Compositio Mathematica **134:** 1–34, 2002.

Katz Correspondence for Quasi-Unipotent Overconvergent Isocrystals

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(Received: 12 January 1999; accepted in final form: 14 August 2001)

Abstract. Let k be a field and $X = \mathrm{Spec}(k[t, t^{-1}])$. Katz proved that a differential equations with coefficients in $k((t^{-1}))$ is uniquely extended to a special algebraic differential equation on X when k is of characteristic 0. He also proved that a finite extension of $k((t^{-1}))$ is uniquely extended to a special covering of X when k is of any characteristic. These theorems are called canonical extension or Katz correspondence. We shall prove a p-adic analogue of canonical extension for quasi-unipotent overconvergent isocrystals. As a consequence, we can show that the local index of a quasi-unipotent overconvergent is equal to its Swan conductor.

Mathematics Subject Classifications (2000). 11S15, 12H25, 14F30.

Key words. overconvergent isocrystal, p-adic analysis, ramification theory.

1. Introduction

Let $\mathbb{G}_{m,\mathbb{C}} = \mathrm{Spec}\,\mathbb{C}[t, t^{-1}]$. Then the inclusion i: $\mathrm{Spec}\,\mathbb{C}((t)) \hookrightarrow \mathbb{G}_{m,\mathbb{C}}$ induces an inverse image functor $i^*: \mathrm{MC}(\mathbb{G}_{m,\mathbb{C}}/\mathbb{C}) \to \mathrm{MC}(\mathbb{C}((t))/\mathbb{C})$ from the category of locally free $\mathcal{O}_{\mathbb{G}_m}$-module with connection to the category of $\mathbb{C}((t))$-vector space with connection. In [20, (2.4.10)], Katz proved that this functor induces an equivalence between the full subcategory of the *special* objects [loc. cit. (2.4.9)] in $\mathrm{MC}(\mathbb{G}_{m,\mathbb{C}}/\mathbb{C})$ and $\mathrm{MC}(\mathbb{C}((t))/\mathbb{C})$. On the other hand, in [19, 1.4.1], Katz also proved its 'covering version'. To be precise, let k be a field of characteristic $p > 0$. Then he proved that the inverse image functor from the category of the special finite étale covering of $\mathbb{G}_{m,k}$ to the category of finite étale coverings of $\mathrm{Spec}\,k((t))$ is an equivalence of categories.

The purpose of this paper is to study p-adic analogue of these theorems. Let k be a field of characteristic $p > 0$ and consider the inclusion i: $\mathrm{Spec}\,k((t)) \hookrightarrow \mathbb{G}_{m,k}$. We shall prove the equivalence between the category of special overconvergent isocrystals (cf. [2]) on $\mathbb{G}_{m,k}$ and the category of 'local' overconvergent isocrystals (Theorem 7.15.). This result is a generalization of the covering version and the analogue of differential equation version of Katz correspondence.

⋆Partially supported by the Grant-in-Aid for Encouragement of Young Scientists.

As an application, we can give another proof of a theorem of Tsuzuki on the equality of the irregularity and the Swan conductor (Theorem 8.6), cf. [24, 29]. Moreover, the existence of canonical extension implies that the category of local quasi-unipotent overconvergent isocrystals has a fiber functor and, hence, it is a neutral Tannakian category.

We only treat the case of quasi-unipotent overconvergent isocrystals in this paper, but it seems to the author that there should be Katz correspondence for a larger category. For example, Garnier has shown that the analogous theorem holds for all the overconvergent isocrystals of rank one [15]. On the other hand, Richard Crew conjectured that that every local overconvergent F-isocrystal is quasi-unipotent, and recently Yves André [1] and Zoghman Mebkhout have announced proofs of this conjecture independently.

Now we explain the contents of this paper. In the first section, we review the basic properties of the Robba ring, which plays the role of a local field at a closed point of a usual algebraic curve in the context of this paper. (It is called *local algebra* in Crew's paper [10, 4].) We also define some terminology for modules with connection. In Section 3 we review the theory of overconvergent isocrystals in our case. In Section 4 and Section 5 we study unipotent objects and étale objects. In Section 6 we define special objects. In Section 7 we prove a local decomposition theorem for quasi-unipotent overconvergent isocrystals, which corresponds to a theorem of Levelt [23]. As a consequence, we get the Katz correspondence. In Section 8 we define the breaks and break-decomposition for a quasi-unipotent overconvergent isocrystal. Then we show the equality of the irregularity defined by Christol and Mebkhout and the swan conductor. As a result, the ramification filtration is compatible with the filtration defined by Christol and Mebkhout [5, 6] (Corollary 8.8.). Let us mention that R. Crew has also given the proof of the above theorem using the canonical extension independently [9].

2. The Robba Ring

In this section, we review the basic properties of the Robba ring (local algebra in Crew's terminology) and prove some lemmas, cf. [5], [10, 4.5], [30, 2].

2.1. DEFINITION

Let k be a perfect field of characteristic $p > 0$. Let K be a complete discrete valuation field of characteristic 0 with residue field k and $\mathcal{O}_K$ its ring of integers. We denote by $|\ |$ the (multiplicative) valuation of K normalized so that $|p| = p^{-1}$.

For an interval I in the set $[0, \infty]$ of nonnegative extended real numbers, we denote by $\mathcal{A}(I)$ the K-algebra of formal Laurent series in the variable t convergent for any number x of the completion of the algebraic closure of K such that $|x| \in I$.
When I is closed, then $\mathcal{A}(I)$ have obvious topologies. When I is open or half-open, we give $\mathcal{A}(I)$ the inverse limit topology arising from the equation

$$\mathcal{A}(I) = \bigcap_{J \subset I, J\text{closed}} \mathcal{A}(J).$$

Then we define *Robba ring* $\mathcal{R}_{K,t}$ to be $\varinjlim_{\lambda<1} \mathcal{A}([\lambda, 1))$ and regard it as topological K-algebra given the inductive limit topology. We observe that

$$\mathcal{R}_{K,t} = \left\{ \sum_{n=-\infty}^{\infty} a_n t^n \;\middle|\; \begin{array}{l} a_n \in K, \\ 0 < \forall\rho < 1, |a_n|\rho^n \to 0\,(n \to \infty), \\ 0 < \exists\lambda < 1, |a_n|\lambda^n \to 0\,(n \to -\infty) \end{array} \right\}.$$

We often denote $\mathcal{R}_{K,t}$ by $\mathcal{R}_K$, $\mathcal{R}$ or $\mathcal{R}_t$ if confusion does not occur.

We define subrings $K\langle t\rangle^\dagger$ and $\mathcal{O}_K\langle t\rangle^\dagger$ of $\mathcal{R}_{K,t}$ by

$$K\langle t\rangle^\dagger = \left\{ \sum_{n=-\infty}^{\infty} a_n t^n \;\middle|\; \begin{array}{l} a_n \in K, \\ \exists C > 0, \forall n > 0, |a_n| < C, \\ 0 < \exists\lambda < 1, |a_n|\lambda^n \to 0\,(n \to -\infty) \end{array} \right\},$$

$$\mathcal{O}_K\langle t\rangle^\dagger = \left\{ \sum_{n=-\infty}^{\infty} a_n t^n \;\middle|\; \begin{array}{l} a_n \in \mathcal{O}_K, \\ \exists C > 0, \forall n > 0, |a_n| < C, \\ 0 < \exists\lambda < 1, |a_n|\lambda^n \to 0\,(n \to -\infty) \end{array} \right\}.$$

Then $K\langle t\rangle^\dagger$ is a Henselian discrete valuation field with the ring of integers $\mathcal{O}_K\langle t\rangle^\dagger$ with respect to the Gauss norm [24, 2]. We define the *residue field* of $\mathcal{R}_{K,t}$ to be that of $\mathcal{O}_K\langle t\rangle^\dagger$. We call K the *coefficient field* of $\mathcal{R}_{K,t}$.

Let λ be a positive number such that $\lambda < 1$. For $\lambda < r \leqslant 1$ and $f = \sum_{n=-\infty}^{\infty} a_n t^n \in \mathcal{A}([\lambda, 1))$, we define $|f|_r = \sup_{n\in\mathbb{Z}} |a_n| r^n$. Note that it is possible that $|f|_1 = \infty$. If $r < 1$, it is a non-Archimedean valuation of $\mathcal{A}([\lambda, 1))$.

LEMMA 2.1. *$K\langle t\rangle^\dagger$ is algebraically closed in $\mathcal{R}_{K,t}$.*

Proof. First note that $f \in K\langle t\rangle^\dagger$ if and only if $\limsup_{r\to 1^-} |f|_r < \infty$. Suppose that $x \in \mathcal{R}_{K,t}$ is algebraic over $K\langle t\rangle^\dagger$. Then there exist $u_i \in K\langle t\rangle^\dagger$ $(1 \leqslant i \leqslant n)$ such that

$$x^n + u_1 x^{n-1} + \cdots + u_n = 0. \tag{1}$$

We can choose $0 < \lambda < 1$ so that $u_1, \ldots, u_n$, and x are all belong to $\mathcal{A}([\lambda, 1))$. Since $\limsup_{r\to 1^-} |u_i|_r$ is bounded for all $1 \leqslant i \leqslant n$, if $\limsup_{r\to 1^-} |x|_r = \infty$, there exists $\lambda < r < 1$ such that $|x|_r$ is larger than any $|u_i|_r$ and 1. Then we have $|x^n + u_1 x^{n-1} + \cdots + u_{n-1} x|_r > |u_n|_r$, which contradicts (1). Therefore $\limsup_{r\to 1^-} |x|_r < \infty$ and hence x belongs to $K\langle t\rangle^\dagger$. □

Let E be the residue field of $\mathcal{O}_K\langle t\rangle^\dagger$ and F a finite separable extension of E. Since $K\langle t\rangle^\dagger$ is Henselian, there exists a finite étale extension $\mathcal{O}_{\mathcal{F}}$ of $\mathcal{O}_K\langle t\rangle^\dagger$ with residue field F. Let $\mathcal{F}$ be the field of fractions of $\mathcal{O}_{\mathcal{F}}$ and K' an unramified extension of K whose residue field is that of F. Then, by [24, 3.4], $\mathcal{F} \simeq K'\langle u\rangle^\dagger$.

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LEMMA 2.2. $\mathcal{F} \otimes_{K\langle t\rangle^\dagger} \mathcal{R}_{K,t} \simeq \mathcal{R}_{K',u}$.

Proof. It is obvious that $K'\langle t\rangle^\dagger \otimes_{K\langle t\rangle^\dagger} \mathcal{R}_{K,t} \simeq \mathcal{R}_{K',t}$. If we replace $K\langle t\rangle^\dagger$ by $K'\langle t\rangle^\dagger$, we can assume that $K' = K$. Then as in [24, 3.4], there exists $b_i \in t\mathcal{O}_K[[t]]$ such that

$$u^m + b_1 u^{m-1} + \cdots + b_m = 0,$$

where m is the degree of F over E. Then the assertion is clear from the previous lemma. □

We denote $\mathcal{F} \otimes_{K\langle t\rangle^\dagger} \mathcal{R}_{K,t}$ by $\mathcal{R}_{K,t}(F)$ or $\mathcal{R}_K(F)$, $\mathcal{R}(F)$ if confusion does not occur.

2.2. FROBENIUS STRUCTURE

Let φ be a Frobenius endomorphism of K. We denote by K_n the subfield of K consisting of the elements fixed by φ^n. We assume that

$$K_1 \text{ contains a uniformizer of } K. \tag{2}$$

Then, if K_0 is the maximal absolutely unramified subfield of K, we have $K_1 \otimes_{\mathbb{Q}_p} K_0 \simeq K$. Let K' be a finite extension of K and K'_0 the maximal absolutely unramified subfield of K'. Suppose that K' satisfies the condition that

$$\begin{aligned}&\text{there exists a subfield } L \text{ of } K' \text{ totally ramified over } K_1\\ &\text{which contains a uniformizer of } K'.\end{aligned} \tag{3}$$

For such an extension K', we extend the Frobenius endomorphism φ so that its action on L is trivial. We denote the Frobenius on K' also by φ. Note that for any finite extension K' of K, there exists a finite extension of K'' of K' which satisfies the condition (3) ([7, 1.8], [30, 2.4.1]).

Let φ be a lifting to $\mathcal{O}_K\langle t\rangle^\dagger$ of the Frobenius endomorphism $x \mapsto x^p$ of the residue field E. In the rest of this paper, we always assume that the restriction of φ to K satisfies the above condition. We shall show that φ extend uniquely to the continuous endomorphism of $\mathcal{R}_K$.

LEMMA 2.3. *Let $\{f_i\}_{i=0}^\infty$ be a sequence of elements of $\mathcal{A}([\lambda, 1))$. If $\lim_{i\to\infty} |f_i|_r = 0$ for any $r \in [\lambda, 1)$, then $\sum_i f_i$ converges to an element of $\mathcal{A}([\lambda, 1))$.*

Proof. Since $\mathcal{A}([\lambda, 1))$ is complete ([5, 2.1], [10, 4.2]), it is clear. □

By Lemma 2.3, we can define a map $\varphi\colon \mathcal{R}_K \to \mathcal{R}_K$ by

$$\varphi\left(\sum_{n\in\mathbb{Z}} a_n t^n\right) = \sum_{n\in\mathbb{Z}} \varphi(a_n)\varphi(t)^n.$$

LEMMA 2.4. *Let π be a uniformizer of K and $q = p^n$. Let $f = \sum a_n t^n \in \mathcal{R}_{K,t}$. Then there exists an $r_0 < 1$ such that, for any r such that $r_0 < r < 1$, $|\varphi^n(f)|_r \leqslant |f|_{r^q}$.*

Proof. Since we can write $\varphi^n(t) = ut^q + h(t)$ with a unit u of $\mathcal{O}_K$ and $h(t) \in \pi\mathcal{O}_K\langle t\rangle^\dagger$, $|\varphi^n(t)|_r = r^q$ for r close enough to 1. Then $|\varphi^n(f)|_r \leqslant \sup_i |\varphi^n(a_i)||\varphi^n(t)|_r^i \leqslant \sup_i |a_i| r^{qi} = |f|_{r^q}$. □

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LEMMA 2.5. *φ: $\mathcal{R}_K \to \mathcal{R}_K$ is continuous.*

Proof. As an fundamental system of neighborhoods of 0 of $\mathcal{A}([\lambda, 1))$, we can take $U_{\varepsilon,I} = \{f \mid \|f\|_r < \varepsilon$ for $\forall r \in I\}$ for closed interval $I \subset [\lambda, 1)$ and $\varepsilon > 0$. By Lemma 2.4, there exists an r_0 such that for any $f \in \mathcal{A}[\lambda, 1)$, $|\varphi(f)|_r \leqslant |f|_{r^p}$ for any r such that $r_0 < r < 1$. Then if $f \in U_{\varepsilon,[r_1,r_2]}$, $\varphi(f) \in U_{\varepsilon,[r_1^p,r_2^p]}$ and, hence, φ: $\mathcal{A}[\lambda, 1) \to \mathcal{A}[\lambda^p, 1)$ is continuous. Taking inductive limit, we can see that φ: $\mathcal{R}_K \to \mathcal{R}_K$ is continuous, because $\varinjlim_{\lambda} \mathcal{A}[\lambda, 1)$ is also the topological inductive limit [10, 4.2]. □

Since $K\langle t\rangle^\dagger$ is dense in $\mathcal{R}_K$ and $\mathcal{R}_K$ is separated, φ is the unique continuous endomorphism of $\mathcal{R}_K$ extending φ on $K\langle t\rangle^\dagger$. We call such an endomorphism a *Frobenius* of $\mathcal{R}_K$.

Let φ be a Frobenius of $\mathcal{R}_{K,t}$ and F a finite separable extension of E. Let $\mathcal{O}_\mathcal{F}$ be a finite étale extension of $\mathcal{O}_K\langle t\rangle^\dagger$ with residue field F and $\mathcal{F}$ its field of fractions. Since $\mathcal{O}_K\langle t\rangle^\dagger$ is Henselian, a Frobenius endomorphism of $K\langle t\rangle^\dagger$ extends uniquely to that of $\mathcal{F}$. Thus we can uniquely extend φ to $\mathcal{R}_{K,t}(F) = \mathcal{F} \otimes_{K\langle t\rangle^\dagger} \mathcal{R}_{K,t}$ so that $\varphi(a \otimes b) = \varphi(a) \otimes \varphi(b)$. This homomorphism is also a Frobenius endomorphism of $\mathcal{R}_{K',u}$.

2.3. CONNECTION AND FROBENIUS

We denote by $\mathrm{MC}(\mathcal{R}_K/K)$ the category of projective $\mathcal{R}_K$-modules M of finite type with K-connection ∇: $M \to M \otimes \Omega_{\mathcal{R}_K}$. Here $\Omega_{\mathcal{R}_K}$ is defined to be $\mathcal{R}_K\,\mathrm{d}t/t$.

With the obvious notions of tensor product and internal hom, $\mathrm{MC}(\mathcal{R}_K/K)$ is a rigid Abelian K-linear tensor category [12].

PROPOSITION 2.6 (R. Crew [10, 6.1]). *If (M, ∇) is an object of $\mathrm{MC}(\mathcal{R}_K/K)$, then M is a free $\mathcal{R}_K$-module.*

For an object (M, ∇) in $\mathrm{MC}(\mathcal{R}_K/K)$, we define its cohomology groups by

$$\mathrm{H}^0_\nabla(M) = \mathrm{Ker}\,\nabla, \qquad \mathrm{H}^1_\nabla(M) = \mathrm{Cok}\,\nabla.$$

A φ^n-structure φ_n on (M, ∇) is a φ^n-linear map φ_n: $M \to M$ which commutes with ∇ such that its linearization

$$\Phi_n = \mathrm{id} \underset{\varphi^n}{\otimes} \varphi_n\colon\ \mathcal{R}_K \underset{\varphi^n}{\otimes} M \to M$$

is an isomorphism of $\mathcal{R}_K$-modules. We say a triple (M, ∇, φ_n) is a φ^n-∇-module over $\mathcal{R}_K$ if (M, ∇) is an object in $\mathrm{MC}(\mathcal{R}_K/K)$ and φ_n is a φ^n-structure on it. A morphism f: $(M, \nabla, \varphi_n) \to (M', \nabla', \varphi'_n)$ of φ^n-∇-modules is an $\mathcal{R}_K$-linear map which commutes with connections and φ^n-structures. We denote the category of φ^n-∇-modules by $\mathrm{MCF}_n(\mathcal{R}_K/K)$. Then $\mathrm{MCF}_n(\mathcal{R}_K/K)$ is also a rigid Abelian K_n-linear tensor category. If confusion does not occur, we often denote an object of $\mathrm{MCF}_n(\mathcal{R}_K/K)$ by M instead of (M, ∇, φ_n) for simplicity. Let K' be a finite extension of K which satisfies the condition (3). We extend the Frobenius endomorphism φ to K' as described in the paragraph following condition (3) and denote also by φ. Let $M' = M \otimes_K K'$.

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Then $\nabla' = \nabla_K \otimes \mathrm{id}$ and $\varphi_n \otimes \varphi^n$ define a connection and a φ^n-structure on M' respectively, so there is a natural functor $\mathrm{MCF}_n(\mathcal{R}_K/K) \to \mathrm{MCF}_n(\mathcal{R}_{K'}/K')$.

LEMMA 2.7. *Let φ be a Frobenius of $\mathcal{R}_K$. Let $\partial = t\,\mathrm{d}/\mathrm{d}t$ and $\mu_n(t) = \partial\varphi^n(t)/\varphi^n(t) = \sum_i c_i t^i$. Then $c_0 = q = p^n$ and $|\mu_n|_r \to |\mu_n|_1 \leqslant |\pi|$ when $r \to 1^-$. Here π is a uniformizer of K.*

Proof. Let $\varphi^n(t) = \sum b_i t^i$, then b_q is a unit and $b_i \equiv 0 (\mathrm{mod}\, \pi)$ if $i \neq q$. Therefore

$$\mu_n = \frac{q + \sum_{i\neq q} i \frac{b_i}{b_q} t^{i-q}}{1 + \sum_{i\neq q} \frac{b_i}{b_q} t^{i-q}} = \frac{q + q\sum_{i\neq q} q\frac{b_i}{b_q} t^{i-q} + \sum_{i\neq q}(i-q)\frac{b_i}{b_q} t^{i-q}}{1 + \sum_{i\neq q} \frac{b_i}{b_q} t^{i-q}}.$$

If we put $g(t) = \sum_{i\neq q}(b_i/b_q)t^{i-q}$, then $g(t) \in \pi\mathcal{O}_K\langle t\rangle^\dagger$ and

$$\mu_n = \frac{q(1+g) + \partial g}{1+g} = q + (\partial g)\left(\sum_{m=0}^{\infty}(-1)^m g^m\right).$$

Since

$$(\partial g)g^m = \frac{1}{m+1}\partial(g^{m+1})$$

has no constant term, the assertion is clear. □

PROPOSITION 2.8. *Let $\omega \in \Omega_{\mathcal{R}}$. If $\varphi^n(\omega) = \omega$, then $\omega = 0$.*

Proof. For simplicity we assume that $n = 1$, but we can prove the general case in the same way. Let us write

$$\omega = \sum_{i\in\mathbb{Z}} b_i t^i \frac{\mathrm{d}t}{t} = g\frac{\mathrm{d}t}{t}.$$

If $\varphi(\omega) = \omega$, then $g = \varphi(g)\mu$ with $\mu = \partial\varphi(t)/\varphi(t)$. We will show that $g = 0$. By Lemma 2.7, there exist $r_0 < 1$ and $0 < C < 1$ such that $g \in \mathcal{A}([r_0, 1))$ and $|\mu|_r < C$ for any r such that $r_0 < r < 1$. It is easy to see that for any such r, there exists an integer i such that $|g|_r = |b_i|r^i$.

Suppose that $g \neq 0$. Then the integer i is bounded, for any fixed r. We denote the maximum of such i by $i_1(r)$ and the minimum by $i_0(r)$. First we consider the case that there exists an r such that $i_1 = i_1(r^p) \geqslant 0$. Since $i_1(r') \geqslant i_1(r)$ if $r' > r$, we can assume that $|\varphi(g)|_r \leqslant |g|_{r^p}$ by Lemma 2.4. Then

$$|\varphi(g)|_r \leqslant |g|_{r^p} = |b_{i_1}|r^{pi_1} \leqslant |b_{i_1}|r^{i_1} \leqslant |g|_r$$

and, hence, $|g|_r = |\mu\varphi(g)|_r < C|g|_r$, a contradiction.

Next assume that $i = i_1(r) < 0$ for any r. Let r_0 be a real number such that the assertion of Lemma 2.4 holds for $n = 1$. Put $i_0 = i_0(r^p)$ for r such that $r_0 < r^p < 1$. Since $i_0(r_0) \leqslant i_0(r^p)$,

$$r^{(p-1)i_0(r_0)} \geqslant r^{(p-1)i_0} = \frac{|b_{i_0}|r^{pi_0}}{|b_{i_0}|r^{i_0}} \geqslant \frac{|b_{i_0}|r^{pi_0}}{|g|_r} = \frac{|g|_{r^p}}{|g|_r}.$$

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Thus if we choose r so that $r^{(p-1)i_0(r_0)} < 1/C$, we have $|g|_{r^p} \leqslant |g|_r/C$ and, hence,

$$|g|_r = |\mu\varphi(g)|_r \leqslant |\mu|_r|g|_{r^p} < C(1/C)|g|_r = |g|_r,$$

a contradiction. □

COROLLARY 2.9. $\mathcal{R}_K^{\varphi^n=1} := \{f \in \mathcal{R}_K \mid \varphi^n(f) = f\} = K_n$ (cf. Section 2.2).

Proof. Let $f \in \mathcal{R}_K^{\varphi^n=1}$. Since $\mathrm{d}\circ\varphi = \varphi\circ\mathrm{d}$, $\mathrm{d}f$ satisfies the assumption of Proposition 2.8 and, hence, $\mathrm{d}f = 0$. Therefore $f \in K$ and the assertion is clear. □

3. Overconvergent Isocrystals

We briefly review the overconvergent isocrystals in our case. Let k and K be as in Section 2. Let $X = \mathbb{G}_{m,k} = \operatorname{Spec} k[t, t^{-1}] \subset \bar{X} = \mathbb{P}_k^1$, and take those formally smooth liftings $\mathcal{X} = \operatorname{Spf} \mathcal{O}_K\{t, t^{-1}\} \subset \bar{\mathcal{X}} = \hat{\mathbb{P}}^1_{\mathcal{O}_K}$. For a smooth formal scheme $\mathcal{Y}$ over $\mathcal{O}_K$, we denote its Raynaud generic fiber by $\mathcal{Y}_K$.

We denote the category of overconvergent isocrystals (resp. overconvergent F-isocrystals) on X [2, (2.3.6), (2.5.1)] by $\mathrm{Isoc}^\dagger(X/K)$ (resp. F-$\mathrm{Isoc}^\dagger(X/K)$). Let $A^\dagger$ be the weak completion of $A = \mathcal{O}_K[t, t^{-1}]$ and $\Omega_{A^\dagger}$ the differential module of $A^\dagger$ in the sense of Monsky–Washnitzer [25, Th. 4.2], [31, (2.3)]. Then we have

$$\begin{aligned} A_K^\dagger &:= A^\dagger \otimes K \simeq \varinjlim_V \Gamma(V, \mathcal{O}_{\bar{\mathcal{X}}_K}) \\ &\simeq \left\{ \sum_{n=-\infty}^{\infty} a_n t^n \;\middle|\; \begin{array}{l} a_n \in K,\ |a_n|\lambda^{|n|} \to 0\,(|n| \to \infty) \\ \text{for some } \lambda > 1 \end{array} \right\}, \\ \Omega_{A_K^\dagger} &:= \Omega_{A^\dagger} \otimes K \simeq \varinjlim_V \Gamma(V, \Omega_V), \end{aligned}$$

where $]X[$ is the tube of X in $\bar{\mathcal{X}}_K$ and V runs through a cofinal set of strict neighborhoods of $]X[$ in $\bar{\mathcal{X}}_K$, cf. [2, §1.1, §1.2]). We denote by $\mathrm{MC}(A_K^\dagger/K)$ (resp. $\mathrm{MC}^\dagger(A_K^\dagger/K)$) the category of $A_K^\dagger$-module projective of finite type with integral connection $\nabla\colon\ M \to M \otimes \Omega_{A_K^\dagger}$ (resp. the full subcategory of $\mathrm{MC}(A_K^\dagger/K)$ of objects with connection whose Taylor series converges on a strict neighborhood of the diagonal in $\bar{\mathcal{X}}_K \times \bar{\mathcal{X}}_K$ (cf. [2, (2.5.2)])). Since $A_K^\dagger$ is a Noetherian ring [14], $\mathrm{Isoc}^\dagger(X/K)$ is equivalent to $\mathrm{MC}^\dagger(A_K^\dagger/K)$.

PROPOSITION 3.1. *If* (M, ∇) *is an object of* $\mathrm{MC}(A_K^\dagger/K)$*, then* M *is a free* $A_K^\dagger$*-module.*

Proof. Proof is almost same with that of [10, 6.1]. □

Let φ be a lifting of Frobenius of $A_0 = A^\dagger \otimes k$. We denote by $\mathrm{MCF}_n(A_K^\dagger/K)$ the category of $A_K^\dagger$-modules projective of finite type M with an integrable connection ∇ and a φ^n-linear endomorphism φ_n of M which commutes with ∇ and its linearization

$$\Phi_n\colon\ (\varphi^{n*}M, \varphi^{n*}\nabla) \to (M, \nabla).$$

is an isomorphism of $A_K^\dagger$-modules.

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THEOREM 3.2 (P. Berthelot [2, (2.5.7)]). *If* (M, ∇, φ_n) *is an object of* $\mathrm{MCF}_n(A_K^\dagger/K)$*, then Taylor series of* ∇ *converges on a strict neighborhood of the diagonal in* $\bar{\mathcal{X}}_K \times \bar{\mathcal{X}}_K$.

Proof. In [2, (2.5.7)], only the case where $n = 1$ is proven, but the same proof is available for any $n \geqslant 2$. □

Thus, there is a forgetful functor from $\mathrm{MCF}_n(A_K^\dagger/K)$ to $\mathrm{MC}^\dagger(A_K^\dagger/K)$.

COROLLARY 3.3. (P. Berthelot [2, (2.5.1)]). *There exists an equivalence of categories between* $F\text{-Isoc}^\dagger(X/K)$ *and* $\mathrm{MCF}_1(A_K^\dagger/K)$.

$\mathrm{MC}(A_K^\dagger/K)$ and $\mathrm{MC}^\dagger(A_K^\dagger/K)$ (resp. $\mathrm{MCF}_n(A_K^\dagger/K)$ for $n \geqslant 1$) are rigid Abelian tensor categories over K (resp. K_n).

The inductive limit

$$\varinjlim_V \Gamma(V \cap]0[, \mathcal{O}_{\bar{\mathcal{X}}_K}),$$

where V runs through the set of strict neighborhoods of $]\bar{X} \setminus \{0\}[$ in $\bar{\mathcal{X}}_K$, is isomorphic to a Robba ring $\mathcal{R}_K$ over K. ($]0[$ and $]\bar{X} \setminus \{0\}[$ denote the tubes of 0 and X in $\bar{\mathcal{X}}_K$, cf. [2, §1.1, §1.2]).

In the rest of this paper we fix a parameter t of $\mathcal{X}$, then we have a canonical injection $A_K^\dagger \hookrightarrow \mathcal{R}_K$. We also fix a Frobenius φ of $\mathcal{R}_K$ such that $\varphi(A_K^\dagger) \subset A_K^\dagger$. Then we have canonical functors

$$\mathrm{MC}(A_K^\dagger/K) \to \mathrm{MC}(\mathcal{R}_K/K),$$
$$\mathrm{MCF}_n(A_K^\dagger/K) \to \mathrm{MCF}_n(\mathcal{R}_K/K).$$

We denote (M, ∇, φ_n) (resp. (M, ∇)) simply by M if confusion does not occur.

4. Unipotent Objects

In this section, we study unipotent objects. R denotes either $\mathcal{R}_K$ or $A_K^\dagger$.

Let (M, ∇) be a free R-module of finite rank with connection. We say (M, ∇) is *unipotent* if it is a successive extension of the trivial object (R, d) by itself. We denote by $\mathrm{MC}^{\mathrm{uni}}(R/K)$ the full subcategory of $\mathrm{MC}(R/K)$ of unipotent objects. The next theorem is classical (cf. [20, (2.4.3)]).

THEOREM 4.1. *The functor* $(V_0, \mathcal{N}) \mapsto (V_0 \otimes_K R, \nabla_{\mathcal{N}})$*, where the connection* $\nabla_{\mathcal{N}}$ *is defined as* $\nabla_{\mathcal{N}}(v \otimes 1) = \mathcal{N} v \otimes \mathrm{d}t/t$*, induces an equivalence of the category of finite-dimensional* K*-vector space with a nilpotent endomorphism and* $\mathrm{MC}^{\mathrm{uni}}(R/K)$.

Proof. Let $\partial = t\mathrm{d}/\mathrm{d}t$ and $R_0 = \left\{\sum_{i \in \mathbb{Z}} a_i t^i \in R \;\middle|\; a_0 = 0\right\}$. Then $\partial : R_0 \to R_0$ is bijective. We denote $\nabla(t\mathrm{d}/\mathrm{d}t)$ also by ∂. Then,

$$(M, \nabla) \mapsto \left(\bigcup_{n>0} \mathrm{Ker}\, \partial^n, \partial\right)$$

gives an inverse functor. □

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COROLLARY 4.2. *The functor* $(M,\nabla)\mapsto(M\otimes_{A_K^\dagger}\mathcal{R}_K,\nabla\otimes 1)$ *induces an equivalence of categories between* $\mathrm{MC}^{\mathrm{uni}}(A_K^\dagger/K)$ *and* $\mathrm{MC}^{\mathrm{uni}}(\mathcal{R}_K/K)$.

LEMMA 4.3. *If* (M,∇) *is unipotent, then there exists a* φ*-structure on* (M,∇).[⋆]

Proof. By Theorem 4.1, we are reduced to the case of $(V_0\otimes\mathcal{R}_K,\nabla_{\mathcal{N}})$ where V_0 is a K-vector space of dimension r and the representation matrix of $\mathcal{N}$ is given by

$$N=\begin{pmatrix}0 & 1 & & 0\\ & \ddots & \ddots & \\ & & 0 & 1\\ 0 & & & 0\end{pmatrix}.$$

for some basis $\boldsymbol{v}=(v_1,\dots,v_r)$ of V_0. Let R_0 be as in the proof of Theorem 4.1 and I the inverse operator of ∂: $R_0\to R_0$. Let $\mu=\sum_{i\in\mathbb{Z}}c_it^i=\partial(\varphi(t))/\varphi(t)$. By Lemma 2.7, the constant term of μ is p. We put $\mu'=\mu-p\in R_0$.

Consider φ-linear morphism φ_1 on (M,∇) determined by $\varphi_1(\boldsymbol{v}\otimes 1)=(\boldsymbol{v}\otimes 1)A$ with

$$A=\begin{pmatrix}f_0 & f_1 & f_2 & f_3 & \cdots\\ & pf_0 & pf_1 & pf_2 & \ddots\\ & & p^2f_0 & p^2f_1 & \ddots\\ & & & \ddots & \ddots\\ 0 & & & & \ddots\end{pmatrix}. \tag{4}$$

Then the condition that ∇ and the above φ-linear morphism commute is equivalent to $\partial A=-NA+\mu A\varphi(N)$. This means that f_i $(i=1,2,\dots,r-1)$ satisfy the following equations:

$$\partial f_0=0,\quad \partial f_i=\mu' f_{i-1}\quad (i=1,2,\dots,r-1). \tag{5}$$

We show that we can define $g_i\in R$ for $i=1,2,\dots,r-1$ inductively by

$$g_1=\mu',\qquad g_i=\mu' I(g_{i-1})\quad (i\geqslant 2).$$

and that $g_i\in R_0$. Then for any n-tuple $\alpha_0,\dots,\alpha_{r-1}$ of elements of K, $f_i=\alpha_0I(g_i)+\cdots+\alpha_{i-1}I(g_1)+\alpha_i$ satisfy (5), and, if $\alpha_0\neq 0$, φ_1 is a φ-structure.

The existence of g_i is trivial if $i=1$. Assume that g_i is well-defined for $1\leqslant i\leqslant j-1$ and belong to R_0. Then the existence of g_j is evident. Since $\partial\big(I(\mu')^{j+1}\big)=(j+1)I(\mu')^j\mu'=(j+1)I(\mu')^jg_1$, $I(\mu')^jg_1\in R_0$. For $i<j$ and $m>0$, we also have

$$\begin{aligned}\partial(I(\mu')^mI(g_i))&=m\mu' I(\mu')^{m-1}I(g_i)+I(\mu')^mg_i\\ &=mI(\mu')^{m-1}g_{i+1}+I(\mu')^mg_i.\end{aligned}$$

This implies that, if $I(\mu')^mg_i\in R_0$, then $I(\mu')^{m-1}g_{i+1}\in R_0$. It follows immediately from these observations that $g_j\in R_0$. □

[⋆]B. Chiarellotto, B. Le Stum and E. Pons have also shown this result, cf. [4, 5.2.2].

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COROLLARY 4.4. *We can regard naturally* $\mathrm{MC}^{\mathrm{uni}}(A_K^\dagger/K)$ *as a full subcategory of* $\mathrm{MC}^\dagger(A_K^\dagger/K)$.

Proof. It follows immediately from Lemma 4.3 and Theorem 3.2. □

Remark 4.5. If M is indecomposable, it is also easy to see by direct calculation that the matrices of above form give the all-Frobenius structures, but there is a better way to see this.

In general, let (P, ∇, ψ_n) be a φ^n-∇-module over $\mathcal{R}$ and ψ_n' another φ^n-structure on (P, ∇). We denote those linearizations by Ψ_n and Ψ_n'. Then $\Psi_n' \circ \Psi_n^{-1}$ gives an automorphism of (P, ∇). Conversely, for any automorphism of f of (P, ∇), $f \circ \Psi_n$ is a φ^n-structure on (P, ∇). Thus we have one-to-one correspondence between the set of φ^n-structures on (P, ∇) and the set of automorphisms of (P, ∇) if at least one φ^n-structure exists.

By the lemma below $\dim \mathrm{Hom}_\nabla(M, M) = n$. Then for $(\alpha_0, \ldots, \alpha_{n-1})$ running through K^n, the above matrices (4) give all the φ^n-linear morphisms on (M, ∇).

LEMMA 4.6. *Let* U_1 *and* U_2 *be indecomposable unipotent R-modules with connection. Then* $\dim \mathrm{Hom}_\nabla(U_1, U_2) = \min(\mathrm{rank}\, U_1, \mathrm{rank}\, U_2)$.

Proof. This follows easily by direct calculation. □

COROLLARY 4.7. *The inverse image functor*

$$\mathrm{MCF}_n^{\mathrm{uni}}(A_K^\dagger/K) \to \mathrm{MCF}_n^{\mathrm{uni}}(\mathcal{R}_K/K)$$

is an equivalence of categories.

Proof. Let (M, ∇) be an object in $\mathrm{MC}^{\mathrm{uni}}(\mathcal{R}_K/K)$. By Corollary 4.2, there exists a sub-$A_K^\dagger$-module with connection $(M^\dagger, \nabla^\dagger)$ in $\mathrm{MC}^{\mathrm{uni}}(A_K^\dagger/K)$ whose inverse image is (M, ∇). It is sufficient to show that every φ^n-structure φ_n of (M, ∇) extends to $(M^\dagger, \nabla^\dagger)$. As in Remark 4.4, φ^n-structures on M (resp. $M^\dagger$) correspond to horizontal automorphisms of M (resp. $M^\dagger$). Since the natural map

$$\mathrm{Hom}_\nabla(M^\dagger, M^\dagger) \to \mathrm{Hom}_\nabla(M, M)$$

is isomorphism by Corollary 4.2, the assertion is clear. □

5. Étale Objects

In this section, we study étale objects. We use the same notation as in Section 3.

5.1. GLOBAL CASE

Let (M, ∇, φ_n) be an object in $\mathrm{MCF}_n(A_K^\dagger/K)$. We say (M, ∇, φ_n) is *unit-root* if there exists a sub-$A^\dagger$-module L of M projective of finite type such that

$$\begin{aligned} &\text{(i) } M \simeq A_K^\dagger \otimes_{A^\dagger} L,\\ &\text{(ii) } \varphi_n(L) \subset L,\\ &\text{(iii) } \Phi_n = \mathrm{id} \otimes \varphi_n : A^\dagger \otimes_{\varphi_n} L \to L \text{ is an isomorphism}\\ &\quad \text{of } A^\dagger\text{-modules.} \end{aligned} \tag{6}$$

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We say that a finite étale covering $U \to X = \mathrm{Spec}(A^\dagger \otimes_{\mathcal{O}_K} k)$ is *special* if it is tame at 0 and if its geometric monodromy group has a unique p-Sylow subgroup (cf. [19, 1.3.1]). Let $U \to X$ be a special Galois covering. Since $(A^\dagger, (\pi))$ is a Henselian couple [24, 2.2], there exists a finite étale Galois extension $B^\dagger$ of $A^\dagger$ such that $\mathrm{Spec}(B^\dagger \otimes_{\mathcal{O}_K} k) \simeq U$ uniquely up to isomorphisms.

We denote by $\mathrm{Rep}^{\mathrm{sp}}_{K_n}(\pi_1(X,*))$ the full subcategory of the category of finite-dimensional continuous representations of $\pi_1(X,*)$ over K_n consisting of objects such that $\pi_1(X,*)$ acts through a finite quotient corresponding to some special Galois covering of X. Let V be an object in $\mathrm{Rep}^{\mathrm{sp}}_{K_n}(\pi_1(X,*))$ and $B^\dagger$ a finite étale Galois extension of $A^\dagger$ which corresponds to a special Galois covering $U \to X$ such that $\pi_1(X,*)$ acts on V through $\mathrm{Gal}(B^\dagger/A^\dagger)$. Let $B^\dagger_K = B^\dagger \otimes_{\mathcal{O}_K} K$. We define

$$D_{A^\dagger_K,n}(V) = (V \otimes_{K_n} B^\dagger_K)^{\pi_1(X,*)},$$

where $\sigma \in \pi_1(X,*)$ acts on $V \otimes_{K_n} B^\dagger_K$ by $\sigma \otimes \sigma$. Note that it does not depend on the choice of $B^\dagger$. We endow $D_{A^\dagger_K}(V)$ with φ^n-structure $\varphi_n = \mathrm{id} \otimes \varphi^n$. Here φ^n on the right-hand side is Frobenius of $A^\dagger_K$ uniquely extended to $B^\dagger_K$.

LEMMA 5.1. *There exists a unique connection* ∇ *on* $M = D_{A^\dagger_K,n}(V)$ *which commutes with* φ_n. *Moreover,* (M, ∇, φ_n) *is a unit-root object in* $\mathrm{MCF}_n(A^\dagger_K/K)$.

Proof. By Galois descent [22, 5.1], M is projective $A^\dagger_K$-module of finite type (and hence free by Proposition 3.1) and

$$M \otimes_{A^\dagger_K} B^\dagger_K \simeq V \otimes_{K_n} B^\dagger_K. \tag{7}$$

The existence and the uniqueness of the connection that commutes with φ_n follows from a similar argument as in [13, A.2.2.4] and Proposition 2.8. We remark that the connection naturally extended to $M \otimes_{A^\dagger_K} B^\dagger_K \simeq V \otimes_{K_n} B^\dagger_K$ is given by $\nabla(x \otimes 1) = 0$ for $x \in V$.

Let $\mathcal{O}_{K_n}$ be the integer ring of K_n and $\mathcal{V}$ a $\mathcal{O}_{K_n}$-lattice of V. Replacing $\mathcal{V}$ by the sum of its translates by the action of $\pi_1(X,*)$, we can assume that $\mathcal{V}$ is stable by $\pi_1(X,*)$. Then $L = (\mathcal{V} \otimes_{\mathcal{O}_{K_n}} \mathcal{B}^\dagger)^{\pi_\infty(\mathcal{X},*)}$ satisfies the three conditions of (6). □

We say an object (M, ∇, φ_n) (resp. (M, ∇)) in $\mathrm{MCF}_n(A^\dagger_K/K)$ (resp. $\mathrm{MC}^\dagger(A^\dagger_K/K)$) is *special unit-root* (resp. *special étale*) if it comes from a representation of the Galois group of a special Galois covering of X in the above way (resp. if there exists a φ^n-structure φ_n on (M, ∇) for some n such that (M, ∇, φ_n) is special unit-root). We denote the full subcategory of $\mathrm{MCF}_n(A^\dagger_K/K)$ (resp. $\mathrm{MC}^\dagger(A^\dagger_K/K)$) of special unit-root (resp. special étale) objects by $\mathrm{MCF}^{\mathrm{spur}}_n(A^\dagger_K/K)$ (resp. $\mathrm{MC}^{\mathrm{se}}(A^\dagger_K/K)$). By Lemma 5.1, we can regard $D_{A^\dagger_K,n}$ as a functor from $\mathrm{Rep}^{\mathrm{sp}}_{K_n}(\pi_1(X,*))$ to $\mathrm{MCF}^{\mathrm{spur}}_n(A^\dagger_K/K)$.

We say an object (M, ∇, φ_n) in $\mathrm{MCF}_n(A^\dagger_K/K)$ is *special étale* if (M, ∇) is special étale as an object in $\mathrm{MC}^\dagger(A^\dagger_K/K)$. We denote the full subcategory of $\mathrm{MCF}_n(A^\dagger_K/K)$ of special étale objects by $\mathrm{MCF}^{\mathrm{se}}_n(A^\dagger_K/K)$.

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Let (M, ∇, φ_n) be an object in $\mathrm{MCF}_n^{\mathrm{spur}}(A_K^\dagger/K)$. Let V be a finite representation of $\pi_1(X,*)$ over K_n such that $M \simeq D_{A_K^\dagger,n}(V)$ in $\mathrm{MCF}_n(A_K^\dagger/K)$. Take a special covering U of X such that $\pi_1(X,*)$ acts on V through $\mathrm{Gal}(U/X)$ and let $B^\dagger$ be the finite étale extension of $A^\dagger$ corresponding to $U \to X$. We choose U so that $W(\mathbb{F}_{p^n}) \subset B^\dagger$. Here $W(\mathbb{F}_{p^n})$ is a Witt ring with residue field $\mathbb{F}_{p^n}$. Let $B_K^\dagger = B^\dagger \otimes K$. Then we define

$$V_{A_K^\dagger,n}(M) = (M \otimes_{A_K^\dagger} B_K^\dagger)^{\varphi_n=1} := \{x \in M \otimes_{A_K^\dagger} B_K^\dagger \mid \varphi_n(x) = x\}.$$

Here φ_n acts on $M \otimes_{A_K^\dagger} B_K^\dagger$ by $\varphi_n \otimes \varphi^n$. $V_{A_K^\dagger,n}(M)$ does not depend on the choice of $B^\dagger$. We endow $V_{A_K^\dagger,n}(M)$ with an action of $\pi_1(X,*)$ by $\mathrm{id} \otimes \sigma$ for $\sigma \in \pi_1(X,*)$. Thus we can define a functor $V_{A_K^\dagger,n}$ from $\mathrm{MCF}_n^{\mathrm{spur}}(A_K^\dagger/K)$ to $\mathrm{Rep}_{K_n}^{\mathrm{sp}}(\pi_1(X,*))$.

LEMMA 5.2. *If $\mathbb{F}_{p^n} \subset k$, $V_{A_K^\dagger,n}$ is a quasi-inverse of $D_{A_K^\dagger,n}$ and, hence,*

$$D_{A_K^\dagger,n}\colon \mathrm{MCF}_n^{\mathrm{spur}}(A_K^\dagger/K) \to \mathrm{Rep}_{K_n}^{\mathrm{sp}}\big(\pi_1(X,*)\big)$$

is an equivalence of categories.

Proof. Suppose that M is isomorphic to $D_{A_K^\dagger,n}(V)$ for some object V in $\mathrm{Rep}_{K_n}^{\mathrm{sp}}(\pi_1(X,*))$. Let $B^\dagger$ be as in the definition of $D_{A_K^\dagger,n}$. Then

$$\begin{aligned} V_{A_K^\dagger,n}(M) &\simeq ((V \otimes_{K_n} B_K^\dagger)^{\pi_1(X,*)} \otimes_{A_K^\dagger} B_K^\dagger)^{\varphi_n=1} \\ &\simeq (V \otimes_{K_n} B_K^\dagger)^{\varphi_n=1} \\ &\simeq V \otimes_{K_n} (B_K^\dagger)^{\varphi^n=1}. \end{aligned}$$

Thus we only have to show that $B_K^{\dagger\,\varphi^n=1} = \{x \in B_K^\dagger \mid \varphi^n(x) = x\} = K_n$. If $x \in B_K^{\dagger\,\varphi^n=1}$, then $\mathrm{d}x = \mathrm{d}\varphi^n(x) = \varphi^n(\mathrm{d}x)$. This implies $\mathrm{d}x = 0$. In fact, we can assume that $x \in B^\dagger$ and in this case it follows from a similar argument as in [13, A.2.2.4], by embedding x in the completion of $B^\dagger$ with respect to the p-adic topology. As a result, x belongs to the maximal unramified extension K' of K in $B_K^\dagger$. Since $\mathbb{F}_{p^n} \subset k$, $K'_n = K_n$. □

5.2. LOCAL CASE

We say an object (M, ∇) in $\mathrm{MC}(\mathcal{R}_K/K)$ is *étale* if there exists a finite separable extension F of the residue field E of $\mathcal{R}_K$ such that $(M \otimes_{\mathcal{R}_K} \mathcal{R}_K(F), \nabla \otimes \mathcal{R}_K(F))$ is trivial in $\mathrm{MC}(\mathcal{R}_K(F)/K_F)$. Here K_F is the algebraic closure of K in $\mathcal{R}(F)$.

We say an object (M, ∇, φ_n) in $\mathrm{MCF}_n(\mathcal{R}_K/K)$ is *unit-root* if there exists a free sub-$\mathcal{O}_K\langle t\rangle^\dagger$-module L of M such that

(i) $M \simeq L \otimes_{\mathcal{O}_K\langle t\rangle^\dagger} \mathcal{R}_K$,
(ii) $\varphi_n(L) \subset L$,
(iii) $1 \otimes \varphi_n : \mathcal{O}_K\langle t\rangle^\dagger \otimes_{\varphi_n} L \to L$ is an isomorphism of $\mathcal{O}_K\langle t\rangle^\dagger$-modules, (8)

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We denote by $\mathrm{MC}^{\mathrm{et}}(\mathcal{R}_K/K)$ (resp. $\mathrm{MCF}_n^{\mathrm{ur}}(\mathcal{R}_K/K)$) the full subcategory of $\mathrm{MC}(\mathcal{R}_K/K)$ (resp. $\mathrm{MCF}_n(\mathcal{R}_K/K)$) of étale (resp. unit-root) objects.

We say an object (M, ∇, φ_n) in $\mathrm{MCF}_n(\mathcal{R}_K/K)$ is étale if (M, ∇) is étale as an object in $\mathrm{MC}(\mathcal{R}_K/K)$. We denote by $\mathrm{MCF}_n^{\mathrm{et}}(\mathcal{R}_K/K)$ the full subcategory of $\mathrm{MCF}_n(\mathcal{R}_K/K)$ of étale objects.

LEMMA 5.3. *If $(M, \nabla) \in \mathrm{MC}(\mathcal{R}_K/K)$ is étale, then (M, ∇) has a unit-root φ^n-structure for sufficiently large n.*

Proof. It suffices to show that (M, ∇) has a unit-root φ^n-structure for some n. Let F be a finite Galois extension of E trivializing (M, ∇) and $G = \mathrm{Gal}(F/E)$. Let V_1 be the kernel of $\nabla \otimes \mathcal{R}(F)$ on $M \otimes \mathcal{R}(F)$. Then V_1 is stable under the action of G and it is a representation of G over K_F. By the theorem of Brauer [27, Theorem 24], if K_n for some n has a mth primitive root of unity for sufficiently large m, V_1 has a K_n-lattice V which is stable under the action of G. Then $M \simeq (V \otimes_{K_n} \mathcal{R}(F))^G$ and the assertion is clear.

In general, there exists a finite Abelian extension K' of K such that $M \otimes K'$ has a unit-root φ^n-structure φ_n for some n. Then φ_n commutes with the action of $\mathrm{Gal}(K'/K)$ and, by Galois descent, M itself has a unit-root φ^n-structure. □

Let E be the residue field of $\mathcal{R}_K$ and $G = \mathrm{Gal}(E^{\mathrm{sep}}/E)$. We denote by $\mathrm{Rep}_{K_n}^{\mathrm{fin}}(G)$ the category of finite-dimensional continuous representations of G on which G acts through finite quotients. For an object V in $\mathrm{Rep}_{K_n}^{\mathrm{fin}}(G)$, take a finite Galois extension F of E such that G acts on V through $\mathrm{Gal}(F/E)$. We define

$$D_{\mathcal{R}_K,n}(V) = \left(V \otimes_{K_n} \mathcal{R}_K(F)\right)^G. \quad \text{(cf. Section 2.1)}$$

Here $\sigma \in G$ acts on $V \otimes \mathcal{R}_K(F)$ by $\sigma \otimes \sigma$. We endow $D_{\mathcal{R}_K,n}(V)$ with φ^n-structure φ_n by $\mathrm{id} \otimes \varphi^n$. It is obvious that $D_{\mathcal{R}_K,n}(V)$ is independent of the choice of F.

LEMMA 5.4. *There exists a unique connection ∇ on $M = D_{\mathcal{R}_K,n}(V)$ which commutes with φ_n. Moreover, (M, ∇, φ_n) is a unit-root object in $\mathrm{MCF}_n(\mathcal{R}_K/K)$.*

Proof. Let F be as in the definition of $D_{\mathcal{R}_K,n}$. As in the proof of Lemma 5.1, we have

$$D_{\mathcal{R}_K,n}(V) \otimes_{\mathcal{R}_K} \mathcal{R}_K(F) \simeq V \otimes_{K_n} \mathcal{R}_K(F) \tag{9}$$

and the assertion follows from Lemma 2.2 and Proposition 2.8. We remark that the connection naturally extended to $D_{\mathcal{R}_K,n}(V) \otimes_{\mathcal{R}_K} \mathcal{R}_K(F) \simeq V \otimes_{K_n} \mathcal{R}_K(F)$ is given by $\nabla(x \otimes 1) = 0$ for $x \otimes 1 \in V \otimes \mathcal{R}_K(F)$. □

We say an object (M, ∇, φ_n) in $\mathrm{MCF}_n^{\mathrm{ur}}(\mathcal{R}_K/K)$ is *finite unit-root* if it is isomorphic to $D_{\mathcal{R}_K,n}(V)$ for some object V in $\mathrm{Rep}_{K_n}^{\mathrm{fin}}(G)$. We denote the full subcategory of $\mathrm{MCF}_n^{\mathrm{ur}}(\mathcal{R}_K/K)$ of finite unit-root objects by $\mathrm{MCF}_n^{\mathrm{fur}}(\mathcal{R}_K/K)$. By Lemma 5.4, we can regard $D_{\mathcal{R}_K,n}$ as a functor from $\mathrm{Rep}_{K_n}^{\mathrm{fin}}(G)$ to $\mathrm{MCF}_n^{\mathrm{fur}}(\mathcal{R}_K/K)$.

LEMMA 5.5. *Let M, N be objects in $\mathrm{MCF}_n^{\mathrm{fur}}(\mathcal{R}_K/K)$. If an endomorphism of $\mathcal{R}_K$-modules from M to N commutes with φ^n-structures, then it also commutes with connections.*

Proof. We reduce to the case that M is $(\mathcal{R}_K, d, \varphi^n)$. Then the set of endomorphisms from $\mathcal{R}_K$ to N that commutes with φ^n-structures is isomorphic to $\{x \in N \mid \varphi_n(x) = x\}$ and we only have to show that it is annihilated by the connection ∇ on N. By Corollary 2.9 and Lemma 5.4, we have $N^{\varphi_n=1} \subset N \otimes_{\mathcal{R}_K} \mathcal{R}_K(F)^{\varphi_n=1} = (V \otimes_{K_n} \mathcal{R}_K(F))^{\varphi_n=1} = V \otimes_{K_n} K'_n$. Then the assertion is clear from the last remark in the proof of Lemma 5.4. □

Remark 5.6. It is easy to see that Lemma 5.5 does not hold without assumption that φ^n-structures are unit-root.

Let (M, ∇, φ_n) be an object in $\mathrm{MCF}_n^{\mathrm{fur}}(\mathcal{R}_K/K)$. By definition, there exists an object V in $\mathrm{Rep}_{K_n}^{\mathrm{fin}}(G)$ such that $M \simeq D_{\mathcal{R}_K,n}(V)$. Take a finite Galois extension F of E such that that G acts on V through $\mathrm{Gal}(F/E)$ and that the residue field contains $\mathbb{F}_{p^n}$. We define

$$V_{\mathcal{R}_K,n}(M) = (M \otimes_{\mathcal{R}_K} \mathcal{R}_K(F))^{\varphi_n=1} := \{x \in M \otimes_{\mathcal{R}_K} \mathcal{R}_K(F) \mid \varphi_n(x) = x\}.$$

Here φ_n acts on $M \otimes_{\mathcal{R}_K} \mathcal{R}_K(F)$ by $\varphi_n \otimes \varphi^n$. We endow $V_{\mathcal{R}_K,n}(M)$ with Galois action by $\mathrm{id} \otimes \sigma$ for $\sigma \in G$. Obviously $V_{\mathcal{R}_K,n}(M)$ is independent of the choice of F.

LEMMA 5.7. *If $\mathbb{F}_{p^n} \subset k$, $V_{\mathcal{R}_K,n}$ is a quasi-inverse functor of $D_{\mathcal{R}_K,n}$ and hence*

$$D_{\mathcal{R}_K,n} : \mathrm{Rep}_{K_n}^{\mathrm{fin}}(G) \to \mathrm{MCF}_n^{\mathrm{fur}}(\mathcal{R}_K/K)$$

is an equivalence of categories.

Proof. We can prove the assertion in a similar way as in the proof of Lemma 5.2 by Corollary 2.9. □

In general, the functor $D_{\mathcal{R}_K,n}$ can be defined for finite-dimensional representations of G over K_n such that the inertia I acts through a finite quotient. We call such a representation *with finite monodromy*.

We briefly review the construction of the functor by Tsuzuki [28, 4] with our notation. Let $\mathcal{E} = K\langle t\rangle^\dagger$ and denote by $\mathcal{O}_\mathcal{E}$ its ring of integers $\mathcal{O}_K\langle t\rangle^\dagger$. For a finite separable extension of F of the residue field E of $\mathcal{O}_\mathcal{E}$, we denote a finite étale extension of $\mathcal{O}_\mathcal{E}$ with residue field F by $\mathcal{O}_\mathcal{F}$ and its field of fractions by $\mathcal{F}$. Let K' be the coefficient field of $\mathcal{R}_K(F)$ (cf. Section 2.1) and let $\bar{\mathcal{F}}$ be the image of $\mathcal{F} \otimes_{K'} \widehat{K^{\mathrm{ur}}}$ in $K_1 \otimes_{\mathbb{Z}_p} W(E^{\mathrm{alg}})$. Here $\widehat{K^{\mathrm{ur}}}$ is the completion of the maximal unramified extension of K, E^{alg} is an algebraic closure of E and $W(E^{\mathrm{alg}})$ is a Witt ring with residue field E^{E}. We define

$$\tilde{\mathcal{E}} = \varinjlim_{F/E} \bar{\mathcal{F}},$$

where F runs through all finite separable extensions over E. Let $G = \mathrm{Gal}(E^{\mathrm{sep}}/E)$. For a representation V of G with finite monodromy over K_n, we define the functor $D_{\mathcal{R}_K,n}$ by

$$D_{\mathcal{R}_K,n}(V) = (V \otimes_{K_n} \tilde{\mathcal{E}})^G \otimes_\mathcal{E} \mathcal{R}_K. \tag{10}$$

It is easy to see that, if V is an object of $\mathrm{Rep}_{K_n}^{\mathrm{fin}}(G)$, then the above functor is compatible with the functor already defined.

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The next theorem means that any overconvergent F-isocrystal has finite monodromy.

THEOREM 5.8 (N. Tsuzuki [28, 4.2.6]). *Any object in* $\mathrm{MCF}^{\mathrm{ur}}_n(\mathcal{R}_K/K)$ *comes from a representation of G with finite monodromy.*

COROLLARY 5.9. *Any object in* $\mathrm{MC}^{\mathrm{et}}(\mathcal{R}_K/K)$ *is isomorphic to the image under the forgetful functor of an object in* $\mathrm{MCF}^{\mathrm{fur}}_n(\mathcal{R}_K/K)$ *for some n.*

Proof. Let (M, ∇) be an object in $\mathrm{MC}^{\mathrm{et}}(\mathcal{R}/K)$. By Lemma 5.3, there is a φ^n-structure φ_n on (M, ∇) such that (M, ∇, φ_n) is an object of $\mathrm{MCF}^{\mathrm{ur}}_n(\mathcal{R}/K)$. Then M can be written as $D_{\mathcal{R},n}(V)$ with a representation V with finite monodromy by Theorem 5.8. We use the same notation as in the paragraph following Lemma 5.7. Since K_n is locally compact, the action of the inertia I of G is trivialized by some finite Galois extension F of E. Let $H = \mathrm{Gal}(E^{\mathrm{sep}}/F)$. Since $I \cap H$ acts trivially on V, we obtain

$$(V \otimes_{K_n} \widehat{K^{\mathrm{ur}}})^H \otimes_{K'} \widehat{K^{\mathrm{ur}}} \simeq V \otimes_{K_n} \widehat{K^{\mathrm{ur}}} \tag{11}$$

(cf. [13, A.1.2.4], [28, 4.2.2]). Thus if we put $\bar{V} = (V \otimes_{K_n} \widehat{K^{\mathrm{ur}}})^H$, then

$$\begin{aligned} M &= (V \otimes_{K_n} \widetilde{\mathcal{E}})^G \otimes_{\mathcal{E}} \mathcal{R} \\ &\simeq (V \otimes_{K_n} \widehat{K^{\mathrm{ur}}} \otimes_{K'} \mathcal{F})^G \otimes_{\mathcal{E}} \mathcal{R}) \\ &\simeq (\bar{V} \otimes_{K'} \widehat{K^{\mathrm{ur}}} \otimes_{K'} \mathcal{F})^G \otimes_{\mathcal{E}} \mathcal{R} \\ &\simeq (\bar{V} \otimes_{K'} \bar{\mathcal{F}})^G \otimes_{\mathcal{E}} \mathcal{R} \\ &\simeq (\bar{V} \otimes_{K'} \mathcal{F})^{G/H} \otimes_{\mathcal{E}} \mathcal{R}. \end{aligned}$$

By (11), $\bar{V}$ is a K'-vector space of dimension equal to $r = \dim_{K_n} V$. We fix an isomorphism $\bar{V} \simeq \bigoplus^r K'$ and define a Frobenius action φ'_n on $\bar{V}$ so that it is compatible with the action $\bigoplus \varphi^n$ on the right-hand side of the above isomorphism. Then φ'_n extends to the φ^n-structure of (M, ∇) by (12) and (M, ∇, φ'_n) is an object of $\mathrm{MCF}^{\mathrm{fur}}_n(\mathcal{R}_K/K)$. □

5.3. KATZ CORRESPONDENCE FOR ÉTALE OBJECTS

By [19, 1.4.7], and Lemma 5.2, Lemma 5.7 above, we have the next proposition.

PROPOSITION 5.10. *If* $\mathbb{F}_{p^n} \subset k$, *the inverse image functor*

$$\mathrm{MCF}^{\mathrm{spur}}_n(A^\dagger_K/K) \to \mathrm{MCF}^{\mathrm{fur}}_n(\mathcal{R}_K/K)$$

is an equivalence of categories.

The forgetful functors

$$\begin{aligned} &\mathrm{MCF}^{\mathrm{spur}}_n(A^\dagger_K/K) \to \mathrm{MC}^{\mathrm{se}}(A^\dagger_K/K), \\ &\mathrm{MCF}^{\mathrm{fur}}_n(\mathcal{R}_K/K) \to \mathrm{MC}^{\mathrm{et}}(\mathcal{R}_K/K) \end{aligned}$$

are faithful but not full if $K_n \subsetneq K$. However, we have the next lemma.

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LEMMA 5.11. *Let M and N be objects in* $\mathrm{MCF}_n^{\mathrm{spur}}(A_K^\dagger/K)$ (*resp.* $\mathrm{MCF}_n^{\mathrm{fur}}(\mathcal{R}_K/K)$). *Then for some finite unramified extension K' of K, we have*

$$\mathrm{Hom}_{\mathrm{MCF}_n^{\mathrm{spur}}(A_{K'}^\dagger/K')}(M', N') \otimes_{K'_n} K' \simeq \mathrm{Hom}_{\mathrm{MC}^{\mathrm{se}}(A_{K'}^\dagger/K')}(M', N')$$

$$(\text{resp. } \mathrm{Hom}_{\mathrm{MCF}_n^{\mathrm{fur}}(\mathcal{R}_{K'}/K')}(M', N') \otimes_{K'_n} K' \simeq \mathrm{Hom}_{\mathrm{MC}^{\mathrm{et}}(\mathcal{R}_{K'}/K')}(M', N')).$$

Here $M' = M \otimes_K K'$ and $N' = N \otimes_K K'$.

Proof. We only give the proof of the case of $\mathcal{R}_K$. The case of $A_K^\dagger$ can be proven in the same way.

It suffices to prove in the case that M is $\mathbf{1} = (\mathcal{R}_K, d, \varphi^n)$. Let $(N, \nabla, \varphi_n) \in \mathrm{MCF}_n^{\mathrm{fur}}(\mathcal{R}_K/K)$ and let V be an object in $\mathrm{Rep}_{K_n}^{\mathrm{fin}}(G)$ such that $N \simeq D_{\mathcal{R}_K,n}(V)$. Take a finite Galois extension F of E such that G acts on V through $\mathrm{Gal}(F/E)$ and that the residue field contains $\mathbb{F}_{p^n}$. Then we have $\mathrm{Hom}_{\mathrm{MCF}_n^{\mathrm{fur}}(\mathcal{R}_K/K)}(\mathbf{1}, N) \simeq N^{\varphi_n=1}$ by Lemma 5.5.

Let K' be the coefficient field of $\mathcal{R}_K(F)$ (cf. Lemma 2.2). We denote $\{x \in L \mid \nabla x = 0\}$ by L^∇ for module with connection (L, ∇). Then we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{MC}^{\mathrm{et}}(\mathcal{R}_K/K)}(\mathbf{1}, N) &\simeq ((V \otimes_{K_n} \mathcal{R}_K(F))^G)^\nabla \\ &= ((V \otimes_{K_n} \mathcal{R}_K(F))^\nabla)^G \\ &= (V \otimes_{K_n} K')^G.\end{aligned}$$

On the other hand, by Lemma 5.5

$$\begin{aligned}\mathrm{Hom}_{\mathrm{MCF}^{\mathrm{fur}}(\mathcal{R}_K/K)}(\mathbf{1}, N) &\simeq ((V \otimes_{K_n} \mathcal{R}_K(F))^G)^{\varphi_n=1} \\ &= ((V \otimes_{K_n} \mathcal{R}_K(F))^{\varphi_n=1})^G \\ &= (V \otimes_{K_n} K'_n)^G.\end{aligned}$$

After extending scalars, we can assume that $K' = K$. Then the assertion is clear. □

COROLLARY 5.12. *The inverse image functor*

$$\iota\colon \mathrm{MC}^{\mathrm{se}}(A_K^\dagger/K) \to \mathrm{MC}^{\mathrm{et}}(\mathcal{R}_K/K)$$

is an equivalence of categories.

Proof. For any positive integer n, we have a diagram

$$\begin{array}{ccc}\mathrm{MCF}_n^{\mathrm{spur}}(A_K^\dagger/K) & \xrightarrow{\iota'} & \mathrm{MCF}_n^{\mathrm{fur}}(\mathcal{R}_K/K) \\ {\scriptstyle\text{forgetful}}\downarrow & & \downarrow{\scriptstyle\text{forgetful}} \\ \mathrm{MC}^{\mathrm{se}}(A_K^\dagger/K) & \xrightarrow[\iota]{} & \mathrm{MC}^{\mathrm{et}}(\mathcal{R}_K/K).\end{array}$$

Here ι' is also an inverse image functor. By Proposition 5.10, if $\mathbb{F}_{p^n} \subset k$, ι' is an equivalence of categories. Since any object in $\mathrm{MC}^{\mathrm{et}}(\mathcal{R}_K/K)$ is an image by forgetful functor of an object in $\mathrm{MCF}_n^{\mathrm{fur}}(\mathcal{R}_K/K)$ for sufficiently large n by Corollary 5.9, ι is essentially surjective. We show that ι is fully faithful. Let us denote $\mathrm{MC}^{\mathrm{et}}(\mathcal{R}_K/K)$ (resp. $\mathrm{MC}^{\mathrm{se}}(A_K^\dagger/K)$) by $\mathcal{C}_{\mathcal{K}}$ and let M, N be objects in $\mathcal{C}_{\mathcal{K}}$. Then it is easy to see that $\mathrm{Hom}_{\mathcal{C}_{\mathcal{K}}}(M,N) \otimes K' \simeq \mathrm{Hom}_{\mathcal{C}_{\mathcal{K}'}}(M \otimes K', N \otimes K')$. On account of this, the full faithfulness follows from Lemma 5.11. $\square$

COROLLARY 5.13. *The inverse image functor*

$$\mathrm{MCF}_n^{\mathrm{se}}(A_K^\dagger/K) \to \mathrm{MCF}_n^{\mathrm{et}}(\mathcal{R}_K/K)$$

is an equivalence of categories.

Proof. It follows from the same argument as in the proof of Corollary 4.7. $\square$

5.4. SCALAR EXTENSION

Let $L = \widehat{K}^{\mathrm{nr}}$ be the completion of the maximal unramified extension of K. We call an object (M, ∇, φ_n) in $\mathrm{MCF}_n(\mathcal{R}_K)$ *geometrically irreducible* if $M \otimes_{\mathcal{R}_K} \mathcal{R}_{L'}$ is irreducible (i.e. it has no proper subobject) in $\mathrm{MCF}_n(\mathcal{R}_{L'}/L')$ for any finite extension L' of L. We call an object (M, ∇) in $\mathrm{MC}(\mathcal{R}_K)$ is *geometrically irreducible* if $M \otimes_K K'$ is irreducible in $\mathrm{MC}(\mathcal{R}_{K'})$ for any finite extension K' of K satisfying (3). We call objects (M, ∇, φ_n) and $(M', \nabla', \varphi_n')$ in $\mathrm{MCF}_n(\mathcal{R}_K)$ are *geometrically isomorphic* if $(M, \nabla, \varphi_n) \otimes_{\mathcal{R}_K} \mathcal{R}_{L'}$ and $(M', \nabla', \varphi_n') \otimes_{\mathcal{R}_K} \mathcal{R}_{L'}$ are isomorphic in $\mathrm{MCF}_n(\mathcal{R}_{L'}/L')$ for any finite extension L' of L satisfying (3). The next lemma shows that, in $\mathrm{MC}(\mathcal{R}_K/K)$, 'geometrically isomorphic' means isomorphic.

LEMMA 5.14. *Let M and N be objects in $\mathrm{MC}(\mathcal{R}_K/K)$, and let L be a finite extension field of K. Then there exists an isomorphism $M \simeq N$ in $\mathrm{MC}(\mathcal{R}_K/K)$ if and only if there exists an isomorphism $M \otimes_K L \simeq N \otimes_K L$ in $\mathrm{MC}(\mathcal{R}_L/L)$.*

Proof. If $M \simeq N$, it is obvious that $M \otimes L \simeq N \otimes L$. We show the converse. Suppose that $M \otimes L \simeq N \otimes L$. We use a similar argument as in [20, (4.1.2)]. Let $r = \mathrm{rank}(M) = \mathrm{rank}(N)$ and consider a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_\nabla(M,N) & \xrightarrow{\det} & \mathrm{Hom}_\nabla(\bigwedge^r M, \bigwedge^r N) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_\nabla(M \otimes L, N \otimes L) & \xrightarrow{\det \otimes L} & \mathrm{Hom}_\nabla(\bigwedge^r M \otimes L, \bigwedge^r N \otimes L). \end{array}$$

Since

$$\mathrm{Hom}_\nabla\left(\bigwedge^r M, \bigwedge^r N\right) \otimes L \simeq \mathrm{Hom}_\nabla\left(\bigwedge^r M \otimes L, \bigwedge^r N \otimes L\right) \simeq L,$$

we have $\mathrm{Hom}_\nabla(\bigwedge^r M, \bigwedge^r N) \simeq K$, and a horizontal map $f\colon M \to N$ is isomorphism if and only if $\det(f)$ is a nonzero element of this K-vector space. We fix a basis $(f_1, \ldots, f_m)$ of $\mathrm{Hom}_\nabla(M,N)$ over K. Then det is a polynomial function on the finite-dimensional K-vector space $\mathrm{Hom}_\nabla(M,N)$. Since

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$$\mathrm{Hom}_\nabla(M,N)\otimes L \simeq \mathrm{Hom}_\nabla(M\otimes L, N\otimes L)$$

$(f_1,\dots,f_m)$ also gives a basis of $\mathrm{Hom}_\nabla(M\otimes L, N\otimes L)$ and for this basis $\det\otimes L$ is represented by the same polynomial. Because K is infinite, if det vanishes identically on $\mathrm{Hom}_\nabla(M,N)$ then it also vanishes on $\mathrm{Hom}_\nabla(M\otimes L, N\otimes L)$. □

LEMMA 5.15. *Let (M,∇,φ_n) be a geometrically irreducible finite unit-root φ_n-∇-module over $\mathcal{R}_K$. Then for $i=0,1$ we have*

$$\mathrm{H}^i_\nabla(M)\simeq\begin{cases} K, & \text{if } M \text{ is the trivial object},\\ 0, & \text{otherwise.}\end{cases}$$

Proof. If $i=0$, the assertion is clear. Consider the case $i=1$. Since

$$\mathrm{H}^i_\nabla(M\otimes_K L)\simeq \mathrm{H}^i_\nabla(M)\otimes_K L$$

for any finite extension L over K (we regard $M\otimes_K L$ as an object in $\mathrm{MC}(\mathcal{R}_L/L)$), we can assume that $\mathbb{F}_{p^n}\subset k$. Let F be a finite Galois extension of the residue field E of $\mathcal{R}_K$ such that $\mathrm{Gal}(E^{\mathrm{sep}}/E)$ acts on $V=V_{\mathcal{R}_K,n}(M)$ through $\mathrm{Gal}(F/E)$. Replacing n by a multiple and K by a finite extension, we can assume that the regular representation $W=K_n[\mathrm{Gal}(F/E)]$ of $\mathrm{Gal}(F/E)$ decomposes as a direct sum $\bigoplus_{i=0}^r V_i$ of representations such that $V_i\otimes_{K_n}K_n^{\mathrm{alg}}$ are all irreducible and that the coefficient field (Section 2.1) of $\mathcal{R}_K(F)$ is also K. Under these condition, $D_{\mathcal{R}_K,n}(W)\simeq\bigoplus_{i=0}^r M_i$ with $M_i=D_{\mathcal{R}_K,n}(V_i)$ and M is isomorphic to some M_i. Let V_0 be a unit representation. Since $\mathrm{H}^1_\nabla(D_{\mathcal{R}_K,n}(W))\simeq \mathrm{H}^1_\nabla(\mathcal{R}_K(F))\simeq K$, we have $\bigoplus_{i=0}^r \mathrm{H}^1_\nabla(M_i)\simeq K$. Since $\mathrm{H}^1_\nabla(M_0)\simeq K$, $\mathrm{H}^1_\nabla(M_i)$ are all 0 for $1\leqslant i\leqslant r$. This proves the assertion. □

6. Special Objects

Combining the results of the last two sections, we can define the notion of special objects and get the equivalence.

DEFINITION 6.1. An object M in $\mathrm{MC}^\dagger(A_K^\dagger/K)$ is called special if it is a finite direct sum of objects of the form $P\otimes U$ where P is special étale and U is unipotent (cf. Corollary 4.4). We call an object (M,∇,φ_n) in $\mathrm{MCF}_n(A_K^\dagger/K)$ special if (M,∇) is special as an object in $\mathrm{MC}^\dagger(A_K^\dagger/K)$. We denote the full subcategory of $\mathrm{MC}^\dagger(A_K^\dagger/K)$ (resp. $\mathrm{MCF}_n(A_K^\dagger/K)$) of special objects by $\mathrm{MC}^{\mathrm{sp}}(A_K^\dagger/K)$ (resp. $\mathrm{MCF}_n^{\mathrm{sp}}(A_K^\dagger/K)$).

DEFINITION 6.2. An object M in $\mathrm{MC}(\mathcal{R}_K/K)$ is called special if it is a finite direct sum of objects of the form $P\otimes U$ where P is étale and U is unipotent. We call an object (M,∇,φ_n) in $\mathrm{MCF}_n(\mathcal{R}_K/K)$ special if (M,∇) is special as an object in $\mathrm{MC}(\mathcal{R}_K/K)$. We denote the full subcategory of $\mathrm{MC}(\mathcal{R}_K/K)$ of special objects by $\mathrm{MC}^{\mathrm{sp}}(\mathcal{R}_K/K)$.

7. Quasi-Unipotent Objects

In this section, we shall show that locally all the quasi-unipotent objects are special. This result is an analogue of the theorem of Levelt [23] for quasi-unipotent overconvergent isocrystals (cf. [20, II]).

7.1. DEFINITION

First we recall the definition of quasi-unipotence.

DEFINITION 7.1. We say an object (M, ∇) in $\mathrm{MC}(\mathcal{R}_K/K)$ is quasi-unipotent if there exists a finite separable extension F of the residue field of $\mathcal{R}_K$ such that $(M \otimes_{\mathcal{R}_K} \mathcal{R}_K(F), \nabla \otimes \mathcal{R}_K(F))$ is unipotent in $\mathrm{MC}(\mathcal{R}_K(F)/K_F)$. Here K_F is the algebraic closure of K in $\mathcal{R}(F)$. We say an object (M, ∇, φ_n) in $\mathrm{MCF}_n(\mathcal{R}_K/K)$ is quasi-unipotent if (M, ∇) is quasi-unipotent as an object in $\mathrm{MC}(\mathcal{R}_K/K)$. We denote the full subcategory of $\mathrm{MC}(\mathcal{R}_K/K)$ (resp. $\mathrm{MCF}_n(\mathcal{R}_K/K)$) of quasi-unipotent objects by $\mathrm{MC}^{\mathrm{qu}}(\mathcal{R}_K/K)$ (resp. $\mathrm{MCF}_n^{\mathrm{qu}}(\mathcal{R}_K/K)$).

7.2. LOCAL DECOMPOSITION THEOREM

It is obvious that the special objects in $\mathrm{MC}(\mathcal{R}_K/K)$ are quasi-unipotent. We will show that the converse is also true. We denote the identity matrix of degree m by I_m. We denote by $N_{m,n}$ a nilpotent matrix of size mn

$$\begin{pmatrix} 0 & I_m & & & 0 \\ & 0 & I_m & & \\ & & \ddots & \ddots & \\ & & & 0 & I_m \\ 0 & & & & 0 \end{pmatrix}.$$

LEMMA 7.2. *Let F be a finite separable extension of E. By Lemma 2.2, there is a finite extension K' of K and $u \in \mathcal{R}_K(F)$ such that $\mathcal{R}_K(F) \simeq \mathcal{R}_{K',u}$. Let $\partial = t\mathrm{d}/\mathrm{d}t$. We extend the operation of ∂ to $\mathcal{R}_K(F)$ by*

$$\partial x = t\frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}u}.$$

Suppose that $Q = (q_{ij}) \in M(m_2n_2, m_1n_1; \mathcal{R}_K(F))$ satisfies the differential equation

$$\partial Q = QN_{m_1,n_1} - N_{m_2,n_2}Q.$$

If $n_1 \geqslant n_2$, then Q can be written in the form

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$$Q = \left(\begin{array}{c|cccc} & Q_1 & Q_2 & \cdots & Q_{n_2} \\ & & \ddots & \ddots & \vdots \\ 0 & & & & \\ & & & \ddots & Q_2 \\ & 0 & & & Q_1 \end{array} \right), \quad Q_i \in M(m_2, m_1; K').$$

Proof. We begin by observing that '$\log t$' does not appear in any 'algebraic' extension of $\mathcal{R}_K$. To be precise, if $f \in \mathcal{R}_K(F)$ and $\partial f \in K'$, then $f \in K'$ and $\partial f = 0$. Replacing K by K', we can assume that $K' = K$. Then by [24, (3.4.1)] t can be written in the form

$$t = u^s(a_0 + a_1 u + \cdots), \qquad a_0 \in \mathcal{O}_K[[t]]^\times, \quad a_i \in \mathcal{O}_K[[t]] \quad \text{for } i > 0.$$

Here s is the degree of F over E. Let $g(u) = \sum_{i>0} (a_i/a_0)\, u^i$, then

$$\frac{1}{t}\partial_u t = \frac{s(1+g) + \partial_u g}{1+g} = s + \partial_u g\left(\sum_{j \geqslant 0}(-g)^j\right).$$

Here $\partial_u = u\, \mathrm{d}/\mathrm{d}u$. Therefore, if $\partial f = c \in K$, then

$$\partial_u f = \frac{c}{t}\partial_u t = c\left(s + \partial_u g\left(\sum_{j \geqslant 0}(-g)^j\right)\right).$$

Since $(\partial_u g)g^j = 1/(j+1)\partial_u g^{j+1}$ has no constant term, c must be 0 and the assertion is clear.

Now we prove the lemma. Let us write $Q = (Q_{i,j})$ with

$$Q_{i,j} \in M(m_2, m_1, \mathcal{R}_K(F)).$$

By the assumption, we have

$$\begin{aligned} &\partial Q_{i,1} = -Q_{i+1,1}, \\ &\partial Q_{i,j} = Q_{i,j-1} - Q_{i+1,j} \quad (1 \leqslant i \leqslant n_1 - 1, 2 \leqslant j \leqslant n), \end{aligned} \tag{13}$$

$$\begin{aligned} &\partial Q_{n_2,1} = 0, \\ &\partial Q_{n_2,j} = Q_{n_2,j-1} \quad (2 \leqslant j \leqslant n_1). \end{aligned} \tag{14}$$

By (14) and the observation above, we see $Q_{n_2,j} = 0$ $(1 \leqslant j < n_1)$ and $Q_{n_2,n_1} \in K'$. Similarly we can prove that

$$\begin{aligned} &Q_{n_2-k,j} = 0 \quad (1 \leqslant j < n_1 - k), \\ &Q_{n_2-k,j} \in K' \quad (n_1 - k \leqslant j \leqslant n_2) \end{aligned} \tag{15}$$

for $1 \leqslant k < n_2$ by induction on k using (13). □

We denote by $\bigoplus_{i=1}^{r} N_{m_i,n_i}$ the nilpotent matrix

$$\begin{pmatrix} N_{m_1,n_1} & & & 0 \\ & N_{m_2,n_2} & & \\ & & \ddots & \\ 0 & & & N_{m_r,n_r} \end{pmatrix}.$$

COROLLARY 7.3. *Let* $N = \bigoplus_{i=1}^{r} N_{m_i,n_i}$ *with* $n_1 > n_2 > \cdots > n_r$ *and* $n = \sum_i n_i m_i$. *Suppose that* $Q = (q_{ij}) \in M(n; \mathcal{R}_K(F))$ *satisfies a differential equation*

$$\partial Q = QN - NQ.$$

Then $q_{ij} = 0$ *for* (i, j) *such that* $m_1 < i$, $1 \leqslant j \leqslant m_1$.

PROPOSITION 7.4. *Suppose* (M, ∇) *is a quasi-unipotent object in* $\mathrm{MC}(\mathcal{R}_K/K)$. *If* M *is irreducible, then* M *is étale.*

Proof. Let (M, ∇) be a quasi-unipotent $\mathcal{R}_K$-module over $\mathcal{R}_K$ of rank n. Let F be a finite Galois extension of E such that $M' = M \otimes \mathcal{R}_K(F)$ is unipotent. We denote its Galois group by G. Then there exists a basis $f = (f_1, \ldots, f_n)$ of M' such that

$$\nabla f = fN \otimes \frac{\mathrm{d}t}{t}, \text{ with } N = \bigoplus_{i=1}^{r} N_{m_i,n_i}, \quad n_1 > n_2 > \cdots > n_r.$$

For $\sigma \in G$, let $\sigma(f) = fQ_\sigma$ with $Q_\sigma \in \mathcal{R}_K(F)$. Since σ and ∇ commute, we have

$$\partial Q_\sigma = Q_\sigma N - NQ_\sigma.$$

By Corollary 7.3, we have $q_{ij} = 0$ for (i, j) such that $m_1 < i$ and $1 \leqslant j \leqslant m_1$. Let M_1' be a sub-$\mathcal{R}_K(F)$-module generated by $f_1, \ldots, f_{m_1}$, then M_1' is stable under the action of G and the connection ∇. By Galois descent, there exists ∇-module M_1 over $\mathcal{R}_K$ such that $M_1' = M_1 \otimes_{\mathcal{R}_K} \mathcal{R}_K(F)$. Therefore if M is irreducible, $n_1 = 0$. This implies $N = 0$, i.e., M is étale. □

For objects M, M' in $\mathrm{MC}(\mathcal{R}_K/K)$, we define

$$\mathrm{Ext}^i_\nabla(M, M') = \mathrm{H}^i_\nabla(\mathrm{Hom}(M, M')).$$

We also denote $\mathrm{Ext}^0_\nabla(M, M')$ by $\mathrm{Hom}_\nabla(M, M')$. Then $\mathrm{Hom}_\nabla(M, M')$ is isomorphic to $\mathrm{Hom}_{\mathrm{MC}(\mathcal{R}_K/K)}(M, M')$ and $\mathrm{Ext}^1_\nabla(M, M')$ is isomorphic to the group of classes of extensions of M' by M in $\mathrm{MC}(\mathcal{R}_K/K)$.

LEMMA 7.5. *Let* (M, ∇) *and* (M', ∇) *be quasi-unipotent geometrically irreducible* ∇-*modules over* $\mathcal{R}_K$. *Then for* $i = 0, 1$,

$$\mathrm{Ext}^i_\nabla(M, M') = \begin{cases} K, & \text{if } M \text{ is isomorphic to } M', \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Suppose that M is isomorphic to M'. We can assume that $M' = M$. By Proposition 7.4, M is étale. Therefore we are able to equip M with finite unit-root

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φ^n-structure for some n. We can replace K by a finite extension of K so that $\mathbb{F}_{p^n} \subset k$. Let $V = V_{\mathcal{R}_K,n}(M)$. Then by Lemma 5.7, V is an irreducible representation of $G = \mathrm{Gal}(E^{\mathrm{sep}}/E)$ such that G acts on V through a finite quotient. Here E is the residue field of $\mathcal{R}_K$ (cf. 2.1). Moreover, by the assumption that M is geometrically irreducible, V is irreducible as a representation of the inertia I of G. By Schur's lemma, $\mathrm{Hom}_K(V, V)$ contains the unit representation with multiplicity one (as a representation of I). Then by Lemma 5.7, $\mathrm{Hom}(M, M)$ is a direct sum of irreducible objects and contains exactly one unit object. Now the assertion follows from Lemma 5.15. The case where M and M' are not isomorphic can be proven in the same way. □

LEMMA 7.6. *Let P and P' be geometrically irreducible objects. Let U and U' be indecomposable unipotent objects in* $\mathrm{MC}^{\mathrm{qu}}(\mathcal{R}_K/K)$. *Then for $i = 0, 1$, we have*

$$\dim_K \mathrm{Ext}^i_\nabla(P \otimes U, P' \otimes U') = \begin{cases} \min(\mathrm{rank}\, U, \mathrm{rank}\, U'), & \text{if } P' \simeq P, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. Suppose that P is not isomorphic to P'. We first consider the case that $U' = \mathbf{1}$. By Theorem 4.1, there is a unique indecomposable unipotent submodule U_1 of U. We may suppose by induction on the rank of U that $\mathrm{Ext}^i_\nabla(P \otimes U_1, P') = 0$. Then the assertion easily follows from the next long exact sequence and Lemma 7.5.

$$\begin{aligned} 0 &\to \mathrm{Hom}_\nabla(P, P') \to \mathrm{Hom}_\nabla(P \otimes U, P') \to \mathrm{Hom}_\nabla(P \otimes U_1, P') \\ &\to \mathrm{Ext}^1_\nabla(P, P') \to \mathrm{Ext}^1_\nabla(P \otimes U, P') \to \mathrm{Ext}^1_\nabla(P \otimes U_1, P') \to 0. \end{aligned}$$

The case that $U' \not\simeq \mathbf{1}$ follows from the case that $U' = \mathbf{1}$ and induction on the rank of U' in a similar way.

Next suppose that $P = P'$. Since $P^\vee \otimes P \simeq \mathbf{1} \oplus Q$ for some direct sum Q of nontrivial irreducible étale objects, the assertion on Ext^0 follows from next calculation and Lemma 4.6.

$$\begin{aligned} &\mathrm{Hom}(P \otimes U, P \otimes U') \\ &\quad \simeq \mathrm{Hom}(\mathbf{1}, P^\vee \otimes P \otimes U^\vee \otimes U') \\ &\quad \simeq \mathrm{Hom}(\mathbf{1}, (\mathbf{1} \oplus Q) \otimes U^\vee \otimes U') \\ &\quad \simeq \mathrm{Hom}(\mathbf{1}, U^\vee \otimes U') \\ &\quad \simeq \mathrm{Hom}(U, U'). \end{aligned}$$

Then the assertion on Ext^1 follows from the above long exact sequence and induction on the ranks of U and U'. □

LEMMA 7.7. *Let (M, ∇) be a quasi-unipotent object in* $\mathrm{MC}(\mathcal{R}_K/K)$. *Then for some finite extension K' of K, $M \otimes K'$ is special.*

Proof. Replacing K by a finite extension, we can assume that $M \otimes_K L$ is indecomposable for any finite extension L of K. We shall show that, for some finite extension K' of K, $M \otimes_K K'$ has the form $P \otimes U$ with a geometrically irreducible étale object P and an indecomposable unipotent object U in $\mathrm{MC}(\mathcal{R}_{K'}/K')$. Let F be a finite separable extension of the residue field E of $\mathcal{R}_K$ such that $M \otimes_{\mathcal{R}_K} \mathcal{R}_K(F)$ is unipotent. Then $\mathcal{R}_K(F)$ is isomorphic to $K''\langle u \rangle^\dagger$ for some finite extension K'' of K. Let K' be a finite extension of K'' satisfying (3) such that the regular representation of $\mathrm{Gal}(F/E)$ decomposes to representations which are irreducible over the algebraic closure of K'. Replacing K by K', we can assume that every irreducible subobject of M is geometrically irreducible.

We use induction on the rank of M. Let P be an irreducible subobject of M and $M' = M/P$. We will show that M' is indecomposable, in which case the assertion of Lemma 7.7 follows immediately from the induction hypothesis and Lemma 7.6. In fact, if M' is decomposable, we can write $M' = \bigoplus_{i \in I} M'_i$ $(|I| > 1)$ with indecomposable components M'_i. By the induction hypothesis, each M'_i can be written in the form $P'_i \otimes U'_i$ with an irreducible object P'_i and an indecomposable unipotent object U'_i. If P'_i is not isomorphic to P, by Lemma 7.6, $\mathrm{Hom}_\nabla(M'_i, M)$ is isomorphic to $\mathrm{Hom}_\nabla(M'_i, M')$ and, hence, M'_i becomes a direct summand of M. Thus all P'_i's are isomorphic to P. Let M_i be the inverse image in M of $M'_i \subset M'$. Then M_i is a nontrivial extension of $M'_i = P \otimes U'_i$ by P and, hence, can be written in the form $P \otimes U_i$ for some indecomposable unipotent object U_i by Lemma 7.6. Consider the exact sequence

$$0 \to \mathrm{Hom}_\nabla(P, P) \to \mathrm{Hom}_\nabla(P, M) \to \mathrm{Hom}_\nabla(P, \oplus M'_i) \to \mathrm{Ext}^1_\nabla(P, P).$$

By Lemma 7.6, $\dim_K \mathrm{Hom}(P, M) > 1$ since $|I| > 1$ and, hence, there is a map $P \to M$ whose projection to some M'_i does not vanish. Then there exists an injection from $P \oplus P \to M_i$. This contradicts Lemma 7.6. □

THEOREM 7.8. *Every quasi-unipotent object in* $\mathrm{MC}(\mathcal{R}_K/K)$ *is special.*

Proof. Let (M, ∇) be a quasi-unipotent object in $\mathrm{MC}(\mathcal{R}_K/K)$. We can assume that M is indecomposable. We use induction on the rank of M. Let Q be an irreducible subobject of M. By Proposition 7.4, Q is étale. Let $N = M/Q$. By the induction hypothesis and the same argument as in the proof of Lemma 7.7, we can see that $N \simeq Q \otimes U_1$ for some indecomposable unipotent object U_1. By Lemma 7.7, there is a finite étale Galois extension K' of K such that $Q' = Q \otimes K'$ decomposes to $\bigoplus_{i \in I} P'_i$ with geometrically irreducible étale objects P'_i's in $\mathrm{MC}(\mathcal{R}_{K'}/K')$. Then we can find a subset S of $G = \mathrm{Gal}(K'/K)$ such that $Q' = \bigoplus_{\sigma \in S} \sigma(P')$. Here P' denotes one of P'_i's. Let $M' = M \otimes K'$ and $U'_1 = U_1 \otimes K'$, then we have a short exact sequence

$$0 \to \bigoplus_{\sigma \in S} \sigma(P') \to M' \to \bigoplus_{\sigma \in S} \sigma(P') \otimes U'_1 \to 0. \tag{16}$$

By the assumption that M is indecomposable, (16) does not split. Therefore, if we decompose M' into a direct sum $\bigoplus_{j \in J} D'_j$ of indecomposable subobjects D'_j, at least

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one of D'_j is isomorphic to $P' \otimes U'$ with an indecomposable unipotent object U' of rank $\mathrm{rank}(U_1)+1$. Denote it by D'. We can assume that $P' \subset D'$. Let $S = \{\sigma_1, \dots, \sigma_s\}$. We shall prove that $\sum_{i=1}^{s} \sigma_i(D') \subset M'$ is actually a direct sum $\bigoplus_{i=1}^{s} \sigma_i(D')$. Suppose it is not and take the minimum number $2 \leqslant i \leqslant s$ such that $\sigma_i(D') \cap \bigoplus_{j=1}^{i-1} \sigma_j(D') \neq 0$. Because D' is indecomposable, $\sigma_i(P')$ is contained in every nontrivial subobject of $\sigma_i(D')$ and, hence, $\sigma_i(P') \cap \bigoplus_{j=1}^{i-1} \sigma_j(D') \neq 0$. Then by the canonical isomorphisms

$$\begin{aligned} &\mathrm{Hom}_\nabla(\sigma_i(P'), \bigoplus_{j=1}^{i-1} \sigma_j(D')) \\ &\quad\simeq \bigoplus_{j=1}^{i-1} \mathrm{Hom}_\nabla(\sigma_i(P'), \sigma_j(D')) \\ &\quad\simeq \bigoplus_{j=1}^{i-1} \mathrm{Hom}_\nabla(\sigma_i(P'), \sigma_j(P')), \end{aligned}$$

we see that $\sigma_i(P') \cap \bigoplus_{j=1}^{i-1} \sigma_j(P') \neq 0$, which contradicts the definition of S. Thus $\bigoplus_{\sigma \in S} \sigma(D') \subset M'$. Comparing the ranks of both sides, we see $\bigoplus_{\sigma \in S} \sigma(D') = M'$ and hence

$$M' \simeq \bigoplus_{\sigma \in S} \sigma(P') \otimes U' \simeq Q' \otimes U'.$$

The assertion follows by Galois descent. □

COROLLARY 7.9. *If an object M in $\mathrm{MC}(\mathcal{R}_K/K)$ is a successive extension of étale objects, then it is special.*

COROLLARY 7.10. *Let (M, ∇) be a quasi-unipotent module with connection over $\mathcal{R}_{K,t}$. Then there exists a φ^n-structure on M for some $n \in \mathbb{Z}$.*

Proof. The assertion follows from Theorem 7.8 and Lemma 4.3.

PROPOSITION 7.11. *Let Q and Q' be objects of $\mathrm{MC}^{\mathrm{et}}(\mathcal{R}_K/K)$ and let U and U' be objects of $\mathrm{MC}^{\mathrm{uni}}(\mathcal{R}_K/K)$. Then we have*

$$\mathrm{Hom}_\nabla(Q, Q') \otimes \mathrm{Hom}_\nabla(U, U') \simeq \mathrm{Hom}_\nabla(Q \otimes U, Q' \otimes U').$$

Proof. Extending scalars, we are reduced to the case that Q and Q' are geometrically irreducible. Then the assertion follows from the calculation in the proof of Lemma 7.6. □

COROLLARY 7.12. *Let (M, ∇, φ_n) be a quasi-unipotent φ^n-∇-module over $\mathcal{R}_K$. By Lemma 7.7, for some finite extension K' of K, we can decompose $M \otimes K'$ into $\bigoplus_{i \in I} P_i \otimes U_i$ as ∇-module over $\mathcal{R}_{K'}$ with geometrically irreducible étale objects P_i and* *unipotent objects U_i such that P_i is not isomorphic to P_j for any $i \neq j$. Then, for some positive integer m, there exist φ^{mn}-structures $\varphi'_{mn,i}$ on P_i and $\varphi''_{mn,i}$ on U_i such that $(\varphi_n)^m = \bigoplus \varphi'_{mn,i} \otimes \varphi''_{mn,i}$.*

Proof. As in Remark 4.5, there is one-to-one correspondence between the set of φ^n-structures on (P, ∇) and the set of automorphisms of (P, ∇) if at least one φ^n-structure exists. By Proposition 7.11, we have

$$\begin{aligned}&\mathrm{Hom}_\nabla\Big(\bigoplus_{i\in I} P_i \otimes U_i, \bigoplus_{j\in I} P_j \otimes U_j\Big)\\ &\quad\simeq \bigoplus_{i,j\in I} \mathrm{Hom}_\nabla(P_i \otimes U_i, P_j \otimes U_j)\\ &\quad\simeq \bigoplus_{i\in I} \mathrm{Hom}_\nabla(P_i, P_i) \otimes \mathrm{Hom}_\nabla(U_i, U_i).\end{aligned}$$

and hence any automorphism of (M, ∇) has the form $\bigoplus g_i \otimes h_i$ with automorphisms g_i of P_i and h_i of U_i. Note that each $\mathrm{Hom}_\nabla(P_i, P_i)$ is one-dimensional vector space over K. On the other hand, if we take as m a number such that all P_i have φ^n-structures, then $M = \bigoplus P_i \otimes U_i$ has a φ^{mn}-structure of the form $\bigoplus \psi'_{mn,i} \otimes \psi''_{mn,i}$ with φ^{mn}-structures $\psi'_{mn,i}$ on P_i and $\psi''_{mn,i}$ on U_i. This means that any φ^{mn}-structure on (M, ∇) has the form $\bigoplus (g \circ \psi'_{mn,i}) \otimes (h_i \circ \psi''_{mn,i})$ and hence the assertion is clear. □

We denote the weak completion of $\mathcal{O}_K[t^{-1}]$ by $\tilde{\mathcal{A}}^\dagger$. Then $\tilde{\mathcal{A}}^\dagger$ is a subring of $A^\dagger$ and we have

$$\begin{aligned}\tilde{\mathcal{A}}^\dagger_K &:= \tilde{\mathcal{A}}^\dagger \otimes K = \Gamma(\hat{\mathbb{P}}^1_K, j^\dagger \mathcal{O}_{\hat{\mathbb{P}}^1_K})\\ &= \left\{ \sum_{i\leqslant 0} a_i t^i \,\middle|\, a_i \in K,\ |a_i|\rho^i \to 0, (i \to -\infty) \ \text{ for some } \rho < 1 \right\}\end{aligned}$$

where $j\colon\ \mathbb{A}^1_k = \operatorname{Spec} k[t^{-1}] \to \mathbb{P}^1_k$ is an inclusion map. We define a differential module with log pole at $t^{-1} = 0$ by

$$\Omega_{\tilde{\mathcal{A}}^\dagger_K} = \tilde{\mathcal{A}}^\dagger_K \frac{\mathrm{d}t}{t}.$$

COROLLARY 7.13. *Let M be a quasi-unipotent module with connection over $\mathcal{R}_K$. Then there exists a free submodule $\tilde{M}$ over $\tilde{\mathcal{A}}^\dagger_K$ such that $\tilde{M} \otimes_{\tilde{\mathcal{A}}^\dagger_K} \mathcal{R}_K \simeq M$ and that $\nabla(\tilde{M}) \subset \tilde{M} \otimes \Omega_{\tilde{\mathcal{A}}^\dagger_K}$.*

Proof. By Theorem 7.8, there is a decomposition on M into $\bigoplus_{i\in I} Q_i \otimes U_i$ with $Q_i \in \operatorname{Ob} \mathrm{MC}^{\mathrm{et}}(\mathcal{R}_K/K)$ and $U_i \in \operatorname{Ob} \mathrm{MC}^{\mathrm{uni}}(\mathcal{R}_K/K)$. By Theorem 4.1, there is a basis e of U_i such that $\nabla e = eC \otimes \mathrm{d}t/t$ with $C \in M(n_i; K)$. Here n_i is a rank of U_i. Thus there are sub-$\tilde{\mathcal{A}}^\dagger_K$-modules $\widetilde{U_i}$ of U_i such that $\nabla(\widetilde{U_i}) \subset \widetilde{U_i} \otimes \Omega_{\tilde{\mathcal{A}}^\dagger_K}$. We can equip each Q_i with finite unit-root φ^n-structure $\varphi_{n,i}$ for some positive integer n. By Galois descent, we can assume that $\mathbb{F}_{p^n} \subset k$. Let F be a finite Galois extension of the residue field E of $\mathcal{R}_K$ such that every $(Q_i, \nabla, \varphi_{n,i}) \otimes \mathcal{R}_K(F)$ is trivial. Let E' be the maximal tamely ramified extension of E in F and k' the residue field of F. Then $E \simeq k'((t'^{-1}))$ for some Nth root

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t' of t and F is a Galois extension of E' such that its Galois group is a p-group. By [19], there is a finite étale Galois covering $\widetilde{B}_0$ of $\widetilde{A}'_0 = k'[t'^{-1}]$ such that $\widetilde{B}_0 \otimes_{\widetilde{A}'_0} E' \simeq F$. Let $\mathcal{O}_{K'}$ be the integer ring of an unramified finite extension K' of K with residue field k' and $\widetilde{A}'^{\dagger}$ a weakly completion of $\mathcal{O}_{K'}[t'^{-1}]$. Here we denote an Nth root of $t \in A^{\dagger}$ also by t'. Since $(\widetilde{A}'^{\dagger}, \pi)$ is a Henselian couple, there exists a finite étale covering $\tilde{\mathcal{B}}^{\dagger}$ of $\widetilde{A}'^{\dagger}$ unique up to isomorphism such that $\tilde{\mathcal{B}}^{\dagger} \otimes_{\mathcal{O}_K} k \simeq \widetilde{B}_0$. Since $\tilde{\mathcal{B}}^{\dagger}$ is integrally closed in $B^{\dagger} = \tilde{\mathcal{B}}^{\dagger} \otimes_{\tilde{\mathcal{A}}^{\dagger}} A^{\dagger}$ and $B^{\dagger}$ is finite étale Galois over $A^{\dagger}$ with Galois group $G = \mathrm{Gal}(F/E)$, $\tilde{\mathcal{B}}^{\dagger}$ is also finite Galois over $\tilde{\mathcal{A}}^{\dagger}$ with the same Galois group. Let $\tilde{\mathcal{B}}^{\dagger}_K = \tilde{\mathcal{B}}^{\dagger} \otimes_{\mathcal{O}_K} K$ and $V_i = V_{\mathcal{R}_K,n}(Q_i)$ for each i. Then we define sub $\tilde{\mathcal{A}}^{\dagger}_K$-module $\widetilde{Q_i}$ of Q_i by

$$\widetilde{Q_i} = (V_i \otimes_{K_n} \tilde{\mathcal{B}}^{\dagger}_K)^G.$$

Let $\Omega_{\tilde{\mathcal{B}}^{\dagger}_K} = \Omega_{\tilde{\mathcal{A}}^{\dagger}_K} \otimes_{\tilde{\mathcal{A}}^{\dagger}} \tilde{\mathcal{B}}^{\dagger}$. It is easy to see that $d\colon \tilde{\mathcal{A}}^{\dagger}_K \to \Omega_{\tilde{\mathcal{A}}^{\dagger}_K}$ naturally extends to $d\colon \tilde{\mathcal{B}}^{\dagger}_K \to \Omega_{\tilde{\mathcal{B}}^{\dagger}_K}$ and hence $\widetilde{Q_i}$ has a connection $\nabla\colon \widetilde{Q_i} \to \widetilde{Q_i} \otimes \Omega_{\tilde{\mathcal{A}}^{\dagger}_K}$ which is compatible with that of Q_i. Then $\tilde{M} = \bigoplus \widetilde{Q_i} \otimes \widetilde{U_i}$ satisfies the condition. □

Remark 7.14. If we choose frobenius structure φ of $A^{\dagger}$ so that $\varphi(\tilde{\mathcal{A}}^{\dagger}) \subset \tilde{\mathcal{A}}^{\dagger}$, we can also equip $\tilde{M}$ with φ_n-structure which commutes with connection.

7.3. KATZ CORRESPONDENCE FOR QUASI-UNIPOTENT OBJECTS

THEOREM 7.15. *The inverse image functors*

$$\mathrm{MC}^{\mathrm{sp}}(A^{\dagger}_K/K) \to \mathrm{MC}^{\mathrm{qu}}(\mathcal{R}_K/K),$$
$$\mathrm{MCF}^{\mathrm{sp}}_n(A^{\dagger}_K/K) \to \mathrm{MCF}^{\mathrm{qu}}_n(\mathcal{R}_K/K)$$

are equivalences of categories.

Proof. The inverse image functor

$$\mathrm{MC}^{\mathrm{sp}}(A^{\dagger}_K/K) \to \mathrm{MC}^{\mathrm{sp}}(\mathcal{R}_K/K)$$

is an equivalence of categories by Corollary 4.2, Corollary 5.12, Lemma 7.11, and the analogous statement for $\mathrm{MC}^{\mathrm{sp}}(A^{\dagger}_K/K)$ of Lemma 7.11, which can be proven in the same way.

Then the equivalence of the first functor in (7.15) follows from Theorem 7.8. The case of the second is proven by the same argument as in the proof of Corollary 4.7. □

We call the quasi-inverse functor

$$\mathrm{MC}^{\mathrm{qu}}(\mathcal{R}_K/K) \to \mathrm{MC}^{\mathrm{sp}}(A^{\dagger}_K/K)$$

the *canonical extension*.

We denote the category of finite-dimensional K-vector spaces by Vect_K.

COROLLARY 7.16. *For any rational point $a \in X = \mathrm{Spec}(A^{\dagger} \otimes k)$ (cf. Section 5.1), the composite*

$$\mathrm{MC}^{\mathrm{qu}}(\mathcal{R}_K/K) \to \mathrm{MC}^{\mathrm{sp}}(A^{\dagger}_K/K) \xrightarrow{\text{fiber at a}} \mathrm{Vect}_K$$

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is a K-valued fiber functor on $\mathrm{MC}^{\mathrm{qu}}(\mathcal{R}_K/K)$*, and hence* $\mathrm{MC}^{\mathrm{qu}}(\mathcal{R}_K/K)$ *is a neutral Tannakian category [12].*

By (7.15), we also have the next functor.

$$\mathrm{MCF}_n^{\mathrm{qu}}(\mathcal{R}_K/K) \to \mathrm{MCF}_n^{\mathrm{sp}}(A_K^\dagger/K) \xrightarrow{\text{fiber at a}} \overset{n}{\varphi} - \mathrm{Vect}_{\mathrm{K}}\,.$$

Here φ^n-Vect_K is the category of finite-dimensional K-vector spaces with injective φ^n-linear morphisms.

8. Swan Conductor and Irregularity

Let the notation and the assumption be as in the previous section.

8.1. BREAKS AND BREAK DECOMPOSITIONS

First we define breaks for the quasi-unipotent local overconvergent isocrystals. Let M be an object in $\mathrm{MC}^{\mathrm{qu}}(\mathcal{R}_K/K)$. By Theorem 7.8, M has a decomposition into a direct sum $\bigoplus_{i\in I} Q_i \otimes U_i$ with irreducible étale objects Q_i and indecomposable unipotent objects U_i. By Jordan–Hölder theory and Krull–Remak–Schmidt theory, the isomorphism classes of $Q_i \otimes U_i$'s and Q_i's are intrinsic invariants of M. By Corollary 5.9, each Q_i has at least one finite unit-root φ^n-structure. By the lemma below, we can define the breaks of M as those of Q_i.

LEMMA 8.1. *Let (Q, ∇) be an object in* $\mathrm{MC}^{\mathrm{et}}(\mathcal{R}_K/K)$ *and let ψ be any finite unit-root φ^n-structure on (Q, ∇). Then the breaks of $V = V_{\mathcal{R}_K,n}(Q, \nabla, \psi)$ is independent of the choice of ψ.*

Proof. We begin by remarking that the breaks do not change by finite extension of K and an unramified extension of the residue field E of $\mathcal{R}_{K,t}$ [21, Chap. 1]. Let ψ_1 (resp. ψ_2) be a finite unit-root φ^n-structure (resp. φ^m-structure) on (Q, ∇) and put $V_i = V_{\mathcal{R}_K,n}(Q, \nabla, \psi_i)$ for $i = 1, 2$. Replacing m, n by those common multiple, we can assume that $m = n$. We can also assume that the residue field of K contains $\mathbb{F}_{p^n}$. Take a finite extension F of the residue field E of $\mathcal{R}_K$ as in Section 5.2. Then we have isomorphisms g_i: $Q \otimes_{\mathcal{R}_K} \mathcal{R}_K(F) \to V_i \otimes_{K_n} \mathcal{R}_K(F)$ compatible with actions of $G = \mathrm{Gal}(F/E)$ and connections. Extending scalars, we can assume that the residue field of F is same with that of E. Put $g = g_2 \circ g_1^{-1}$. Taking kernels of connection, we have an isomorphism $g|_{V_1 \otimes_{K_n} K}$: $V_1 \otimes_{K_n} K \to V_2 \otimes_{K_n} K$ which is compatible with Galois actions. Thus the breaks of V_1 coincide with those of V_2. □

We say that M is *purely of break* x if either $M = 0$ or if all the breaks of M are equal to x.

LEMMA 8.2. *Let M be an object in* $\mathrm{MC}^{\mathrm{qu}}(\mathcal{R}_K/K)$*, then M has a unique direct-sum decomposition $M = \bigoplus_{x \geqslant 0} M(x)$ into subobjects $M(x)$ in* $\mathrm{MC}^{\mathrm{qu}}(\mathcal{R}_K/K)$*, indexed by* *real numbers* $x \geqslant 0$*, such that* $M(x)$ *is purely of break* x*. Moreover, this decomposition is compatible with finite scalar extensions.*

Proof. By Theorem 7.8, M has a decomposition into $\bigoplus_{i \in I} Q_i \otimes U_i$ with étale irreducible objects Q_i and unipotent objects U_i. For some finite extension K' of K satisfying (3), $Q_i \otimes K'$ decomposes into the direct sum $\bigoplus_{j \in J} P_{i,j}$ of geometrically irreducible objects $P_{i,j}$. It is easy to see that $P_{i,j} \simeq P_{i,j'}$ for any $j, j' \in J$, and hence the breaks of the corresponding representation is pure (cf. [21, Chap. 1]). Thus taking the tensor product of the break decomposition of the individual Q_i and U_i, we can see that M has a decomposition $\bigoplus_x M(x)$ of the desired sort. Since breaks do not change after a finite scalar extension, the decomposition also does not change.

Next we show uniqueness. After extending scalars, we can assume that any irreducible subobjects are geometrically irreducible. Each $M(x)$ has a decomposition into $\bigoplus_{i \in I_x} P_i \otimes U_i$ with irreducible objects P_i and unipotent objects U_i. Let $\bigoplus_x M'(x)$ be another decomposition of M such that $M'(x)$ has purely of break x. It suffices to show that $M'(x) \subset M(x)$ for any x. We show that any indecomposable subobject D' of $M'(x)$ is contained in $M(x)$. Let P' be an irreducible subobject of D'. If $x' \neq x$, $M(x')$ has no irreducible subobject isomorphic to P', and hence

$$\mathrm{Hom}_{\nabla}(D', M) = \bigoplus_{i \in I_x} \mathrm{Hom}_{\nabla}(D', P_i \otimes U_i)$$

by Lemma 7.6. This implies that $D' \subset M(x)$. □

DEFINITION 8.3. We call the decomposition $M = \bigoplus M(x)$ of the above lemma the break-decomposition of M. We define the Swan conductor of M as $\sum_x x \dim M(x)$ and denote it by $\mathrm{sw}(M)$.

It follows from the Hasse-Arf theorem and the next lemma that $\mathrm{sw}(M)$ is a non-negative integer. Cf. [26, IV, §3], [27, III, §19] and [21, 1.9].

By definition and [21, Chap. 1], we have the following lemmas.

LEMMA 8.4. *Let*

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of quasi-unipotent modules with connections over $\mathcal{R}_K$*, then we have* $\mathrm{sw}(M) = \mathrm{sw}(M') + \mathrm{sw}(M'')$ *and*

$$(\textit{breaks of } M) = (\textit{breaks of } M') \cup (\textit{breaks of } M'').$$

LEMMA 8.5. *Let* M *and* N *be quasi-unipotent modules with connections over* $\mathcal{R}_K$ *and* $M = \bigoplus M(x)$ *and* $N = \bigoplus N(x)$ *their break-decompositions. Then,*

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$$M(x)\otimes N(y)\subset (M\otimes N)(\sup(x,y))\quad \textit{if } x\neq y,$$
$$M(x)\otimes N(x)\subset \sum_{y\leqslant x}(M\otimes N)(y),$$
$$\mathrm{Hom}(M(x),N(y))\subset \mathrm{Hom}(M,N)(\sup(x,y))\quad \textit{if } x\neq y,$$
$$\mathrm{Hom}(M(x),N(x))\subset \sum_{y\leqslant x}\mathrm{Hom}(M,N)(y),$$

and if we denote the dual of M *by* $M^\vee$,

$$M^\vee(x)=M(x)^\vee.$$

We denote by $\mathrm{MC}^{\mathrm{qu}}_{(\leqslant x)}(\mathcal{R}_K/K)$ the full subcategory of $\mathrm{MC}^{\mathrm{qu}}(\mathcal{R}_K/K)$ of objects all whose breaks are $\leqslant x$. Similarly we denote by $\mathrm{MC}^{\mathrm{qu}}_{(<x)}(\mathcal{R}_K/K)$ the full subcategory of $\mathrm{MC}^{\mathrm{qu}}(\mathcal{R}_K/K)$ of objects all of whose breaks are $<x$. By the above lemmas, both $\mathrm{MC}^{\mathrm{qu}}_{(\leqslant x)}(\mathcal{R}_K/K)$ and $\mathrm{MC}^{\mathrm{qu}}_{(<x)}(\mathcal{R}_K/K)$ are stable by tensor product, internal hom, and subquotient.

8.2. RELATION WITH THE CHRISTOL–MEBKHOUT THEORY

When k is a finite field, Christol and Mebkhout defined the filtration with respect to irregularity and proved an index formula [5, 6].

Our decomposition and Swan conductor are compatible with their filtration and irregularity. More precisely, we have the next theorem.

THEOREM 8.6. *Let M be an object in* $\mathrm{MC}^{\mathrm{qu}}(\mathcal{R}_K/K)$. *Then the Swan conductor of M coincides with the irregularity of M in the sense of Christol and Mebkhout* [5, Def. 8.3-8].

Proof. This is an immediate consequence of Tsuzuki's theorem [29], but we give another proof using the canonical extension.

Let $\tilde{M}$ and $\tilde{\mathcal{A}}_K^\dagger$ be as in Corollary 7.13. Note that $\tilde{M}$ is free $\tilde{\mathcal{A}}_K^\dagger$-module since $\tilde{\mathcal{A}}_K^\dagger$ is principal ideal domain. (By [14], $\tilde{\mathcal{A}}_K^\dagger$ is noetherian and the same argument as in the proof of [10, 6.1] shows that it is a Bezout ring.) We denote the index of $\tilde{M}$ by

$$\chi(\tilde{M})=\dim \mathrm{Ker}(\nabla\colon \tilde{M}\to \tilde{M}\otimes\Omega_{\tilde{\mathcal{A}}_K^\dagger})-\dim \mathrm{Cok}(\nabla\colon \tilde{M}\to \tilde{M}\otimes\Omega_{\tilde{\mathcal{A}}_K^\dagger}).$$

Then $-\chi(\tilde{M})$ coincides with the irregularity of M. Cf. [5, Def. 8.2-9]. They use projection operators to define the generalized index, but it coincides with the index above because $\nabla(\tilde{M})\subset \tilde{M}\otimes\Omega_{\tilde{\mathcal{A}}_K^\dagger}$. We regard $M^\dagger=\tilde{M}\otimes_{\tilde{\mathcal{A}}_K^\dagger}A_K^\dagger$ as an overconvergent isocrystal on $\mathbb{G}_k$ and denote the alternating sum of the dimensions of the rigid cohomology groups $\mathrm{H}^i_{\mathrm{rig}}(\mathbb{G}_m, M^\dagger)$ by $\chi(M^\dagger)$. Since index and Swan conductor are both additive, we can reduce the assertion to the case where M is irreducible and hence finite étale. Therefore we can assume that $M^\dagger=D_{A^\dagger,n}(V)$ for some integer n and $V\in \mathrm{Rep}^{\mathrm{sp}}_{K_n}(\pi_1(A_k^\dagger,*))$ with some φ^n-structure.

We first prove that

$$\chi(\tilde{M})=\chi(M^\dagger). \tag{17}$$

We denote the composition

$$\tilde{M} \xrightarrow{\nabla} \tilde{M} \otimes \Omega_{\tilde{\mathcal{A}}_K^\dagger} \xrightarrow{\mathrm{id}\otimes\mathrm{Res}} \tilde{M}$$

$$(\text{resp. } M^\dagger \xrightarrow{\nabla} M^\dagger \otimes \Omega_{\tilde{\mathcal{A}}_K^\dagger} \xrightarrow{\mathrm{id}\otimes\mathrm{Res}} M^\dagger)$$

by $\nabla(\partial)$. Then the index of $\nabla(\partial)\colon \tilde{M} \to \tilde{M}$ (resp. $\nabla(\partial)\colon M^\dagger \to M^\dagger$) is of course equal to $\chi(\tilde{M})$ (resp. $\chi(M^\dagger)$). Let N be the cokernel of $\tilde{M} \to M^\dagger$. If we denote by $\chi(N)$ the index of the induced endomorphism on N from $\nabla(\partial)\colon M^\dagger \to M^\dagger$, we have $\chi(\tilde{M}) = \chi(M^\dagger) + \chi(N)$, and hence it suffices to show that $\chi(N) = 0$. Let $M_\infty = M^\dagger \otimes \mathcal{R}_{t^{-1}}$. The cokernel of $\tilde{\mathcal{A}}_K^\dagger \to A_K^\dagger$ is isomorphic to the K-vector space

$$\mathcal{H}_{t^{-1}}^\dagger = \left\{ \sum_{i>0} a_i t^i \in K[[t]] \;\middle|\; \lambda > 1,\ |a_i|\lambda^i \to 0\ (i \to \infty) \right\}.$$

Let r be the rank of M and fix an isomorphism $\bigoplus^r \mathcal{R}_{t^{-1}} \simeq M_\infty$. $\chi(N)$ can be regarded as the generalized index $\chi(\nabla(\partial), \mathcal{H}_{t^{-1}}^\dagger)$ of Christol and Mebkhout [5, Def. 8.2-9] and hence it does not depend on the choice of basis (see remark following Definition 8.2-7 of loc. cit.). Since V is tamely ramified at ∞, (M_∞, ∇) is trivialized by tensoring $\mathcal{R}_{t^{-1}}(F)$ for some tamely ramified extension F of the residue field E of $\mathcal{R}_{t^{-1}}$. After extending scalars, we can assume that $F = E(t^{1/l})$ for some integer l prime to p and that k contains the lth root of unity. Then we can prove $\chi(N) = 0$ by direct calculation (cf. [24, Lemma 5.3]).

Next let U be a special Galois covering of $\mathbb{G}_{m,k}$ with Galois group G which trivializes V and $B^\dagger$ the corresponding Galois extension of $A^\dagger$. We denote $B_K^\dagger = B^\dagger \otimes K$. We claim that

$$\mathrm{H}^i_{\mathrm{rig}}(\mathbb{G}_{m,k}, M^\dagger) = (\mathrm{H}^i_{\mathrm{rig}}(U/K) \otimes_{K_n} V)^G. \tag{18}$$

Indeed, $\mathrm{H}^i_{\mathrm{rig}}(\mathbb{G}_{m,k}, M^\dagger)$ (resp. $\mathrm{H}^i_{\mathrm{rig}}(U/K)$) are cohomology groups of a complex

$$M^\dagger \to M_K^\dagger \otimes_{A_K^\dagger} \Omega_{A_K^\dagger} \quad (\text{resp. } B_K^\dagger \to B_K^\dagger \otimes_{A_K^\dagger} \Omega_{A_K^\dagger}).$$

Since the dual $V^\vee$ of V is projective $K_n[G]$-module, the functor from $K_n[G]$-vector space to K_n-vector space

$$P \mapsto (V \otimes_{K_n} P)^G = \mathrm{Hom}_{K_n[G]}(V^\vee, P)$$

is an exact functor. Since $M^\dagger = (V \otimes_{K_n} B^\dagger)^G$, the assertion is clear.

Let $\mathrm{H}^i_{\mathrm{c,rig}}(U/K)$ be compact support rigid cohomology groups of U. We denote the alternating sum of the trace of the action of σ of G on $\mathrm{H}^i_{\mathrm{rig}}(U/K)$ (resp. $\mathrm{H}^i_{\mathrm{c,rig}}(U/K)$) by

$$\mathrm{tr}(\sigma : \mathrm{H}^*_{\mathrm{rig}}(U/K)) := \sum_{i=0}^{2} (-1)^i \mathrm{tr}(\sigma : \mathrm{H}^i_{\mathrm{rig}}(U/K)),$$

$$(\text{resp. } \mathrm{tr}(\sigma : \mathrm{H}^*_{\mathrm{c,rig}}(U/K)) := \sum_{i=0}^{2} (-1)^i \mathrm{tr}(\sigma : \mathrm{H}^i_{\mathrm{c,rig}}(U/K))).$$

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By Poincaré duality [10, 9.5],

$$\operatorname{tr}(\sigma : \mathrm{H}^*_{\mathrm{rig}}(U/K)) = \operatorname{tr}(\sigma : \mathrm{H}^*_{\mathrm{c,rig}}(U/K)). \tag{19}$$

On the other hand, by the fact that crystalline cohomology is a Weil cohomology [16] and by the comparison theory of rigid cohomology and crystalline cohomology [3, Prop. 1.9], we can show the Weil formula:

$$\operatorname{tr}(\sigma : \mathrm{H}^*_{\mathrm{c,rig}}(U/K)) = -\mathrm{sw}_0(\sigma). \tag{20}$$

in the same way as in [17] (cf. [18]). Here sw_0 is a Swan character at 0. By (19) and (20), we have

$$\sum_{i=0}^{2}(-1)^i \dim \mathrm{H}^i_{\mathrm{rig}}(\mathbb{G}_{m,k}, M^\dagger) = -\frac{1}{|G|}\sum_{\sigma} \mathrm{sw}_0(\sigma)\operatorname{tr}(\sigma : V) = -\mathrm{sw}(V).$$

Here $\mathrm{sw}(V)$ denotes the Swan conductor of V as an representation of $\mathrm{Gal}(k((t))^{\mathrm{sep}}/k((t)))$ via natural injection $\mathrm{Gal}(k((t))^{\mathrm{sep}}/k((t))) \hookrightarrow \pi_1(\mathbb{G}_{k,m}, *)$. By definition, it is nothing other than $\mathrm{sw}(M)$. Thus, by (17) and (18), we have $\chi(\tilde{M}) = -\mathrm{sw}(M)$. This completes the proof. □

Remark 8.7. Richard Crew proved the same result in case that M is étale [9].

COROLLARY 8.8. *If we denote the filtration of Christol and Mebkhout by $M_{>\gamma}$ as in* [5], *we have $M_{>\gamma} = \bigoplus_{x>\gamma} M(x)$.*

Proof. We first remark that for any subobject $N \subset M$, we have $M_{>\gamma} \cap N = N_{>\gamma}$ since $(-)_{>\gamma}$ is an exact functor [5, Prop. 6.3-1]. Therefore, if $M = \bigoplus M_i$, then $M_{>\gamma} = \bigoplus (M_i)_{>\gamma}$, and it suffices to show the assertion in case that $M = M(x)$. Since $\mathrm{Gr}_\gamma M(x)$ is pure with respect to both the break filtration defined above and the Christol–Mebkhout filtration, it coincides with $M(x)$ if $\gamma = x$ by Theorem 8.6, [5, Prop. 8.3-1] and [6, Prop. 2.1-2], and hence it is 0 if $\gamma \neq x$. This proves the assertion. □

8.3. DIFFERENTIAL GALOIS GROUPS

We fix a K-rational point $a \in \mathbb{G}_m$ and denote by ω the fiber functor defined in Corollary 7.16.

We denote

$$W = W(\mathcal{R}_K/K) = \mathrm{Aut}^{\otimes}(\omega).$$

This is an analogue of local Galois group of a local field of positive characteristic. We also define as in [20, 2.5] the upper numbering filtration $W^{(x)} = W(\mathcal{R}_K/K)^{(x)}$ (resp. $W^{(x+)} = W(\mathcal{R}_K/K)^{(x+)}$) to be the kernel of the fully faithful homomorphism

$$W \to \mathrm{Aut}^{\otimes}(\omega|_{\mathrm{MC}^{\mathrm{qu}}_{(<x)}(\mathcal{R}_K/K)})$$

$$(\text{resp. } W \to \mathrm{Aut}^{\otimes}(\omega|_{\mathrm{MC}^{\mathrm{qu}}_{(\leqslant x)}(\mathcal{R}_K/K)})).$$

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Then, for $0 < x < y$, we have

$$W \supset W^{(0+)} \supset W^{(x)} \supset W^{(x+)} \supset W^{(y)} \supset W^{(y+)}.$$

We can regard $W/W^{(0+)}$ as the regular singular part of W.

For an object $M = (M, \nabla)$ of $\mathrm{MC}(\mathcal{R}_K/K)$, we denote by $\langle M \rangle$ the full subcategory of $\mathrm{MC}(\mathcal{R}_K/K)$ whose objects are all the subquotient of all finite direct sums of the objects $M^{\otimes n} \otimes (M^\vee)^{\otimes m}$ for all $n, m \geqslant 0$, i.e., the smallest rigid tensor subcategory of $\mathrm{MC}(\mathcal{R}_K/K)$ containing M. If M is quasi-unipotent, $\langle M \rangle$ is a neutral Tannakian category. We denote the group scheme $\mathrm{Aut}^{\otimes}(\omega|\langle M \rangle)$ by $\mathrm{DGal}(M)$.

Let K' be a finite extension of K, then there exists a fiber functor $\omega_{K'}$ from $\mathrm{MC}^{\mathrm{qu}}(\mathcal{R}'_K/K')$. We denote $\mathrm{Aut}^{\otimes}(\omega_{K'}|\langle M \otimes K' \rangle)$ by $\mathrm{DGal}(M \otimes K')$, then there is a natural closed immersion $\mathrm{DGal}(M \otimes_K K') \hookrightarrow \mathrm{DGal}(M) \otimes_K K'$.

PROPOSITION 8.9. *For a finite extension K' of K, we have*

$$\mathrm{DGal}(M \otimes_K K') \simeq \mathrm{DGal}(M) \otimes_K K'.$$

Proof. The proof is almost same as Gabber's proof of [20, (1.3.2), (2.4.15)]. Cf. [8, 2.1]. □

We denote $I = I(\mathcal{R}_K/K) = \mathrm{Aut}^{\otimes}(\omega|_{\mathrm{MC}^{\mathrm{et}}(\mathcal{R}_K/K)})$. By Proposition 7.11, we have

$$\mathrm{MC}^{\mathrm{qu}}(\mathcal{R}_K/K) \simeq \mathrm{MC}^{\mathrm{et}}(\mathcal{R}_K/K) \otimes \mathrm{MC}^{\mathrm{uni}}(\mathcal{R}_K/K)$$

and hence

$$W = I \times \mathbb{G}_a$$

(cf. [11, 5.13] and [11, 6.21]).

CONJECTURE 8.10. *Let E be as in Section* 2.1 *and P the constant pro-algebraic group associated with pro-p-part of* $\mathrm{Gal}(E^{\mathrm{sep}}/E)$. *Then there is an exact sequence of pro-algebraic groups*

$$1 \to P \to I \to \varinjlim_{p \nmid N} \mu_N \to 1.$$

Acknowledgements

A part of this work was done during a stay at Padova University in the spring of 1997. The author would like to thank Francesco Baldassarri and Bruno Chiarellotto for their hospitality and advice. The author also would like to thank Nobuo Tsuzuki for useful discussion.

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