Multiplicity-Free Schubert Calculus

Hugh Thomas and Alexander Yong

Abstract. Multiplicity-free algebraic geometry is the study of subvarieties $Y \subseteq X$ with the "smallest invariants" as witnessed by a multiplicity-free Chow ring decomposition of $[Y] \in A^{\star}(X)$ into a predetermined linear basis.

This paper concerns the case of Richardson subvarieties of the Grassmannian in terms of the Schubert basis. We give a nonrecursive combinatorial classification of multiplicity-free Richardson varieties, *i.e.,* we classify multiplicity-free products of Schubert classes. This answers a question of W. Fulton.

1 Introduction

1.1 The Main Result

Let $\mathrm{Gr}(\ell, \mathbb{C}^n)$ denote the Grassmannian of ℓ-dimensional subspaces in $\mathbb{C}^n$. The cohomology ring $\mathrm{H}^{\star}(\mathrm{Gr}(\ell, \mathbb{C}^n), \mathbb{Z})$ has an additive basis of Schubert classes σ_λ, indexed by Young diagrams $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0)$ contained in the $\ell \times k$ rectangle where $k = n - \ell$ (we denote this by $\lambda \subseteq \ell \times k$). The product of two Schubert classes in $\mathrm{H}^{\star}(\mathrm{Gr}(\ell, \mathbb{C}^n), \mathbb{Z})$ is given by

$$(1.1) \qquad \sigma_\lambda \cdot \sigma_\mu = \sum_{\nu \subseteq \ell \times k} c_{\lambda,\mu}^{\nu} \sigma_\nu,$$

where $c_{\lambda,\mu}^{\nu}$ is the classical Littlewood–Richardson coefficient (see [3,7]).

The expansion (1.1) is *multiplicity-free* if $c_{\lambda,\mu}^{\nu} \in \{0, 1\}$ for all $\nu \subseteq \ell \times k$. In this paper, we give a nonrecursive, combinatorial answer to the following question of W. Fulton.

Question 1 When is $\sigma_\lambda \cdot \sigma_\mu$ multiplicity-free?

Famously, Littlewood–Richardson coefficients also arise as decomposition multiplicities of the tensor product $V^\lambda \otimes V^\mu = \bigoplus_\nu (V^\nu)^{\oplus c_{\lambda,\mu}^\nu}$ of irreducible polynomial representations of $\mathrm{GL}(\ell)$. Earlier, J. Stembridge [9] solved the analogous question in this context; the above question was motivated by this work. (The $\mathrm{GL}(\ell)$ problem may be regarded as the special case in the "$k \to \infty$ limit" of Fulton's question; Stembridge's classification is expressed inside our solution below.)

For partitions $\lambda, \mu \subseteq \ell \times k$, place λ against the upper left corner of the rectangle. Then rotate μ 180 degrees and place it in the lower right corner. We refer to rotate(μ) as the resulting subshape of $\ell \times k$. The *boxes* of these shapes refer to their configuration inside $\ell \times k$.
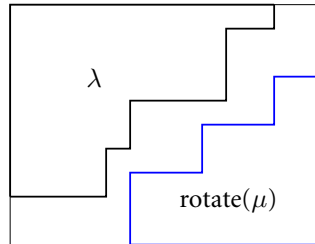
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Figure 1: λ and rotate(μ) inside $\ell \times k$.

A boring reason for multiplicity-freeness is that $\lambda \cap$ rotate(μ) $\neq \varnothing$, since the (intersection) product $\sigma_\lambda \cdot \sigma_\mu$ is merely zero. Geometrically, this reflects the fact that the *Richardson variety* $X_\lambda^\mu = X_\lambda(F_\bullet) \cap X_\mu(F_\bullet^{\mathrm{opp}})$ is empty. This variety is the scheme-theoretic intersection of Schubert varieties indexed by λ and μ; these Schubert varieties are in general position when defined with respect to opposite complete flags $F_\bullet$ and $F_\bullet^{\mathrm{opp}}$.

Nevertheless, our main idea is to apply a simple extension of this classic geometric condition. Define a *Richardson quadruple* to be the datum $(\lambda, \mu, \ell \times k)$ where $\lambda \cap$ rotate(μ) $= \varnothing$. If $\lambda \cup$ rotate(μ) does not contain a full ℓ-column or k-row, call this Richardson quadruple *basic*. Otherwise, after removing all full columns and/or rows, we obtain a Richardson quadruple $\widetilde{\mathfrak{R}} = (\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\ell} \times \widetilde{k})$ for smaller partitions $\widetilde{\lambda}, \widetilde{\mu} \subseteq \widetilde{\ell} \times \widetilde{k}$. This latter quadruple is basic, and we call it the *basic demolition* of $\mathfrak{R}$.

Example 1.1   Figure 1 depicts a basic Richardson quadruple.

The non-basic Richardson quadruple

$$\mathfrak{R} = ((6, 5, 4, 3, 2, 1, 1), (7, 6, 6, 6, 5, 2), 7 \times 9)$$

has three full columns (1, 4, and 5) and two full rows (3 and 4). Its basic demolition is $\widetilde{\mathfrak{R}} = ((3, 2, 1), (5, 4, 4, 2), 5 \times 6)$:
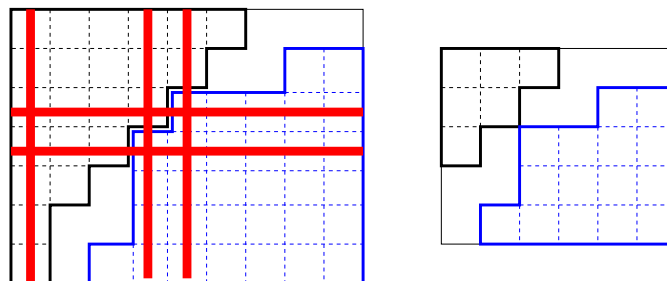

Figure 2: A non-basic Richardson quadruple and its basic demolition.

Conceptually, this type of combinatorics is inspired by a geometric comparison of Richardson varieties, $X_\lambda^\mu \subseteq \mathrm{Gr}(\ell, \mathbb{C}^{\ell+k})$ and $X_{\widetilde{\lambda}}^{\widetilde{\mu}} \subseteq \mathrm{Gr}(\widetilde{\ell}, \mathbb{C}^{\widetilde{\ell}+\widetilde{k}})$.

In order to state our main result, we need a little more notation and terminology, some nonstandard. A rectangle is a Young shape with exactly one distinct part size. A *fat hook* is a shape whose partition λ has two distinct part sizes. Furthermore, $\lambda \subseteq \ell \times k$ naturally defines a lattice path from the southwest to the northeast corner points of the rectangle. A *segment* of this lattice path is a maximal consecutive sequence of north or east steps. The *shortness* of this lattice path is the length of its shortest segment. Lastly, a *multiplicity-free Richardson quadruple* is a Richardson quadruple such that the corresponding intersection product $\sigma_\lambda \cdot \sigma_\mu$ is multiplicity-free.

We are now ready to state our main result.

Theorem 1.2 *A Richardson quadruple is multiplicity-free if and only if its basic demolition is multipli-city-free. A basic Richardson quadruple $\mathfrak{R} = (\lambda, \mu, \ell \times k)$ is multiplicity-free if and only if one of the following conditions holds.*

(I)   *either λ or μ is a rectangle of shortness 1;*
(II) *λ is a rectangle of shortness 2, μ is a fat hook (or vice versa);*
(III) *λ is a rectangle, μ is a fat hook of shortness 1 (or vice versa);*
(IV) *both λ and μ are rectangles.*

We remark that once the idea of basic demolitions is found, the solution to Fulton's question becomes relatively straightforward by exploiting [9, Theorem 3.1]; in the basic case, our classification is the same as Stembridge's classification, although the proof requires further combinatorial analysis. However, as is often the case in combinatorics (and we believe here), the central difficulty of a problem often turns on precisely such an observation. In fact, after formulating this technique for this work, we found it to be crucial (in a more general form) for our arguments in [12].

Example 1.3 $\mathfrak{R} = ((4, 4, 2, 2), (3, 3, 3), 6 \times 6)$ is *not* multiplicity-free:

$$\sigma_{(4,4,2,2)} \cdot \sigma_{(3,3,3)} = 2\sigma_{(6,5,4,3,2,1)} + \text{multiplicity-free terms,}$$

despite the fact that it is a product corresponding to a fat hook with a rectangle. This is since the lattice path defined by λ has shortness 2. Meanwhile, $((4, 4, 2, 2, 2), (3, 3, 3), 6 \times 6)$ is multiplicity-free, by (III).

Another example is $((4, 3, 2, 1), (4, 4, 2, 2, 1), 5 \times 5)$. This is not basic, but it is multiplicity-free since its basic demolition is $((1), (1), 2 \times 2)$. However, $((4, 3, 2, 1), (4, 4, 2, 2, 1), 6 \times 5)$ is not multiplicity-free; the tensor product of $GL(6)$ irreducible representations

$$V^{(4,3,2,1)} \otimes V^{(4,4,2,2,1)} = (V^{(5,4,4,4,4,2)})^{\oplus 3} \oplus \cdots$$

is not multiplicity-free either.

Thus, complicated pairs of partitions can give rise to (nonzero) multiplicity-freeness in the Schubert calculus context. This is not true for the $GL(\ell)$-problem (or for basic quadruples): one of the shapes involved must be a rectangle.

1.2　Extensions

Investigation of multiplicity-freeness in geometry and representation theory is of interest, see [2, 9, 10] and the references therein.

In relation to [2], this report may be viewed as part of "multiplicity-free algebraic geometry". By this term we mean the study of algebraic subvarieties $Y \subseteq X$ that have "the smallest invariants" according to the decomposition of their class into a predetermined linear basis of the Chow ring $A^\star(X)$ (in our case, the class of a Richardson variety decomposed into Schubert classes for the Grassmannian).

A feature of our methods is that they extend naturally to other (classical) Lie types. In future work, we plan to study multiplicity-free Schubert calculus on cominuscule flag manifolds, a natural generalization of "Grassmannian", with the following goal.

Question 2　Give a (uniform) characterization of multiplicity-free products of Schubert classes for cominuscule flag manifolds.

For the main interesting cases of Lagrangian and even orthogonal Grassmannians, the Schubert calculus is determined by the "shifted tableaux" combinatorics of Schur P, Q polynomials [5,6,8]. Briefly, partitions with distinct parts contained in the staircase $\rho_n = (n, n - 1, \ldots, 3, 2, 1)$ index the Schubert classes. There is a standard simultaneous placement of shifted shapes λ and μ into the shifted staircase ρ_n. When the shapes overlap, the product of their Schubert classes is zero. Otherwise, whenever there is a full i-th column and $i + 1$-th row, the situation is a non basic Richardson triple. Removal of all such hooks is a basic Richardson triple.

This allows us to prove multiplicity-free characterizations (omitted here) with similar arguments to those found below. Results of C. Bessenrodt [1] can replace the role of [9]. (She studies the problem of multiplicity-freeness of Schur P polynomials in connection to projective outer products of spin characters.)

Finally, after a version of this paper was made available, C. Gutschwager [4] answered the question of determining multiplicity-free skew characters. As is explained there, this problem is equivalent to Fulton's question.

2　Demolitions

In this section, we develop demolition techniques that we will use in the proof of the main theorem.

2.1　The Littlewood–Richardson Rule

We use a standard formulation of the Littlewood–Richardson rule: $c_{\lambda,\mu}^\nu$ counts the number of semistandard fillings of the skew-shape ν/λ of content μ such that the right to left, top to bottom reading word $w_1 w_2 \cdots w_{|\mu|}$ is a *ballot sequence, i.e.,* the number of appearances of i in $w_1 w_2 \cdots w_j$ is at most the number of appearances of $i - 1$, for $i \geq 2$ and $1 \leq j \leq |\mu|$. We call these *LR fillings*.

We say a row or column of $\ell \times k$ is *empty* if it neither contains a box of λ nor of μ. The following *emptiness demolition* is a simple application of the Littlewood–

Richardson rule.

Lemma 2.1 *Suppose $\mathfrak{R}$ contains an empty row. Then $\sigma_\lambda \cdot \sigma_\mu \in \mathrm{H}^\star(\mathrm{Gr}(\ell, \mathbb{C}^{\ell+k}))$ is the same as $\sigma_\lambda \cdot \sigma_\mu \in \mathrm{H}^\star(\mathrm{Gr}(\ell-1, \mathbb{C}^{\ell+k-1}))$. In particular, one product has multiplicity if and only if the other does. A similar statement holds in the presence of an empty column.*

Proof This follows the above Littlewood–Richardson rule (and conjugation) that any ν that contributes $c^\nu_{\lambda,\mu} \neq 0$ satisfies $\ell(\nu) \leq \ell(\lambda) + \ell(\mu) \leq \ell - 1$ (the latter inequality being the empty row assumption). So empty rows (or columns) do not affect the expansion. ∎

Example 2.2 The third row in $\mathfrak{R} = ((2,1),(2),4 \times 3)$ is the only empty row or column. The expansion

$$\sigma_{(2,1)} \cdot \sigma_{(2)} = \sigma_{(2,2,1)} + \sigma_{(3,1,1)} + \sigma_{(3,2)}$$

is the same in both $\mathrm{H}^\star(\mathrm{Gr}(4, \mathbb{C}^7), \mathbb{Z})$ and $\mathrm{H}^\star(\mathrm{Gr}(3, \mathbb{C}^6), \mathbb{Z})$.

2.2 Basic Demolitions

The following lemma explicates the demolition technique from the statement of the main theorem.

Lemma 2.3 *Suppose $\mathfrak{R}$ is not basic. Then*

(a) $\widetilde{\mathfrak{R}}$ *is basic;*
(b) $\sigma_\lambda \cdot \sigma_\mu$ *is multiplicity-free in $\mathrm{H}^\star(\mathrm{Gr}(\ell, \mathbb{C}^{\ell+k}))$ if and only if $\sigma_{\widetilde{\lambda}} \cdot \sigma_{\widetilde{\mu}}$ is multiplicity-free in $\mathrm{H}^\star(\mathrm{Gr}(\widetilde{\ell}, \mathbb{C}^{\widetilde{\ell}+\widetilde{k}}))$.*

Proof For (a), suppose that $\widetilde{\mathfrak{R}}$ contains a full row (the column case is similar). Since the boxes in that full row were not eliminated by the demolition, the corresponding row in $\mathfrak{R}$ was not full. Rather these boxes were "clamped" together after removing the full columns. But then these boxes, together with the boxes of the removed full columns from $\mathfrak{R}$, form a full row in $\mathfrak{R}$, and all would have been removed by the basic demolition, a contradiction.

For (b), it is enough to prove the case when $\widetilde{\lambda}$ and $\widetilde{\mu}$ are obtained by removing either one full row or one full column, since then the stated claim will follow by iterating this case until the basic situation is reached.

By conjugation, it suffices to prove the case that there is a full column in $\lambda \cup$ rotate(μ). Now consider the skew shape $\alpha = \mathrm{rotate}(\mu)^c / \lambda$ where rotate$(\mu)^c$ is the complement in $\ell \times k$ of rotate(μ). Removal of the full column gives a skew shape $\widetilde{\alpha}$ which has the same number of boxes as α, and also whose boxes are in the same relative position as those of α. Thus, any LR filling of α of content β is also an LR filling of $\widetilde{\alpha}$ of content β, and conversely. Moreover, notice, by the Littlewood–Richardson rule, that in either kind of filling,

$$(2.1) \qquad\qquad \beta \subseteq \ell \times (k-1).$$

Finally, we use the well-known symmetry

$$(2.2) \qquad\qquad c_{\lambda,\mu}^{\nu} = c_{\lambda,\mathrm{rotate}(\nu)^c}^{\mathrm{rotate}(\mu)^c}.$$

Now assume this number has multiplicity, witnessed by LR fillings of α with content β, as above. Set $\nu = \mathrm{rotate}(\beta^c)$ where rotation and complementation are done with respect to $\ell \times k$. In view of (2.1) we can select $\widetilde{\nu}$ such that $\beta = \mathrm{rotate}(\widetilde{\nu})^c$, with respect to $\ell \times (k-1)$. Now by (2.2) we see that if $c_{\lambda,\mu}^{\nu} \geq 2$, then $c_{\widetilde{\lambda},\widetilde{\mu}}^{\widetilde{\nu}} \geq 2$, and so if $\sigma_\lambda \cdot \sigma_\mu$ is not multiplicity-free, then neither is $\sigma_{\widetilde{\lambda}} \cdot \sigma_{\widetilde{\mu}}$. The converse argument is similar, starting with LR fillings of $\widetilde{\alpha}$ of content β.   ∎

2.3   Stembridge Demolitions

We use the following demolition technique throughout the proof of the main theorem. Suppose that $\mathfrak{R}$ contains a row or column of $\ell \times k$ containing boxes of λ, or alternatively $\mathrm{rotate}(\mu)$, but not both. Then define a *Stembridge demolition* to be the Richardson quadruple $\overline{\mathfrak{R}} = (\overline{\lambda}, \overline{\mu}, \overline{\ell} \times \overline{k})$ corresponding to (sequentially) removing such rows or columns. (We emphasize that this differs from a basic demolition, which applies when a row or column contains *only* boxes of (possibly both) λ and/or μ.)

Example 2.4 $\mathfrak{R} = ((6,6,4,2),(4,3,2,2),5 \times 8)$ is a Richardson quadruple with multiplicity. The columns 1,2,3,4,7, and 8 are can be demolished, as can rows 1 and 5. If we remove columns 1, 2, and 3 and row 5, we obtain $\overline{\mathfrak{R}}$ having multiplicity. However, if we further remove row 1, the demolition is multiplicity-free.

Stembridge demolitions are useful in one direction.

Lemma 2.5   *If $\sigma_{\overline{\lambda}} \cdot \sigma_{\overline{\mu}} \in \mathrm{H}^\star(\mathrm{Gr}(\overline{\ell}, \mathbb{C}^{\overline{\ell}+\overline{k}}))$ has multiplicity, then so does $\sigma_\lambda \cdot \sigma_\mu \in \mathrm{H}^\star(\mathrm{Gr}(\ell, \mathbb{C}^{\ell+k}))$.*

Proof   By conjugation, it suffices to handle the case that $\overline{\lambda} \cup (r) = \lambda$ (*i.e.*, deleting a row of λ gives $\overline{\lambda}$), $\overline{\mu} = \mu$, $\overline{\ell} = \ell - 1$ and $\overline{k} = k$. By assumption, there exists $\nu \subseteq (\ell-1) \times k$ such that $c_{\lambda,\mu}^{\nu} \geq 2$. Moreover, the Littlewood-Richardson rule implies $c_{\lambda,\mu}^{\nu \cup (r)} \geq c_{\lambda,\mu}^{\nu}$, see [9, Lemma 2.2]. Since $\nu \cup (r) \subseteq \ell \times k$, the claim follows.   ∎

2.4   A Reformulation of Theorem 1.2, Inductive Stembridge Demolitions

We find it useful to give an "inverse" formulatation of Theorem 1.2, *i.e.*, in terms of when multiplicity appears:

Theorem 1.3′   *If $\mathfrak{R}$ is basic, then $\sigma_\lambda \cdot \sigma_\mu$ has multiplicity if and only if one of the following cases holds:*

(I′)   *λ and μ both have at least two different part sizes,*
(II′)   *λ has at least three different part sizes and μ is a rectangle of shortness at least 2,*
(III′)   *λ is a fat hook of shortness at least 2 and μ is a rectangle of shortness at least 3,*

(IV′) *cases* (II′) *or* (III′) *with the roles of* λ *and* μ *interchanged.*

Otherwise if $\Re$ *is not basic, we conclude as in the statement of Theorem* 1.2.

This formulation is useful for induction on $\ell, k \geq 1$. For this, we are interested in the situations where a Stembridge demolition takes a basic $\Re$ satisfying (I′), (II′), (III′) or (IV′) of Theorem 1.2′ to a basic $\overline{\Re}$ that falls into the same case. We call these *inductive Stembridge demolitions.*

Example 2.6 Consider $\Re = ((4, 4, 2, 2), (3, 3, 3), 7 \times 6)$, which lies in case (III′). This has 5 column and 7 row Stembridge demolitions available. However, none are inductive. On the other hand, for $\Re = ((4, 4, 2, 2), (4, 4, 4), 7 \times 7)$ one can remove columns 5, 6, or 7 as a inductive Stembridge demolition.

It is also possible to combine Stembridge and basic demolitions in inductive arguments, so long as we start and end in the same case (I′)–(IV′).

2.5 Well-Ordering Corners of Fat Hooks

A box of λ is a *corner* if there are no boxes of λ below or to the right of it. Similarly, a *corner* of rotate(μ) is box of this shape without others above or to the left of it.

Now suppose λ and μ are fat hooks. Then both have precisely two corners. Let A be the lowest/leftmost corner of λ, and B the highest/rightmost. Let X and Y be the lowest (equivalently) leftmost corner and rightmost (equivalently highest) corner of rotate(μ), respectively. Let $\mathtt{row}(A)$ denote the row index of A (as in matrix notation). Define $\Re$ to be *well-ordered*[1] if

$$(2.3) \qquad \mathtt{row}(A) < \mathtt{row}(X) \text{ and } \mathtt{row}(B) < \mathtt{row}(Y).$$
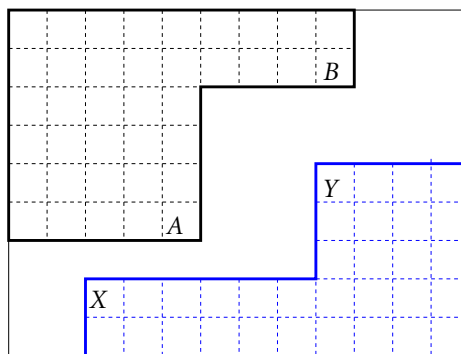

Figure 3: Fat hooks and their well-ordered corners.

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[1]An apology: this has nothing to do with the usual mathematical notion of being well-ordered.

There is an isomorphism between $\mathrm{Gr}(\ell, \mathbb{C}^n)$ and $\mathrm{Gr}(k, \mathbb{C}^n)$ reflected by the operation of *conjugating* the rectangle $\ell \times k$ to $k \times \ell$ (and the shapes within). This sends σ_λ to $\sigma_{\lambda'}$. We record the following fact for later use.

Proposition 2.7 *If $\mathfrak{R}$ is basic and λ and μ both have at least two distinct part sizes, then one of the following holds:*

(i)　$\mathfrak{R}$ *has an inductive Stembridge demolition,*
(ii)　λ *or μ is a hook,*
(iii)　λ *and μ are fat hooks, and either $(\lambda, \mu, \ell \times k)$ or $(\lambda', \mu', k \times \ell)$ is well-ordered.*

Proof Basicness implies that the rightmost column of $\ell \times k$ contains only boxes from $\mathrm{rotate}(\mu)$; removing that column is a Stembridge demolition. However, there are two reasons why this might fail to be an *inductive* Stembridge demolition (for case (I$'$) of Theorem 1$'$):

- μ may no longer have at least two distinct parts;
- $\overline{\mathfrak{R}} = (\lambda, \overline{\mu}, \ell \times (k-1))$ may not be basic because the top row of $\ell \times (k-1)$ consists entirely of boxes from λ.

(Similar analysis applies to the other three edges of $\ell \times k$.)

If μ has at least three distinct parts, then the first of these possibilities cannot occur. Thus failure of inductiveness must be blamed on λ extending to the $(k-1)$-th column. Similarly, if removing the bottom row is not inductive, then λ extends all the way to the $(\ell-1)$-th row.

Assume further that λ is a fat hook. If λ is not a hook, then the previous paragraph, together with the basicness implies either removing the top row or the leftmost column is an inductive Stembridge demolition.

Thus, suppose λ has at least three distinct parts. We may remove the leftmost column of $\ell \times k$ to obtain an inductive Stembridge demolition unless μ extends all the way to the second column of $\ell \times k$. In this case, remove both the leftmost column and the bottom row. Since μ has at least three parts by assumption, the result of applying these two Stembridge demolitions is inductive.

We have just disposed of the case where μ (or symmetrically λ) has at least three distinct parts. Assume now that both λ and μ have exactly two distinct parts.

By (iii) we are done if $\mathfrak{R}$ is well-ordered, so assume otherwise. Thus, at least one of $\mathrm{row}(X) \leq \mathrm{row}(A)$ or $\mathrm{row}(Y) \leq \mathrm{row}(B)$ holds. If both of these are true, then the partition will be well-ordered after conjugating $\ell \times k$. If $\mathrm{row}(X) \leq \mathrm{row}(A)$ but $\mathrm{row}(Y) > \mathrm{row}(B)$, then it follows that either λ is a hook or there is an inductive Stembridge demolition removing either the top row or the leftmost column. Finally, if $\mathrm{row}(X) > \mathrm{row}(A)$ but $\mathrm{row}(Y) \leq \mathrm{row}(B)$, then either μ is a hook or there is an inductive Stembridge demolition obtained by removing either the bottom row or the rightmost column. ∎

3　Proof of the Main Result

We will actually prove Theorem 1.2$'$, the reformulation of the main result given in Section 2. The equivalence of these two statements is straighforward to check.

In each of the first three subsections below, we assume basicness throughout and apply the demolitions of Section 2 to induct on $\ell, k \geq 1$. In the base cases where no induction is possible, we describe a skew shape $\nu/\lambda \subseteq \ell \times k$ having two LR fillings of content μ, thus showing $c^\nu_{\lambda,\mu} \geq 2$.

3.1 Proof of Multiplicity in Case (I′)

Let λ, μ have at least two distinct nonzero part sizes. By Proposition 2.7, one of its scenarios (i), (ii), or (iii) occurs. We induct if (i) happens.

Thus consider (ii). Let $\mu = (b, 1^a)$ be a hook. Let r be the smallest index such that $\lambda_r < \lambda_1$. Let s be the smallest index such that $\lambda_s < \lambda_r$.

For the first filling, add a horizontal strip of boxes (emph i.e., no two in the same column) of size b with each box labeled 1 such that

- at least one box appears in the first row, and in the r-th row;
- the maximal possible number of boxes is *not* placed in row s (so possibly no boxes occur there).

(Note that by basicness, $b < k$ and $\lambda'_1 < \ell$, so such a horizontal strip exists.) Now add a boxes, no two in the same row, with a box at the right-hand end of all rows 2 through $a + 2$, except row r. Label the box in row i with i if $i < r$, and with $i - 1$ if $i > r$. This clearly gives a LR filling of a skew shape ν/λ with content μ, as desired.

We modify the above to obtain a second LR filling: replace the rightmost 1 in row r with an r. The column below the leftmost 1 in row r consists of boxes labeled by r to $s - 2$. Increase all of these labels by 1. Finally, replace the box labeled $s - 1$ in row s by a 1.
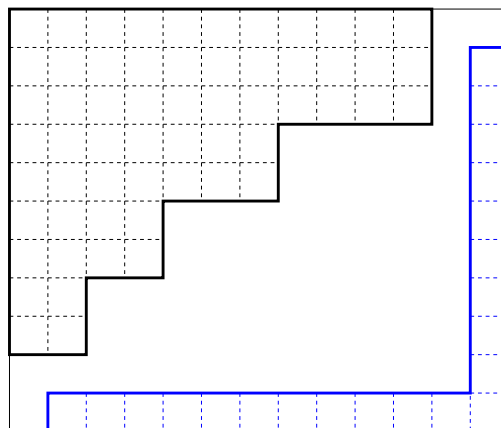

Figure 4: Find two LR fillings.

Example 3.1 Being the first of several such arguments, let us take this one in slow-motion. Consider $\mathfrak{R} = ((11, 11, 11, 7, 7, 4, 4, 2, 2), (12, 1^9), 11 \times 13)$. Here $r = 4, s = 6$. See Figure 4. In general there is choice in constructing the two fillings, but not in this example. We invite the reader to pencil in the first LR filling, followed by marking in the modification. The first reading word is

$$1, 1, 2, 3, 1, 1, 1, 1, 4, 5, 1, 1, 6, 7, 1, 1, 8, 9, 1, 1, 10.$$

The second is

$$1, 1, 2, 3, 4, 1, 1, 1, 5, 1, 1, 1, 6, 7, 1, 1, 8, 9, 1, 1, 10.$$

How does the filling vary as a and b change?

Now we turn to (iii). Assume without loss of generality that $\mathfrak{R}$ is well-ordered. The example given in Figure 3 provides a running example.

To describe our LR fillings, think of the boxes of rotate(μ) as movable tiles, labeled by their row number in μ (*i.e.*, by $\ell - \texttt{row}(\cdot) + 1$).

Remove the tiles in the columns below X and Y (including X and Y) and set them aside. Shift all the remaining tiles of rotate(μ) one square to the left. Note that since $\mathfrak{R}$ is basic, the tiles still do not overlap λ.

Using only columns $k - \mu_1$ to $k - 1$, move the $\mu_1 - 2$ remaining columns of tiles (possibly left or right, but maintaining their relative order) so that the columns below A and B are empty.[2] Next slide these columns up so they are immediately below (the lattice path defined by) λ, and reverse the order of the tiles in each column. The only worry in this last step is that there might not be enough room to fit each column. It is straightforward to check that this cannot happen, because of the well-ordered assumption.

Example 3.2 Following the running example defined by Figure 3, there is actually no choice in this case which columns to use in the "slide the columns up" step. The result at that stage is depicted below in Figure 5.

Let $a = \ell - \texttt{row}(A)$ and $b = \ell - \texttt{row}(B)$ count the number of boxes strictly below A and B respectively. Also, let $x = \ell - \texttt{row}(X) + 1$, $y = \ell - \texttt{row}(Y) + 1$ be the number of boxes in rotate(μ) weakly below X and Y. The tiles that we set aside consist of two copies of 1 to x, together with one copy of $x + 1$ to y. We wish to use these to fill the columns below A and B and the rightmost column.

For our first filling, put 1 in the rightmost column. Put 2 to $y - (a - x)$ in the column below B, and put 1 to x followed by $y - (a - x) + 1$ to y in the column below A. There is space to do this: under A we use precisely a labels, while under B we use fewer than b elements, precisely because of well-orderedness (2.3).

By construction, the associated word is a ballot sequence, because this is true for the word coming from the columns of labels we have inserted *not* below A and B. (To

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[2]Depending on the exact well-ordered configuration of A, B, X, and Y, there can be choice, but these all give the same end result. To be precise, we just insist that the same columns be used for both fillings described.
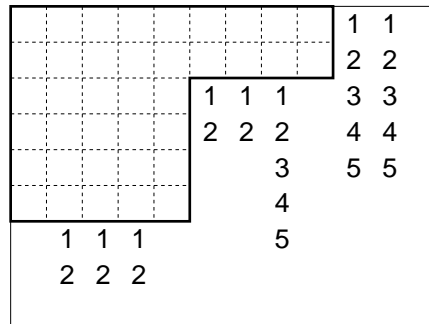
Figure 5: In process filling for Example 3.2.

each i, there is an $i-1$ north of it in its column, which appears first in the reading word.) This property is clearly maintained after adding the three special columns. The only complaint is that this filling may not yet describe a skew-shape. We fix this by sliding tiles to the left (in the same row) as necessary. This does not alter the reading word. Lastly, it is straightforward to verify that the filling is semistandard.

Example 3.3 Continuing our running example, the reader may wish to fill in the extra three columns. Since $a = 3, b = 7, x = 2, y = 5$, we need to place a column of $2, 3, 4$ under column 9, a column of $1, 2, 5$ under column 5, and a single 1 in the rightmost column. The final shape after shifting to the left is $(12, 11, 11, 11, 9, 7, 5, 4, 1)/\lambda$.

The second LR filling is similar: put 1 in the rightmost column, 1 to $y-(a-x)-1$ in the column below B, and 2 to x followed by $y-(a-x)$ to y in the column below A. One also checks that this satisfies the necessary conditions and leads to an LR filling of the same shape as the first filling (even before the final left justificiation).

Example 3.4 Erasing what we did to columns 9 and 5 in the running example last time, we put $1, 2, 3$ under column 9, and $2, 4, 5$ under column 5.

3.2 Proof of Multiplicity in Case (II′)

Let λ have at least three part sizes and μ be a rectangle of shortness at least two. Thus, $\mu = (g^h)$ where $2 \le g \le k-2$ and $2 \le h \le \ell - 2$.

If g or h equals 2, by conjugation, we may assume that $h = 2$. Note that the basicness assumption together with the fact that λ contains at least three distinct parts, implies one can add to λ a horizontal strip of length $g + 2$ which uses (at least) four different rows. Call the rightmost boxes of these four rows B_1, B_2, B_3 and B_4 where $\text{row}(B_i) < \text{row}(B_j)$ when $i < j$.

If λ does not extend to the $(\ell - 1)$-th row, then this horizontal strip can be done in any way possible. Otherwise some extra care is required. In this latter situation, by inductive Stembridge reductions (removing the first few columns), we may as-

sume that λ has at least one column reaching the $(\ell - 1)$-th row. Thus, choose our horizontal strip to include a (single) box in row ℓ, and make that box B_4.

In either case, label the other $g - 2$ boxes with 1s. Place $g - 2$ boxes labeled 2 under these boxes we have just labeled. These boxes give a skew shape ν/λ after sliding the boxes along each row to the left against the lattice path defined by λ. See Figure 6 below.
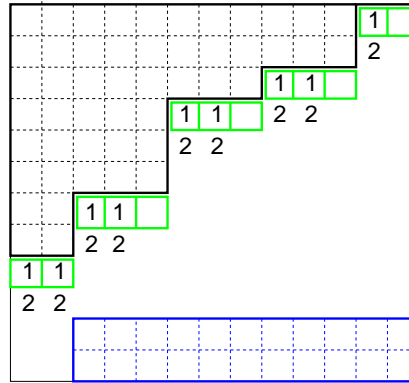

Figure 6: A partial LR filling with blanks B_1, B_2, B_3, B_4 to fill.

One LR filling of ν/λ is given by $B_1 = B_2 = 1$, $B_3 = B_4 = 2$; another is given by $B_1 = B_3 = 1$, $B_2 = B_4 = 2$. Thus, the result holds if g or h equals 2.

Now assume that $\mu = (g^h)$ where $g, h \geq 3$. After applying inductive Stembridge demolitions one can assume (after conjugating) that $\lambda = (k-1, 2^a, 1^{\ell-a-2})$ for some $a > 0$. (Proof: applying inductive Stembridge demolitions to rows and columns of λ we can reduce to the case that λ has exactly three part sizes. If neither the bottom row nor the rightmost column of λ can be removed by an inductive Stembridge reduction, λ must extend all the way to the $(k - 1)$-th column and $(\ell - 1)$-th row. Further inductive Stembridge demolitions to the first few columns/rows of λ imply that $\lambda = (k-1, b^a, 1^{\ell-a-2})$. Finally, inductive Stembridge demolitions to columns 2 through b, or rows 2 through a (depending on the position of μ) allows us to deduce $b = 2$ or $a = 2$. Conjugating we may assume the former.)

If $a + h + 1 > \ell$, then the Stembridge demolition obtained by removing the third column is clearly inductive if $g < k-3$. So we may assume that $g = k-3$. But then we can still Stembridge-reduce by removing the third column. The result is non-basic (and thus not inductive). But removing the full rows that result gives us a smaller case of (II$'$), so we can still induct nonetheless.

Example 3.5 A minimal example is given by $\mathfrak{R} = ((4, 2, 2, 1), (2, 2, 2), 5 \times 5)$. Removing the third column is a noninductive Stembridge demolition, giving

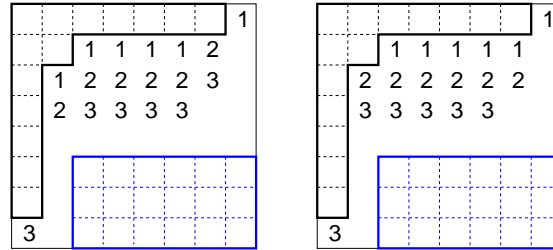$$\overline{\mathfrak{R}} = ((3, 2, 2, 1), (2, 2, 2), 5 \times 4),$$

Figure 7: Two LR fillings.

which is not basic because of the full third row. The basic demolition results in the final quadruple $\widetilde{\mathfrak{R}} = ((3, 2, 1), (2, 2), 4 \times 4)$, which is in case (II$'$).

Thus suppose $a + h + 1 \leq \ell$. We have an inductive Stembridge demolition by removing the third column unless $g = k - 2$. Then removing the second row is an inductive Stembridge demolition unless $a = 1$.

Summarizing, we have reduced to the case that $\lambda = (k - 1, 2, 1^{\ell-3})$ and $\mu = ((k - 2)^h)$. Here is the first LR filling of a skew shape ν/λ:

$$\text{Column } k : 1$$
$$\text{Column } k - 1 : 2, \ldots, h$$
$$\text{Column } i, 3 \leq i \leq k - 2 : 1, \ldots, h$$
$$\text{Column } 2 : 1, \ldots, h - 1$$
$$\text{Column } 1 : h$$

For the second filling, interchange columns 2 and $k - 2$. See Figure 7 for an example.

3.3 Proof of Multiplicity in Case (III$'$)

Let $\lambda = (c^d, a^b)$ be a fat hook of shortness at least 2 and let μ be a rectangle of shortness at least 3. Thus, $\mu = (g^h)$ where $3 \leq g \leq k - 3$ and $3 \leq h \leq \ell - 3$.

Note that Lemma 2.1 implies that if $g > k - a$, then we can use emptiness demolitions and basic demolitions to obtain a basic situation thatwhich also lies in case (III$'$), so that we can conclude by induction. Similarly, we may assume from now on that $h < \ell - d$, since a similar argument holds if $h > \ell - d$, and since the $h = \ell - d$ reduces to the $g = k - a$ case handled below, after conjugation.

Next suppose $g = k - a - 1$ (we will deal with $g = k - a$ after this). If moreover $3 \leq h \leq b$, we have a filling F_h of a skew shape ν/λ with content $\mu = (g^h)$:

$$\text{Column } 1 : h - 1, h$$
$$\text{Column } 2 : h$$
$$\text{Column } c - 1 : 1, \dots, h - 2, h$$
$$\text{Column } c : 2, \dots, h - 1$$
$$\text{Column } k - 1 : 1, \dots, h - 1$$
$$\text{Column } k : 1$$
$$\text{All other columns } i > a : 1, \dots, h$$

Our assumptions ensure that the insertion of columns gives rise to a skew-shape automatically. It is also easy to see that the corresponding reading word is a ballot sequence. Finally, when checking that the semistandard conditions hold, the only worry is between columns c and $c+1$. But notice, by the assumption that all segments of the lattice path defined by λ are of length at least 2, the first comparison (in row $d+1 > 2$), between a 2 in column c and $d+1$ in column $c+1$, is satisfactory. Since the labels in both columns increment by 1 as we go down, all desired inequalities between these two columns are satisfied. The second filling G_h is obtained by interchanging the bottom-most entries of columns $c - 1$ and c. It is easy to see that this is also an LR filling. In the remaining cases, we invite the reader to consider how the fillings from the example above are adjusted.

Example 3.6   Consider $\mathfrak{R} = ((8^2, 3^5), (7^5), 12 \times 11)$ so $c = 8, d = 2, a = 3, b = 5$. Here $h = 5 = b$ and the filling given in Figure 8.
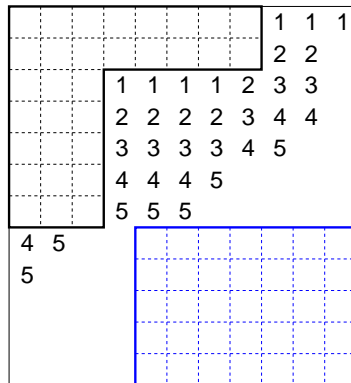

Figure 8: The LR filling F_5.

If $h = b + 1$, then we apply the instructions above, but when we place the entries in the columns as instructed, we see that we do not obtain a skew shape. However, we do obtain a skew shape if we push filled boxes to the left. Call the resulting fillings F_{h+1} and G_{h+1}. It is easy to verify that they are again LR fillings of the same shape, of content μ.

If $h > b + 1$, define F_h and G_h by taking F_{b+1} and G_{b+1} and then adding $b + 2, \ldots, h$ onto the first g columns. (Note that since $h < \ell - d$, there is always sufficient room to do this.) Now F_h and G_h clearly serve our purpose.

Next, when $g = k - a$, note that $a \geq 3$, so we can apply the same procedures as above, except we move the first two columns labeled with $h - 1, h$ and h to columns 2 and 3, and insert into column 1 the labels $1, \ldots, h$.

Finally, consider what happens when $g < k - a - 1$. If we assume further that $h \leq b$, then simply remove some columns whose content is $1, \ldots, h$ from the fillings F_h, G_h defined above, and then move all boxes to the left so as to form a skew shape. We apply a similar procedure when $h = b + 1$, although we must remove the columns whose content is $1, \ldots, h$ prior to sliding any of the squares over. When $h > b + 1$, build the fillings just described for $h = b + 1$, and then add $b + 2, \ldots, h$ to the first g columns.

In each case, we produce the two desired LR fillings.

3.4 Conclusion of the Proof of Theorems 1.2 and 1.2$'$

Having reduced nonbasic quadruples to the basic case, by Lemma 2.3, we have just shown that the cases (I$'$)–(IV$'$) imply multiplicity. Conversely, suppose that $\mathfrak{R}$ is basic, but this quadruple does not satisfy any of (I$'$)–(IV$'$), then we need to show that $\sigma_\lambda \cdot \sigma_\mu$ is multiplicity-free.

Since we are not in case (I$'$), one of λ and μ has only one part size; so let us assume that μ is a rectangle.

For a partition $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s > 0) \subseteq \ell \times k$, Stembridge [9] defines

$$\alpha^* = (\alpha_1 - \alpha_\ell \geq \alpha_1 - \alpha_{\ell-1} \geq \cdots \geq \alpha_1 - \alpha_1).$$

By conjugation symmetry, we may assume that μ (as placed in the bottom right corner of $\ell \times k$) is closer to the top edge of $\ell \times k$ than the leftmost edge. We consider the possibilities for what this distance is. By the basicness assumption, it is at least one unit.

Now, if the distance is precisely one unit, then μ^* is a single row partition, so the product is multiplicity-free by [9, Theorem 3.1(i)].

Next assume that μ is exactly two units from the top edge of $\ell \times k$. If μ is a one-line rectangle (*i.e.,* has only one nonzero part), the product is multiplicity-free by [9, Theorem 3.1(i)], so assume otherwise. If λ has at least three part sizes, then we are in case (II$'$). Therefore, λ has only one or two part sizes. Note that in either case, λ is a rectangle or fat hook and μ^* is a two-line rectangle (*i.e.,* it has exactly two equal nonzero part sizes). So [9, Theorem 3.1(ii, iv)] allows us to conclude multiplicity-freeness here also.

Finally, assume that μ is at least three units from the top edge of $\ell \times k$. As before, if μ is a one-line rectangle, the product is multiplicity-free, so assume otherwise. As above, to avoid being in case (II$'$), λ has at most two part sizes. If μ were a two-line rectangle, the product would be multiplicity-free by [9, Theorem 3.1(ii, iv)], so assume otherwise. Now avoiding (III$'$) means that we are multiplicity-free, by

[9, Theorem 3.1(iii)] (λ or, after conjugating if necessary, λ^* is a "near rectangle" in the terminology of [9]). ∎

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Department of Mathematics and Statistics, University of New Brunswick, Fredericton, New Brunswick E3B 5A3
e-mail: hugh@math.unb.ca

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA
e-mail: ayong@illinois.edu