$.

(2) *Let $\theta > 0$. Then (4.2) with $\alpha(r) = cr^{-\theta}$, which is equivalent to $\|P_t\|_{\infty\to 2} \leqslant c' t^{-1/\theta}$ for some $c' > 0$ and all $t > 0$, implies that*

$$\mu(|f|^p)^{2/p} \leqslant C\mathcal{E}(f,f), \quad f \in \mathcal{D}(\mathcal{E}), \tag{4.3}$$

for any $p \in (0, 2/(1+\theta))$ and some constant C depending on p. On the other hand, (4.3) with $p = 2/(1+\theta)$ implies (4.2) with $\alpha(r) = cr^{-\theta}$ for some $c > 0$.

Proof. Simply note that in the present case the formula (2.3) in the proof of Theorem 1.3 (1) reduces to

$$\mu(\Phi(\bar{f}_N)) \leqslant \tfrac{1}{4} + C_N\mathcal{E}(\bar{f}_N, \bar{f}_N), \quad f \in \mathcal{D}(\mathcal{E}),\ \|f\|_\Phi = 1, \tag{4.4}$$

and the remainder of the proof of Theorem 1.3 is valid without the last assumption in (Φ). □

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Proposition 4.2. *Assume (Φ') but without the convexity of Φ. Proposition 3.4 holds for (4.1) in place of (1.4) and, for any $p \in (0,2)$, the result in Proposition 3.5 holds for (4.3) in place of (1.5).*

To prove this proposition, one needs only to use (4.1) and (4.3) to replace (1.4) and (1.5), respectively, in the proofs of Propositions 3.4 and 3.5. Therefore we omit the proof.

Finally, we present below two examples in which, unlike in the situation for (1.5), (4.3) can also be described for $p < 1$.

Example 4.3. We use the situation of Example 1.5 but set $E = [0,\infty)$ and $\mathcal{D}(\mathcal{E}) = \{f \in W^{2,1}([0,\infty);\mu) : f(0) = 0\}$. Then, for any $\delta > 0$, (4.3) holds for $p \in (0, 2\sigma/(\sigma+2))$ but does not hold for $p = 2\sigma/(\sigma+2)$.

Proof. Let us extend functions in $\mathcal{D}(\mathcal{E})$ onto $\mathbb{R}$ by setting $f|_{(-\infty,0]} = 0$ and letting $\tilde{\mu}$ be the probability measure in Example 1.5. By (1.6) with $\alpha(r) = cr^{-2/\sigma}$ indicated in Example 1.5 and noting that $\tilde{\mu}(f)^2 \leqslant \frac{1}{2}\tilde{\mu}(f^2)$ for $f \in \mathcal{D}(\mathcal{E})$, we obtain (4.2) for $\alpha(r) = c' r^{-2/\sigma}$ for some $c' > 0$. The desired assertions are then direct consequences of Theorem 4.1 and Proposition 4.2. □

Example 4.4. We use the situation of Example 1.7 (1) but consider $\delta \in (1,2)$ and the Dirichlet form with $\mathcal{D}(\mathcal{E}) = \{f \in L^2(\mu) : \mathcal{E}(f,f) < \infty,\ f(0) = 0\}$. Then (4.3) holds for all $p \in (0, 2(\delta-1))$ but fails when $p = 2(\delta-1)$. The proof is similar to that of Example 4.3 by noting that in the present setting

$$\mu(f)^2 \leqslant \mu(f^2)(1-\mu_0).$$

5. Applications to porous media equations

Let $p \in (0,2)$. Consider the differential equation

$$\partial_t u(t,\cdot) = L\{u(t,\cdot)^m\}, \quad u(0,\cdot) = f, \tag{5.1}$$

where $m > 1$, f is a bounded measurable function on E and $u^m := \operatorname{sgn}(u)|u|^m$. In particular, if $L = \Delta$ on $\mathbb{R}^d$ or a regular domain with Dirichlet or Neumann boundary condition, then (5.1) is called the porous medium equation (see, for example, [**3**, **13**] and references within).

We call $T_t f := u(t,\cdot)$ a solution to (5.1) if $u(t,\cdot)^m \in \mathcal{D}(L)$ for $t > 0$ and $u^m \in L^1_{\mathrm{loc}}([0,\infty) \to \mathcal{D}(\mathcal{E}); \mathrm{d}t)$ such that, for any $g \in \mathcal{D}(\mathcal{E})$,

$$\mu(u(t,\cdot)g) = \mu(fg) - \int_0^t \mathcal{E}(g, u(s,\cdot)^m)\,\mathrm{d}s, \quad t > 0.$$

In general, since $(L, \mathcal{D}(L))$ is a Dirichlet operator, one has (see [**7**, p. 242])

$$\mu(f^l L f^m) \leqslant -\frac{4lm}{(l+m)^2}\mathcal{E}(f^{(l+m)/2}, f^{(l+m)/2}) \tag{5.2}$$

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for $l,m>0$ and f such that $f^m\in\mathcal{D}(L)$ and $f^{(l+m)/2}\in\mathcal{D}_{\mathrm{e}}(\mathcal{E})$, where $\mathcal{D}_{\mathrm{e}}(\mathcal{E})$ is the extended domain (see [**8**]). If, in particular, L is a second-order elliptic differential operator on a Riemannian manifold such that $\mathcal{E}(f,g)=\mu(\langle\nabla f,\nabla g\rangle)$ for $f,g\in\mathcal{D}(\mathcal{E})$, then the equality in (5.2) holds by the chain rule.

Proposition 5.1. *Assume that, for any bounded $f\in\mathcal{D}(L)$, Equation (5.1) has a unique solution, T_tf.*

(1) *Let $p\in(1,2)$ and $m=3-p$. If (1.5) holds, then*

$$\mu(|T_tf|^p)\leqslant\left\{\mu(|f|^p)^{(p-2)/p}+\frac{(p-1)(2-p)(3-p)t}{C}\right\}^{-p/(2-p)},\quad t\geqslant 0,\ \mu(f)=0. \tag{5.3}$$

If the equality in (5.2) holds, then the converse is true.

(2) *Let $p\in(1,2)$ and $m=3-p$. Assertions in (1) hold for (4.3) in place of (1.5) and for (5.3) without the restriction $\mu(f)=0$.*

(3) *Let $p\in(0,2)$ and $m=(4-p)/p$. Then (4.3) implies that*

$$\mu((T_tf)^2)\leqslant\left\{\mu(f^2)^{(p-2)/p}+\frac{(4-p)(2-p)}{2C}t\right\}^{-p/(2-p)},\quad t\geqslant 0, \tag{5.4}$$

and the converse is true, provided that the equality in (5.2) holds.

Proof. We first prove (1). Let $f\in\mathcal{D}(L)$ with $\mu(f)=0$. By (1.5) and (5.2), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu(|T_tf|^p)&=p\mu((T_tf)^{p-1}L\{(T_tf)^{3-p}\})\\&\leqslant -p(p-1)(3-p)\mathcal{E}(T_tf,T_tf)\\&\leqslant -\frac{p(p-1)(3-p)}{C}\mu(|T_tf|^p)^{2/p}.\end{aligned}$$

This implies (5.3). Since the equality in (5.3) holds at $t=0$, the inequality remains true by taking derivatives with respect to t at $t=0$ for both sides. Combining with the equality in (5.2), we obtain (1.5).

Since the proof of (2) is completely similar to that of (1), it remains to prove (3). By (4.3) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu((T_tf)^2)&=-2\mathcal{E}(T_tf,(T_tf)^{(4-p)/p})\\&\leqslant -\frac{(4-p)p}{2}\mathcal{E}((T_tf)^{2/p},(T_tf)^{2/p})\\&\leqslant -\frac{(4-p)p}{2C}\mu((T_tf)^2)^{2/p}.\end{aligned}$$

This implies (5.4). If the equality in (5.2) holds, then, as explained above, one may take the derivative at $t=0$ to derive the converse. □

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