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RATLIFF–RUSH CLOSURE OF IDEALS IN INTEGRAL DOMAINS

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Abstract. This paper studies the Ratliff–Rush closure of ideals in integral domains. By definition, the Ratliff–Rush closure of an ideal I of a domain R is the ideal given by $\tilde{I} := \cup(I^{n+1} :_R I^n)$, and an ideal I is said to be a Ratliff–Rush ideal if $\tilde{I} = I$. We completely characterise integrally closed domains in which every ideal is a Ratliff–Rush ideal, and we give a complete description of the Ratliff–Rush closure of an ideal in a valuation domain.

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1. Introduction. Let R be a commutative ring with identity and I a regular ideal of R; that is I contains a non-zero divisor. The ideals of the form $(I^{n+1} :_R I^n) := \{x \in R | xI^n \subseteq I^{n+1}\}$ increase with n. In the case in which R is a Noetherian ring, the union of this family is an interesting ideal, first studied by Ratliff and Rush in [**23**]. In [**13**], W. Heinzer, D. Lantz and K. Shah called the ideal $\tilde{I} := \cup(I^{n+1} :_R I^n)$ the Ratliff–Rush closure of I or the Ratliff–Rush ideal associated with I. An ideal I is said to be a Ratilff–Rush ideal, or Ratliff–Rush closed, if $I = \tilde{I}$. Among the interesting facts of this ideal is that for any regular ideal I in a Noetherian ring R, there exists a positive integer n such that for all $k \geq n$, $I^k = \widetilde{I^k}$, that is all sufficiently high powers of a regular ideal are Ratliff–Rush ideals, and a regular ideal is always a reduction of its Ratliff–Rush closure in the sense of Northcoot and Rees (see [**18**]), that is $I(\tilde{I})^n = (\tilde{I})^{n+1}$ for some positive integer n. Also the ideal $\tilde{I}$ is always between I and the integral closure I' of I, that is $I \subseteq \tilde{I} \subseteq I'$, where $I' := \{x \in R | x$ satisfies an equation of the form $x^k + a_1x^{k-1} + \cdots + a_k = 0$, where $a_i \in I^i$ for each $i \in \{1, \ldots, k\}\}$. Therefore, integrally closed ideals, i.e. ideals such that $I = I'$, are Ratliff–Rush ideals. Since then, many investigations of the Ratliff–Rush closure of ideals in a Noetherian ring have been carried out (for instance see [**12**, **13**, **17**, **24**], among others). The purpose of this paper is to extend the notion of Ratliff–Rush closure of ideals to an arbitrary integral domain and examine ring-theoretic properties of this kind of closure. In the second section, we give an answer to a question raised by B. Olberding [**21**] about the classification of integral domains for which every ideal is a Ratliff–Rush ideal in the context of integrally closed domains. This leads us to give a new characterisations of Prüfer and strongly discrete Prüfer domains. Specifically, we prove that 'a domain R is a Prüfer (respectively strongly discrete Prüfer) domain if and only if R is integrally closed and each non-zero finitely generated (respectively each non-zero) ideal of R is a Ratliff–Rush ideal (Theorem 2.6). It turns out that a

Ratliff–Rush domain (i.e. a domain such that each non-zero ideal is a Ratliff–Rush ideal) is a quasi-Prüfer domain; that is, its integral closure is a Prüfer domain. As an immediate consequence, we recover a characterisation of Noetherian Ratliff–Rush domains due to Heinzer, Lantz and Shah (Corollary 2.8). The third section deals with valuation domains. Here, we give a complete description of the Ratliff–Rush closure of a non-zero ideal in a valuation domain (Proposition 3.2), and we state necessary and sufficient condition under which the Ratliff–Rush closure preserves inclusion (Proposition 3.3). We also extend the Ratliff–Rush closure to arbitrary non-zero fractional ideals of a domain R, and we investigate its link to the notions of star operations. We prove that 'for a valuation domain V, the Ratliff–Rush closure is a star operation if and only if every non-zero non-maximal prime ideal of V is not idempotent, and in this case it coincides with the v-closure (Theorem 3.5).

Throughout, R denotes an integral domain, $qf(R)$ its quotient field and R' and $\overline{R}$ its integral closure and complete integral closure respectively. For a non-zero (fractional) ideal I of R, the inverse of I is given by $I^{-1} = (R : I) := \{x \in qf(R) | xI \subseteq R\}$. The v-closure and t-closure are defined respectively by $I_v = (I^{-1})^{-1}$ and $I_t = \cup J_v$, where J ranges over the set of f.g. subideals of I. We say that I is divisorial (or a v-ideal) if $I = I_v$ and a t-ideal if $I = I_t$. Unreferenced material is standard as in [**11**] or [**16**].

2. Ratliff–Rush ideals in an integral domain.

Let R be an integral domain. A non-zero ideal I of R is L-stable (here L stands for Lipman) if $R^I := \cup(I^n : I^n) = (I : I)$. The ideal I is stable (or Sally–Vasconcelos stable) if I is invertible in its endomorphisms ring $(I : I)$ [**25**]. A domain R is L-stable (respectively stable) if every non-zero ideal of R is L-stable (respectively stable). We recall that a stable domain is L-stable [**1**, Lemma 2.1], and for recent developments on stability (in settings different than originally considered), we refer the reader to [**1**, **19**–**22**]. We start this section with the following definition which extends the notion of Ratliff–Rush closure to non-zero integral ideals in an arbitrary integral domain.

Definition 2.1. Let R be an integral domain and I a non-zero integral ideal of R. The Ratliff–Rush closure of I is the (integral) ideal of R given by
$\tilde{I} = \cup(I^{n+1} :_R I^n)$. An integral ideal I of R is said to be a Ratliff–Rush ideal, or Ratliff–Rush closed, if $I = \tilde{I}$, and R is said to be a Ratliff–Rush domain if each non-zero integral ideal of R is a Ratliff–Rush ideal.

The following useful lemma treats the Ratliff–Rush closure of some particular classes of ideals.

Lemma 2.2. *Let R be an integral domain. Then*

(1) *all stable (and thus all invertible) ideals are Ratliff–Rush.*
(2) $\tilde{I} = R$ *if I is a non-zero idempotent ideal of R.*

Proof. (1) Let I be a stable ideal of R and set $T = (I : I)$. Then $I(T : I) = T$. Now, let $x \in \tilde{I}$. Then $x \in R$ and $xI^s \subseteq I^{s+1}$ for some positive integer s. Composing the two sides with $(T : I)$ and using the fact that $I(T : I) = T$, we obtain $xI^{s-1} \subseteq I^s$. Iterating this process, we get $xT \subseteq I$. Hence $x \in I$ and therefore $I = \tilde{I}$, as desired.

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(2) Let I be a non-zero idempotent ideal of R. Then for each n, $I^n = I$. So $(I^{n+1} :_R I^n) = (I :_R I) = (I : I) \cap R = R$. Hence $\tilde{I} = R$. □

The next proposition relates Ratliff–Rush closure to L-stability.

PROPOSITION 2.3. *Let R be an integral domain. If R is a Ratliff–Rush domain, then R is L-stable.*

Proof. Assume that R is a Ratliff–Rush domain. Let I be a non-zero (integral) ideal of R and let $x \in R^I$. Then there exists a positive integer n such that $xI^n \subseteq I^n$. Let $0 \neq d \in R$ such that $dx \in R$. Then $xI^{n+1} \subseteq I^{n+1}$ implies that $dxI(dI)^n = d^{n+1}xI^{n+1} \subseteq d^{n+1}I^{n+1} = (dI)^{n+1}$. Hence $dxI \subseteq ((dI)^{n+1} : (dI)^n)$. Since $dxI \subseteq R$, $dxI \subseteq \widetilde{(dI)} = dI$ (since R is Ratliff–Rush) and so $xI \subseteq I$. Hence $x \in (I : I)$ and therefore $R^I = (I : I)$. So I is L-stable, and therefore R is L-stable, as desired. □

It is easy to see that for a finitely generated ideal I of a domain R, in particular if R is Noetherian, $\tilde{I} \subseteq I'$. However, this is not the case for an arbitrary ideal of an integral domain. Indeed, let V be a valuation domain with maximal ideal M such that $M^2 = M$, $0 \neq a \in M$, and set $I = aM$. It is easy to see that $\tilde{I} = a(M : M) \cap V = aV$ and $I = I'$ (since all ideals of a Prüfer domain are integrally closed). The next theorem establishes a connection between stable domains, Ratliff–Rush domains and domains for which $\tilde{I} \subseteq I'$ for all ideals I. For this, we need the following crucial lemma.

LEMMA 2.4. *Let R be an integral domain. If $\tilde{I} = I$ for every finitely generated ideal I of R, then R' is a Prüfer domain.*

Proof. Let N be a maximal ideal of R'. To show that R'_N is a valuation domain, let $x = \frac{a}{b} \in qf(R)$, where $a, b \in R \setminus \{0\}$. Let J be the ideal (a^4, a^3b, ab^3, b^4) of R. Then $a^2b^2J = (a^6b^2, a^5b^3, a^3b^5, a^2b^6) \subseteq J^2 = (a^8, a^7b, a^5b^3, a^4b^4, a^6b^2, a^3b^5, a^2b^6, ab^7, b^8)$. So $a^2b^2 \in (J^2 :_R J) \subseteq \tilde{J} = J$. Thus $a^2b^2 = g_1a^4 + g_2a^3b + g_3ab^3 + g_4b^4$ for some g_1, g_2, g_3 and g_4 in R. Dividing by b^4, we get $0 = g_1x^4 + g_2x^3 - x^2 + g_3x + g_4$. By the u, u^{-1} theorem [**16**, Theorem 67], we get that either $x \in R'_N$ or $x^{-1} \in R'_N$, as desired. □

THEOREM 2.5. *Let R be an integral domain. Consider the following:*

(1) *R is stable;*
(2) *R is Ratliff–Rush;*
(3) *$\tilde{I} \subseteq I'$ for each non-zero ideal I of R;*
(4) *R has no non-zero idempotent prime ideals.*

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4). *Moreover, if R is a semi-local Prüfer domain, then* (4) $\Longrightarrow$ (1).

Proof. (1) $\Longrightarrow$ (2) by Lemma 2.2.
(2) $\Longrightarrow$ (3) is clear.
For (3) $\Longrightarrow$ (4), assume that P is a non-zero idempotent prime ideal of R. Now if $I = aP$ with $0 \neq a \in P$, then for all $n \geq 1$, $(I^{n+1} :_R I^n) = (I^{n+1} : I^n) \cap R = (a^{n+1}P : a^nP) \cap R = a(P : P) \cap R = a(P : P)$ (since $a(P : P) \subseteq P(P : P) = P \subseteq R$). So $a \in a(P : P) = \tilde{I}$. Suppose $a \in I' = (aP)'$. Then $a^k + c_1a^{k-1} + \cdots + c_k = 0$, where $c_i = a^ib_i \in I^i = a^iP$ for each $i \in \{1, \ldots, k\}$. So $a^k + b_1a^k + b_2a^k + \cdots + b_ka^k = 0$ with $b_i \in P$. Thus $a^k(1 + b) = 0$ with $b \in P$, a contradiction.
(4) $\Longleftrightarrow$ (1) if R is a semi-local Prüfer domain by [**1**, Theorem 2.10]. □

We are now ready to announce the main theorem of this section. It gives a classification of the integral domains for which every ideal is a Ratliff–Rush ideal in the context of integrally closed domains and states a new characterisation of Prüfer and strongly discrete Prüfer domains. Recall that a Prüfer domain is said to be strongly discrete if $P \neq P^2$ for each non-zero prime ideal P of R.

THEOREM 2.6. *Let R be an integrally closed domain. The following statements are equivalent:*

(1) $\tilde{I} = I$ *for every finitely generated (respectively every) non-zero ideal I of R;*

(2) *R is Prüfer (respectively strongly discrete Prüfer).*

Proof. (1) $\Longrightarrow$ (2) By Lemma 2.4, R is a Prüfer domain. Moreover, if each ideal is a Ratliff–Rush ideal, by Theorem 2.5, R is strongly discrete.

(2) $\Longrightarrow$ (1). Let R be a Prüfer domain. Then every finitely generated ideal is invertible and therefore a Ratliff–Rush ideal by Lemma 2.2. Assume that R is a strongly discrete Prüfer domain. Let I be a non-zero ideal of R and let $x \in \tilde{I}$. Then $x \in R$ and $xI^s \subseteq I^{s+1}$ for some positive integer s. Let M be a maximal ideal of R. If $I \nsubseteq M$, then $x \in R \subseteq R_M = IR_M$. Assume that $I \subseteq M$. Since $x \in R_M$ and $xI^sR_M \subseteq I^{s+1}R_M$, $x \in \widetilde{IR_M}$. Since R is strongly discrete, R_M is a strongly discrete valuation domain. By Theorem 2.5, $\widetilde{IR_M} = IR_M$. Hence $x \in IR_M$. So $x \in \bigcap\{IR_M / M \in Max(R)\} = I$. Hence $I = \tilde{I}$, as desired. □

The following example shows that the above theorem is not true if R is not integrally closed.

EXAMPLE 2.7. Let $\mathbb{Q}$ be the field of rational numbers, X an indeterminate over $\mathbb{Q}$ and $V = \mathbb{Q}(\sqrt{2})[[X]] = \mathbb{Q}(\sqrt{2}) + M$. Set $R = \mathbb{Q} + M$. Then R is stable. Let I be a non-zero (integral) ideal of R. Since R is local with maximal ideal M, $I \subseteq M$. If I is an ideal of V, then $I = cV$ for some $c \in I$. If I is not an ideal of V, then $I = m(W + M)$, where $\mathbb{Q} \subseteq W \subsetneqq \mathbb{Q}(\sqrt{2})$ is a $\mathbb{Q}$-vector space. Since $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, $\mathbb{Q} = W$, and so $I = cR$. Therefore R is stable and then Ratliff–Rush by Theorem 2.5. However, R is not a Prüfer domain [**4**, Theorem 2.1].

Our next corollary recovers a characterisation of Noetherian Ratliff–Rush domains due to Heinzer, Lantz and Shah [**13**].

COROLLARY 2.8. (*cf.* [**13**, Proposition 3.1 and Theorem 3.9]) *Let R be a Noetherian domain. Then R is a Ratliff–Rush domain if and only if R is stable.*

Proof. Since R is Noetherian, $R' = \bar{R}$ is a Krull domain. By Lemma 2.4, R' is a Prüfer domain. Hence R' is a Dedekind domain and therefore $dimR = dimR' = 1$. By Proposition 2.3, R is L-stable and therefore stable by [**1**, Proposition 2.4]. □

We recall that a domain R is said to be strong Mori if R satisfies the ascending chain conditions on w-ideals [**8**]. Trivially, a Noetherian domain is strong Mori and a strong Mori domain is Mori. The next corollary shows that the Ratliff–Rush property forces a strong Mori domain to be Noetherian.

COROLLARY 2.9. *Let R be a strong Mori domain. If R is a Ratliff–Rush domain, then R is Noetherian.*

Proof. By Lemma 2.4, R' is a Prüfer domain. Hence every maximal ideal of R is divisorial (see [**6**, Corollary 2.5] and [**7**, Theorem 2.6]). Now, let M be a maximal ideal of R. Since $M = M_v$, R_M is Noetherian [**8**, Theorem 3.9]. Hence $R'_M = (R_M)' = \overline{R_M}$ is a Krull domain. But since R' is Prüfer, so is R'_M. Hence R'_M is Dedekind and so $htM = dimR_M = dimR'_M = 1$. Then $dimR = 1$ and therefore R is Noetherian [**8**, Corollary 3.10]. □

Recall that R is semi-normal if for each $x \in qf(R)$, $x^2, x^3 \in R$ implies that $x \in R$. Our next corollary states some conditions under which a Ratliff–Rush Mori domain has dimension one.

COROLLARY 2.10. *Let R be a Mori domain such that either $(R : \overline{R}) \neq 0$ or R is semi-normal. If R is a Ratliff–Rush domain, then $dimR = 1$.*

Proof. Assume that R is a Ratliff–Rush domain. By Lemma 2.4, R' is a Prüfer domain.
(1) If $(R : \overline{R}) \neq (0)$, then $\overline{R}$ is a Krull domain [**2**, Corollary 18]. Since $R' \subseteq \overline{R}$, $\overline{R}$ is a Prüfer domain and therefore Dedekind. Hence $\dim(\overline{R}) = 1$. By [**3**, Corollary 3.4], $\dim(R) = 1$, as desired.
(2) Assume that R is semi-normal. If $\dim(R) \geq 2$, then R has a maximal ideal M such that $htM \geq 2$. Set $B = (MR_M)^{-1} = (MR_M : MR_M)$. Since R_M is a local Mori domain which is semi-normal and $htMR_M = htM \geq 2$, B contains a non-divisorial maximal ideal N contracting to MR_M [**3**, Lemma 2.5]. Since R' is a Prüfer domain (Lemma 2.4) and combining [**6**, Corollary 2.5] and [**7**, Theorem 2.6], we get that every maximal ideal of B is a t-ideal and so a v-ideal, since B is Mori, which is absurd. Hence $\dim(R) = 1$, as desired. □

3. Ratliff–Rush ideals in a valuation domain. It is well known that the maximal ideal M of a valuation domain V is either principal or idempotent; any non-zero prime ideal P of V is a divided prime ideal, that is, $PV_P = P$; and any idempotent ideal is a prime ideal. Also we recall that a valuation domain is a TP domain, that is for each non-zero ideal I of V, either $II^{-1} = V$ or $II^{-1} = Q$ is a prime ideal of V [**9**, Proposition 2.1], and for each positive integer n, $I^nI^{-n} = II^{-1}$ [**14**, Remark 2.13(b)]. We will often use these facts without explicit mention. Finally V is strongly discrete if it has no non-zero idempotent prime ideal [**10**, Chapter 5.3].

LEMMA 3.1. *Let V be a valuation domain and I a non-zero ideal of V and assume that $\tilde{I} \neq V$. Then $(I : I) \subseteq (\tilde{I} : \tilde{I})$.*

Proof. Let I be a non-zero ideal of V, and assume that $\tilde{I} \neq V$. If $II^{-1} = V$, then $I = \tilde{I}$ by Lemma 2.2 and therefore $(I : I) = (\tilde{I} : \tilde{I})$. Assume that $II^{-1} = Q$ is a prime ideal of V. Since V is a valuation domain, V is L-stable. So $(I : I) = (I^n : I^n)$ for each positive integer n. Let $x \in (I : I)$ and $z \in \tilde{I}$. Then $z \in V$ and $zI^r \subseteq I^{r+1}$ for some positive integer r. Since $(I : I) = (I^{r+1} : I^{r+1})$, $xzI^r \subseteq xI^{r+1} \subseteq I^{r+1}$. Hence $xz \in (I^{r+1} : I^r)$. To show that $xz \in \tilde{I}$, it suffices to prove that $xz \in V$. Suppose that $xz \notin V$. Then $(xz)^{-1} \in V$. Since $z \in \tilde{I}$, $x^{-1} = (xz)^{-1}z \in \tilde{I}$. So $x^{-1} \in V$ and $x^{-1}I^s \subseteq I^{s+1}$ for some positive integer s. Hence $I^s \subseteq xI^{s+1} \subseteq I^{s+1}$ (since $(I : I) = (I^{s+1} : I^{s+1})$) and therefore $I^s = I^{s+1}$. Hence $I^s = I^{2s}$, and therefore $I = P$ is an idempotent prime ideal of V. By Lemma 2.2, $\tilde{I} = \tilde{P} = V$, which is absurd. Hence $xz \in V$. So $xz \in \tilde{I}$ and then $x\tilde{I} \subseteq \tilde{I}$. Hence $x \in (\tilde{I} : \tilde{I})$ and therefore $V_Q = (I : I) \subseteq (\tilde{I} : \tilde{I})$. □

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The next proposition describes the Ratliff–Rush closure of a non-zero integral ideal in a valuation domain.

PROPOSITION 3.2. *Let I be a non-zero integral ideal of a valuation domain V. Then*
(1) *$\tilde{I} = V$ if and only if I is an idempotent prime ideal of V.*
(2) *assume that $\tilde{I} \subsetneq V$; now either $\tilde{I} = I$ or $\tilde{I} = (IQ :_V Q)$ for some non-zero prime ideal Q of V.*

Proof. (1) If I is an idempotent prime ideal of V, by Lemma 2.2, $\tilde{I} = V$. Conversely, assume that $\tilde{I} = V$. Then there exists a positive integer n such that $I^n \subseteq I^{n+1}$. Hence $I^n = I^{n+1}$. By induction, $(I^n)^2 = I^n$. So I^n is an idempotent ideal of V. Hence $I^n = P$ is a prime ideal of V. Then $I \subseteq P \subseteq I$ and therefore $I = P$, as desired.

(2) Assume that $\tilde{I} \subsetneq V$. If $II^{-1} = V$, then $I = \tilde{I}$ by Lemma 2.2. Assume that $II^{-1} = Q \subsetneq V$ is a prime ideal. Then $(I : I) = V_Q$ and for each positive integer n, $I^n I^{-n} = Q$ since V is a TP domain. Let $x \in \tilde{I}$. Then $x \in V$ and $xI^n \subseteq I^{n+1}$ for some positive integer n. So $xQ = xI^n I^{-n} \subseteq xI^{n+1} I^{-n} = IQ$. Hence $x \in (IQ :_V Q)$ and therefore $\tilde{I} \subseteq (IQ :_V Q)$. Now, assume that $I \subsetneq \tilde{I} \subsetneq V$.
To complete the proof, we will show that $\tilde{I} = (IQ :_V Q)$. Since $V_Q = (I : I) \subseteq (\tilde{I} : \tilde{I})$ (Lemma 3.1), $\tilde{I}$ is an ideal of V_Q. Suppose that $\tilde{I} \subsetneq (IQ :_V Q)$. Let $x \in (IQ :_V Q) \setminus \tilde{I}$. Since V is a valuation domain, $\tilde{I} \subsetneq xV$. So $x^{-1}\tilde{I} \subsetneq V \subseteq V_Q$. Hence $x^{-1}\tilde{I}$ is a proper ideal of V_Q. So $x^{-1}\tilde{I} \subseteq Q$ ($Q = QV_Q$ is the maximal ideal of V_Q). Hence $\tilde{I} \subseteq xQ \subseteq IQ \subseteq I \subsetneq \tilde{I}$, a contradiction. It follows that $\tilde{I} = (IQ :_V Q)$, as desired. □

Our next proposition shows that the Ratliff–Rush closure of an ideal I in a valuation domain is itself a Ratliff–Rush ideal and gives necessary and sufficient condition for preserving the Ratliff–Rush closure under inclusion.

PROPOSITION 3.3. *Let I be a non-zero ideal of a valuation domain V. Then*
(1) $\tilde{\tilde{I}} = \tilde{I}$.
(2) *$\tilde{I} \subseteq \tilde{J}$ for all ideals $I \subseteq J$ if and only each non-zero non-maximal prime ideal of V is not idempotent.*

Proof. (1) If $I = \tilde{I}$ or $\tilde{I} = V$, then clearly $\tilde{\tilde{I}} = \tilde{I}$. Assume that $I \subsetneq \tilde{I} \subsetneq V$. By Proposition 3.2, $\tilde{I} = (IQ :_V Q)$, where $Q = II^{-1}$ is a prime ideal of V (note that $II^{-1} \subsetneq V$, otherwise $I = \tilde{I}$, by Lemma 2.2). For simplicity, we set $J = \tilde{I}$. Our aim is to prove that $J = \tilde{J}$. If $JJ^{-1} = V$, then $J = \tilde{J}$ by Lemma 2.2. Assume that $JJ^{-1} \subsetneq V$. By Lemma 3.1, $V_Q = (I : I) \subseteq (\tilde{I} : \tilde{I}) = (J : J) = V_P$, where $P = JJ^{-1}$. So $P \subseteq Q$. Let $x \in \tilde{J}$. Then $x \in V$ and $xJ^n \subseteq J^{n+1}$ for some positive integer n. Composing the two sides with J^{-n} and using the fact that $P = JJ^{-1} = J^n J^{-n}$, we obtain $xP \subseteq JP$. Hence $\tilde{J}P \subseteq JP \subseteq JQ = \tilde{I}Q = IQ$. Now, if $P \subsetneq Q$, then let $a \in Q \setminus P$. Since V is a valuation domain, $P \subsetneq aV$. So $a^{-1}P \subsetneq V$. Hence $a^{-1} \in (V : P) = (P : P) = V_P = (J : J)$ [**15**]. So $a^{-1}J \subseteq J$. Then $J \subseteq aJ \subseteq QJ = QI \subseteq I \subsetneq J$, a contradiction. Hence $P = Q$. So $\tilde{J}P = \tilde{J}Q = JQ = IQ$. Hence $\tilde{J} \subseteq (IQ :_V : Q) = \tilde{I} = J$, as desired.

(2) Assume that $\tilde{I} \subseteq \tilde{J}$ for every ideals $I \subseteq J$. Suppose that there is a non-zero non-maximal prime ideal P of V such that $P^2 = P$. Let $a \in M \setminus P$, where M is the maximal ideal of V. Since V is a valuation domain, $P \subsetneq aV = I$. By Lemma 2.2 and the hypothesis, $V = \tilde{P} \subseteq \tilde{I} = aV \subseteq M$, which is absurd.

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Conversely, assume that each non-zero non-maximal prime ideal of V in not idempotent, and let $I \subseteq J$ be ideals of V. If $I = \tilde{I}$ or $\tilde{J} = V$, then clearly $\tilde{I} \subseteq \tilde{J}$. If $\tilde{I} = V$, by Proposition 3.2, $I = P$ is an idempotent prime ideal of V. By the hypothesis, $I = M$. So $M = I \subseteq J \subseteq M$. Then $I = J = M$ and so $\tilde{I} = \tilde{J}$. Hence we may assume that $I \subsetneq \tilde{I} \subsetneq V$ and $\tilde{J} \subsetneq V$. By Proposition 3.2, $\tilde{I} = (IQ :_V Q)$, where $Q = II^{-1}$. Now, suppose that $\tilde{I} \nsubseteq \tilde{J}$. Then let $x \in \tilde{I} \setminus \tilde{J}$. Since V is a valuation domain, $\tilde{J} \subsetneq xV$. So $x^{-1}I \subseteq x^{-1}J \subseteq x^{-1}\tilde{J} \subsetneq V \subseteq V_Q$. Since I is an ideal of $(I : I) = V_Q$, $x^{-1}I \subseteq Q$. So $I \subseteq xQ \subseteq \tilde{I}Q = IQ \subseteq I$. Therefore $I = xQ$. If Q is non-maximal, by the hypothesis, $Q^2 \subsetneq Q$. Hence $Q = aV_Q$ for some non-zero $a \in Q$ (since Q is the maximal ideal of V_Q). Hence $I = xQ = xaV_Q = xa(I : I)$. So I is stable, and by Lemma 2.2, $\tilde{I} = I$, which is absurd. Hence $Q = M$ and $I = xM$. If M is principal in V, then so is I, and therefore $\tilde{I} = I$, which is absurd. Hence $M = M^2$. So $\tilde{I} = (IM :_V M) = (xM^2 :_V M) = (xM :_V M) = x(M : M) = xV$. Let $b \in J \setminus I$. Then $xM = I \subsetneq bV$. Hence $xb^{-1}M \subseteq M$. So $xb^{-1} \in (M : M) = V$. Hence $x = (xb^{-1})b \in J \subseteq \tilde{J}$, which is absurd. It follows that $\tilde{I} \subseteq \tilde{J}$, as desired. □

Now, we extend the Ratliff–Rush closure to arbitrary non-zero fractional ideals, and we study its link to the notion of star operations. Our motivation is [**13**, Example 1.11], which provided an example of a Noetherian domain R with a non-zero ideal I such that $\widetilde{aI} \neq a\tilde{I}$ for some $0 \neq a \in R$. First, we recall that a star operation on R is a map $* : F(R) \longrightarrow F(R)$, $E \mapsto E^*$, where $F(R)$ denotes the set of all non-zero fractional ideals of R, with the following properties for each $E, F \in F(R)$ and each $0 \neq a \in K$:
(E_1) $R^* = R$ and $(aE)^* = aE^*$;
(E_2) $E \subseteq E^*$, and if $E \subseteq F$, then $E^* \subseteq F^*$;
(E_3) $E^{**} = E^*$.
For more details on the notion of star operations, we refer the reader to [**11**].

Definition 3.4. Let R be an integral domain with quotient field K, and let I be a non-zero fractional ideal of R.
(1) The generalised Ratliff–Rush closure of I is defined by $\hat{I} := \{x \in K | xI^n \subseteq I^{n+1}$, for some $n \geq 1\}$. Clearly $\tilde{I} = \hat{I} \cap R$ for any non-zero integral ideal I of R.

It is easy to see that for a non-zero fractional ideal I of a domain R, $\hat{I}$ is an R-module which is a fractional ideal if $(R : R^I) \neq 0$. In particular if R is conducive (i.e. the conductor $(R : T) \neq (0)$ for each overring $T \subsetneq qf(R)$ of R [**5**]) or L-stable, then $\hat{I}$ is always a fractional ideal of R. The next theorem gives necessary and sufficient conditions for the generalised Ratliff–Rush closure to be a star operation on a valuation domain.

Theorem 3.5. *Let V be a valuation domain. The generalised Ratliff–Rush closure on V is a star operation if and only if each non-zero non-maximal prime ideal P of V is not idempotent. In this case, it coincides with the v-operation.*

Proof. Assume that the generalised Ratliff–Rush closure is a star operation. Then, by Proposition 3.3, each non-zero non-maximal prime ideal of V is not idempotent. Conversely, assume that each non-zero non-maximal prime ideal of V is not idempotent.

Claim. For each integral ideal I of V, $\tilde{I} = \hat{I}$. Indeed, it suffices to show that $\hat{I} \subseteq V$. If $II^{-1} = V$, then $\hat{I} = I$, as desired. Assume that $II^{-1} = Q$ is a prime ideal of V. Then

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$(I : I) = V_Q$. Let $x \in \hat{I}$. Then $xI^n \subseteq I^{n+1}$ for some positive integer n. Since $I^n I^{-n} = Q$, we get $xQ \subseteq IQ$. Now, if $Q = M$, then $xM \subseteq IM \subseteq M$. So $x \in (M : M) = V$. If $Q \subsetneq M$, by hypothesis, Q is not idempotent. Hence $Q = aV_Q$ (since Q is the maximal ideal of V_Q). So $xaV_Q \subseteq aIV_Q = aI$ (here I is an ideal of $(I : I) = V_Q$). Hence $xV_Q \subseteq I$ and therefore $x \in I \subseteq V$, as desired.

Now, we prove the three properties of star operations. Let I and J be non-zero fractional ideals of V and $o \neq a \in qf(V)$.

(1) (E_1): $x \in \widehat{aI}$ if and only if $x(aI)^n \subseteq (aI)^{n+1}$ for some positive integer n, if and only if $xa^{-1} \in (I^{n+1} : I^n) \subseteq \hat{I}$, if and only if $x \in a\hat{I}$.

(2) (E_2): Let $o \neq d \in V$ such that $dI \subseteq dJ \subseteq V$. By (E_1), Proposition 3.3(2) and the claim, $d\hat{I} = \widehat{dI} = \widetilde{dI} \subseteq \widetilde{dJ} = \widehat{dJ} = d\hat{J}$. Hence $\hat{I} \subseteq \hat{J}$.

(3) (E_3): Clearly $I \subseteq \hat{I}$ and by (E_1) and Proposition 3.3(1), $\hat{\hat{I}} = \hat{I}$.

To complete the proof, we prove that $\tilde{I} = I_v$ for each non-zero fractional ideal I of V. Since the v-operation is the largest star operation on V, $\hat{I} \subseteq I_v$. Suppose that $\hat{I} \subsetneq I_v$ for some ideal I of V. Then I is not divisorial in V. Hence $I = aM$ for some $a \in qf(V)$ and $M = M^2$. Since M is idempotent, M is not divisorial. So $M_v = V$. Hence $I_v = aM_v = aV = \hat{I}$ (note that by (E_1) and Lemma 2.2 $\hat{I} = a\hat{M} = a\tilde{M} = aV$), which is absurd. □

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REFERENCES

1. D. D. Anderson, J. A. Huckaba and I. J. Papick, A notes on stable domains, *Houston J. Math.* **13** (1) (1987), 13–17.

2. V. Barucci, Strongly divisorial ideals and complete integral closure of an integral domain, *J. Algebra* **99** (1986), 132–142.

3. V. Barucci and E. Houston, On the prime spectrum of a Mori domain, *Comm. Algebra* **24** (11) (1996), 3599–3622.

4. E. Bastida and R. Gilmer, Overrings and divisorial ideals of rings of the form D+M, *Michigan Math. J.* **20** (1992), 79–95.

5. D. Dobbs and R. Fedder, Conducive integral domains, *J. Algebra* **86** (1984), 494–510.

6. D. Dobbs, E. Houston, T. Lucas, M. Roitman and M. Zafrullah, On t-linked overrings, *Comm. Algebra* **20** (1992), 1463–1488.

7. D. Dobbs, E. Houston, T. Lucas and M. Zafrullah, t-linked overrings and Prüfer v-multiplication domains, *Comm. Algebra* **17** (1989), 2835–2852.

8. W. Fangui and R. L. McCasland, On strong Mori domains, *J. Pure Appl. Algebra* **135** (1999), 155–165.

9. M. Fontana, J. Huckaba and I. Papick, Domains satisfying the trace property, *J. Algebra* **107** (1987), 169–182.

10. M. Fontana, J. Huckaba and I. Papick, *Prüfer domains*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 203 (Marcel Dekker, New York, 1997).

11. R. Gilmer, *Multiplicative ideal theory*, Pure and Applied Mathematics, no. 12. (Marcel Dekker, New York, 1972).

12. W. Heinzer, Johnston, D. Lantz and K. Shah, *The Ratliff–Rush ideals in a Noetherian ring: A survey in methods in module theory*, vol. 140 (Marcel Dekker, New York, 1992), 149–159.

13. W. Heinzer, D. Lantz and K. Shah, The Ratliff–Rush ideals in a Noetherian ring, *Comm. Algebra* **20** (1992), 591–622.

14. W. Heinzer and I. Papick, The radical trace property, *J. Algebra* **112** (1988), 110–121.

15. J. A. Huckaba and I. J. Papick, When the dual of an ideal is a ring, *Manuscripta Math.* **37** (1982), 67–85.

16. I. Kaplansky, *Commutative rings* (University of Chicago Press, Chicago, 1974).

17. J. C. Liu, Ratliff–Rush closures and coefficient modules, *J. Algebra* **201** (1998), 584–603.

18. D. G. Northcoot and D. Rees, Reductions of ideals in local rings, *Proc. Camb. Phil. Soc.* **50** (1954), 145–158.

19. B. Olberding, Globalizing local properties of Prüfer domains, *J. Algebra* **205** (1998), 480–504.

20. B. Olberding, On the classification of stable domains, *J. Algebra* **243** (2001), 177–197.

21. B. Olberding, Stability of ideals and its applications, in *Ideal theoretic methods in commutative algebra* (Anderson D. D. and Papick I. J., Editors), Lecture Notes in Pure and Applied Mathematics, vol. 220 (Marcel Dekker, New York, 2001), 319–341.

22. B. Olberding, On the structure of stable domains, *Comm. algebra* **30**(2) (2002), 877–895.

23. L. J. Ratliff, Jr, and D. E. Rush, Two notes on reductions of ideals, *Indiana Univ. Math. J.* **27** (1978), 929–934.

24. Rossi and I. Swanson, *Notes on the behavior of the Ratliff–Rush filtration*, Commutative Algebra, Contemporary Mathematics, vol. 331 (American Mathematical Society, Providence RI, 2003), 313–328.

25. J. D. Sally and W. V. Vasconcelos, Stable rings and a problem of Bass, *Bull. Amer. Math. Soc.* **79** (1973), 574–576.